Dax's Work on Embedding Spaces

David Gabai
Princeton University

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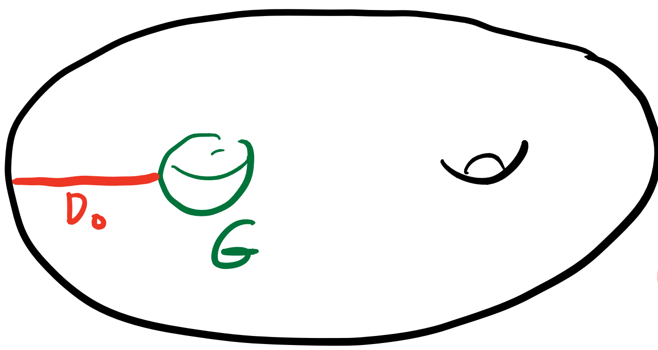
Question

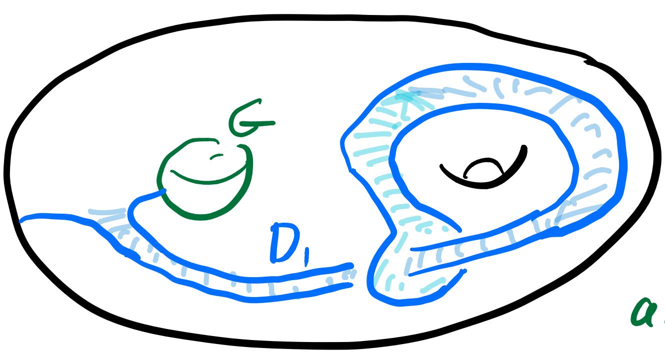
$M^4 =$



$S^2\times D^2 \natural S^1\times B^3$

$\times [-1,1]$

$D_0 =$ std vertical disc



$D_1 =$ self-referential disc

$= D_0$ tubed to an unknotted S^2

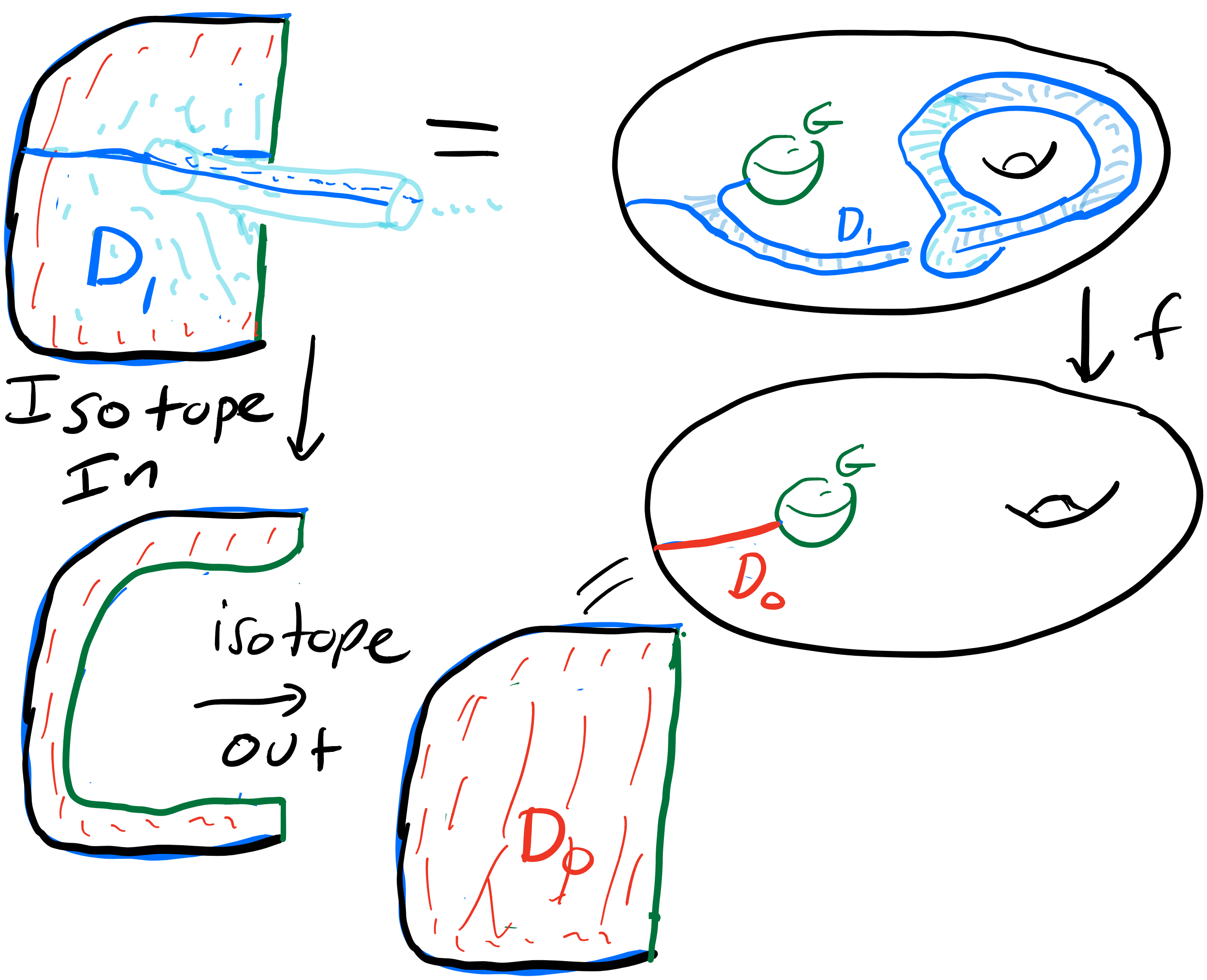
Is D_1 isotopic to D_0 via an isotopy fixing ∂D_1 pointwise?

