Congruence Subgroups of the Braid Group

Wade Bloomquist

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Vague Big Picture

- We will look at a representation of the braid group (the integral Burau representation)
- I claim this is a natural example for us
- I'll also give you a different perspective

Shout Out

Joint with Nancy Scherich and Peter Patzt

"Quotients of braid groups by their congruence subgroups"–arxiv:2209.09889

$\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Mapping Class Group acts on Homology

$$\mathrm{MCG}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma, \mathbb{Z}))$$

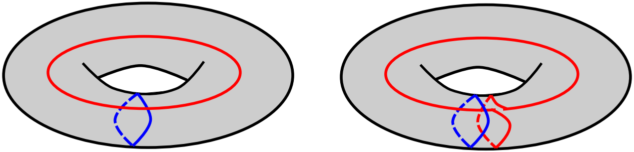
$$f : \Sigma \to \Sigma$$

$$f_* : H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$$

$$f_*([\gamma]) = [f(\gamma)]$$

Back to $\mathrm{SL}(2,\mathbb{Z})$

$H_1(T^2,\mathbb{Z}) \cong \mathbb{Z}^2$ (meridian and longitude)

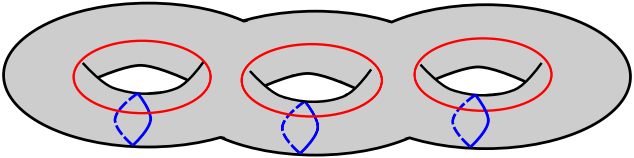


$\mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Back to Genus g

$$H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$



$\mathrm{Aut}(\mathbb{Z}^{2g})$ is $\mathrm{GL}(2g, \mathbb{Z})$

$$\mathrm{MCG}(\Sigma_g) \to \mathrm{GL}(2g, \mathbb{Z})$$

But where in $GL(2g, \mathbb{Z})$

$$I : H_1(\Sigma_g, \mathbb{Z}) \times H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{Z}$$

algebraic intersection is preserved

$$\hat{\rho} : \mathrm{MCG}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

$$\hat{\rho}(T_b([a])) = [a] + I(a, b)[b]$$

The Symplectic Representation

Theorem (Burkhardt (1890))

The map

$$\hat{\rho} : \mathrm{MCG}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

is surjective.

Immediate corollary

$$\mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Congruence Subgroups of Symplectic Groups

$$r_m : \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Level m subgroup

$$\mathrm{Sp}(2g, \mathbb{Z})[m] := ker(r_m)$$

$$Id_{2g} + mX \in \mathrm{Sp}(2g, \mathbb{Z})$$

$$\mathrm{Sp}(2g, \mathbb{Z})/\mathrm{Sp}(2g, \mathbb{Z})[m] \cong \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Congruence Subgroups of Mapping Class Groups

$$\hat{\rho} : \mathrm{MCG}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

Level *m* subgroup

$$\mathrm{MCG}(\Sigma_g)[m] := ker(r_m \circ \hat{\rho})$$

$$\hat{\rho}(f) = Id_{2g} + mX \in \mathrm{Sp}(2g, \mathbb{Z})$$

$$\mathrm{MCG}(\Sigma_g)[m] = (\hat{\rho})^{-1}(\mathrm{Sp}(2g, \mathbb{Z})[m])$$

The mod m Symplectic Representation

$$\mathrm{Im}(r_m \circ \hat{\rho}) = \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

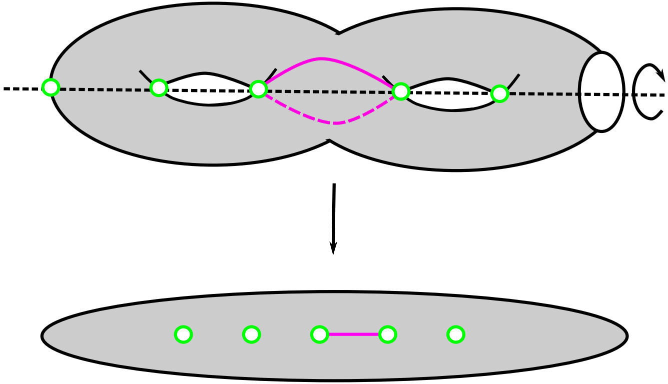
$$\mathrm{MCG}(\Sigma_g)/\mathrm{MCG}(\Sigma_g)[m] \cong \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Braid Groups

You were promised braid groups!

Hyperelliptic involution and Birman-Hilden

$$B_{2g+1} \hookrightarrow \mathrm{MCG}(\Sigma_g^1)$$



Integral Burau Representation and Congruence Subgroups

$$\rho : B_{2g+1} \hookrightarrow \mathrm{MCG}(\Sigma_g^1) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z})$$

The level m braid group

$$B_{2g+1}[m] := ker(r_m \circ \rho)$$

$$B_{2g+1}[m] = \rho^{-1}(\mathrm{Sp}(2g, \mathbb{Z})[m])$$

The Hurdle

$$\rho : B_{2g+1} \to \mathrm{Sp}(2g, \mathbb{Z})$$

is **not** surjective

So what is the image? or

$$B_{2g+1}/B_{2g+1}[m] \cong \quad ?$$

The Image

Theorem (B.-Patzt-Scherich 2022)

For $g \geq 2$ and ℓ odd

$$B_{2g+1}/B_{2g+1}[2^k\ell] \cong \mathcal{Z}_{g,k} \times \mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$$

where $\mathcal{Z}_{g,k}$ is a non-split extension of S_n by $\mathrm{Sp}(2g,\mathbb{Z})[2]/\mathrm{Sp}(2g,\mathbb{Z})[2^k]$.

Our paper covers the $g = 1$ and n even cases as well

Theorem (B.-Patzt-Scherich 2022)

We describe $B_n/B_n[m]$.

Split it up

Lemma

For $(m, \ell) = 1$

$$B_{2g+1}/B_{2g+1}[m\ell] \cong B_{2g+1}/B_{2g+1}[m] \times B_{2g+1}/B_{2g+1}[\ell].$$

- $B_n \cong B_n[m] \cdot B_n[\ell] \qquad \sigma_i = \sigma_i^{am+b\ell} = (\sigma_i^m)^a(\sigma_i^\ell)^b$
- $B_n[m\ell] = B_n[m] \cap B_n[\ell]$
- $HK/(H \cap K) \cong HK/H \times HK/K$

So

$$B_{2g+1}/B_{2g+1}[p^k] \cong \quad ?$$

What was known?

- $B_{2g+1}/B_{2g+1}[2] \cong S_n$ (Arnol'd 1968)
- $B_{2g+1}/B_{2g+1}[p] \cong \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ (A'Campo 1979)
- $B_{2g+1}/B_{2g+1}[4] \cong$ a non-split extension of S_n by $(\mathbb{Z}/2\mathbb{Z})^{\binom{n}{2}}$ (Kordek-Margalit 2019)

And a slightly different key tool

- $B_{2g+1}[2] \twoheadrightarrow Sp(2g, \mathbb{Z})[2]$ (Brendle-Margalit 2018)
- $B_{2g+1}[2\ell] \twoheadrightarrow Sp(2g, \mathbb{Z})[2\ell]$

The Five Lemma

$$
\begin{array}{ccccc}
B_{2g+1}[p]/B_{2g+1}[p^k] & \hookrightarrow & B_{2g+1}/B_{2g+1}[p^k] & \twoheadrightarrow & B_{2g+1}/B_{2g+1}[p] \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle ACampo} \\
\mathrm{Sp}(2g,\mathbb{Z})[p]/\mathrm{Sp}(2g,\mathbb{Z})[p^k] & \hookrightarrow & \mathrm{Sp}(2g,\mathbb{Z}/p^k\mathbb{Z}) & \twoheadrightarrow & \mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})
\end{array}
$$

$$
\begin{array}{ccccc}
B_{2g+1}[2]/B_{2g+1}[2^k] & \hookrightarrow & B_{2g+1}/B_{2g+1}[2^k] & \twoheadrightarrow & B_{2g+1}/B_{2g+1}[2] \\
 & & & & \downarrow {\scriptstyle ACampo} \\
\Big\downarrow & & \Big\downarrow & & S_{2g+1} \\
 & & & & \hookdownarrow \\
\mathrm{Sp}(2g,\mathbb{Z})[2]/\mathrm{Sp}(2g,\mathbb{Z})[2^k] & \hookrightarrow & \mathrm{Sp}(2g,\mathbb{Z}/2^k\mathbb{Z}) & \twoheadrightarrow & \mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z})
\end{array}
$$

Summary- mod m integral Burau

$$r_m \circ \rho : B_{2g+1} \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

We described the image of this map

The mod p integral Burau representation

The villian revealed

Untwisted Dijkgraaf-Witten theory with dihedral gauge group