Fact There exists

$$f:(M,D_1)\longrightarrow(M,D_0)$$

such that $f\simeq id$ and $f|\partial M = id$

Idea of Proof (Based on H. Schwartz argument for spheres using Cerf-Palais (1960))



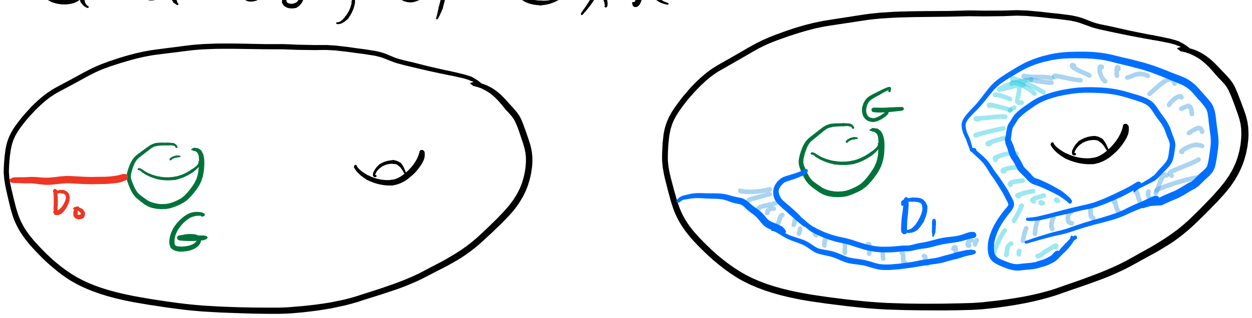
Theorem (Cerf, Palais 1960) If M is connected, then any two k-balls are ambiently isotopic. If $k=n$, then assume their orientations are induced from M

Key idea If $f:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(0)=0$ then

$$f_t(x) = \begin{cases} \dfrac{f(tx)}{t} & 0<t<1 \\ df_0(x) & t=0 \end{cases}$$

is an isotopy of f to a linear map

Fact If $M_1 = M \cup 2$-handle attached along $\partial D_0 = \partial D_1$ and S_0, S_1 extensions of



D_0, D_1 to spheres, then S_1 is isotopic to S_0 via an isotopy fixing G pointwise.

Theorem (G, JAMS 2020)
4-D Lightbulb theorem

If S_0, S_1 homotopic embedded 2-spheres in M^4 with common dual sphere G and $\pi_1(M)$ has no nontrivial elements of order 2 then S_1 is isotopic to S_0 via an isotopy fixing G pointwise.

G dual sphere means

1) $|G \pitchfork S_i| = 1$

2) G has a trivial normal bundle

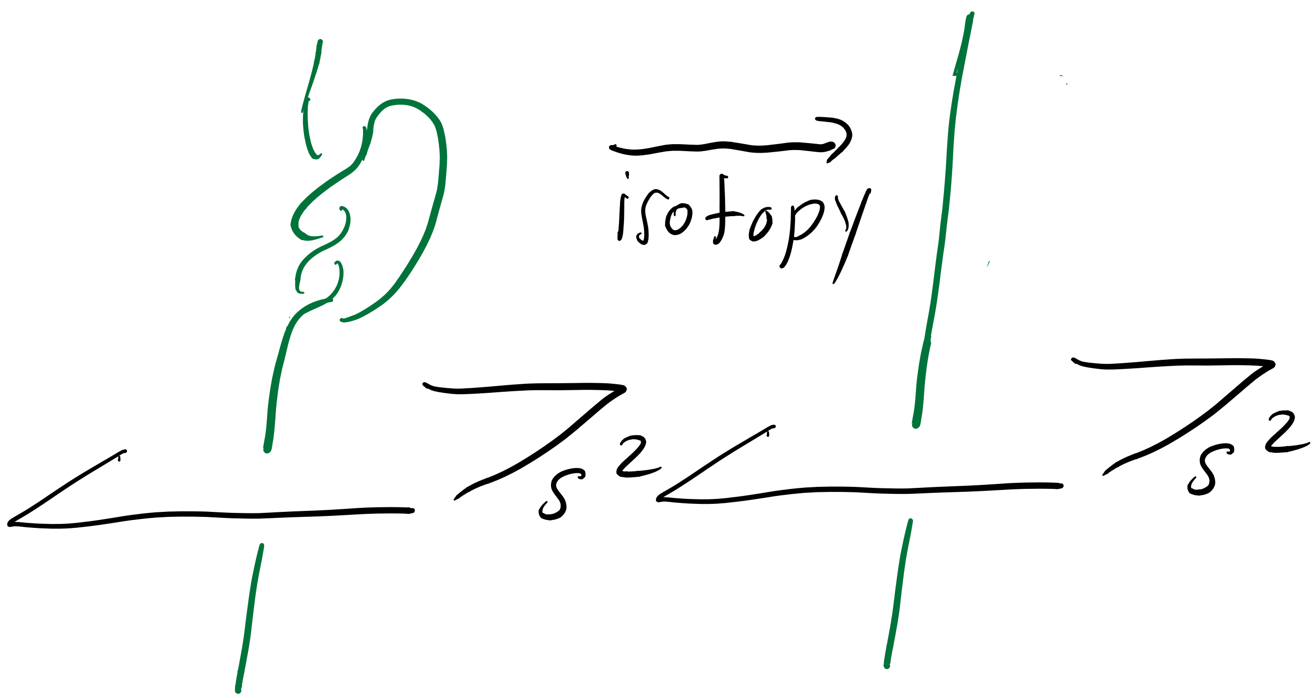
3D Lightbulb Theorem

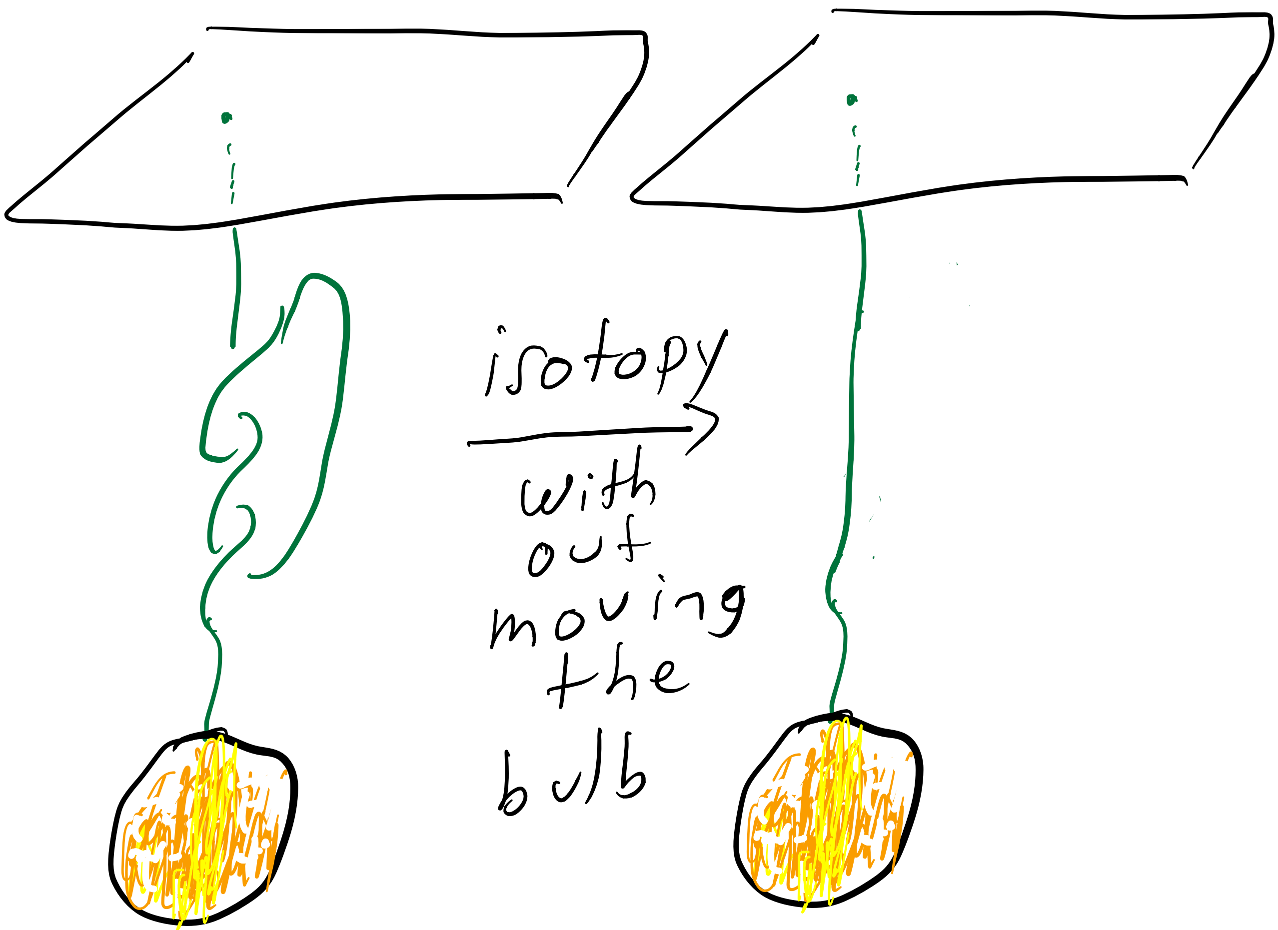
K knot in $S^1 \times S^2$

$K \cap (pt \times S^2) = 1\ pt$ then

K is isotopic to $S^1 \times pt$

by isotopy fixing $pt \times S^2$ pointwise

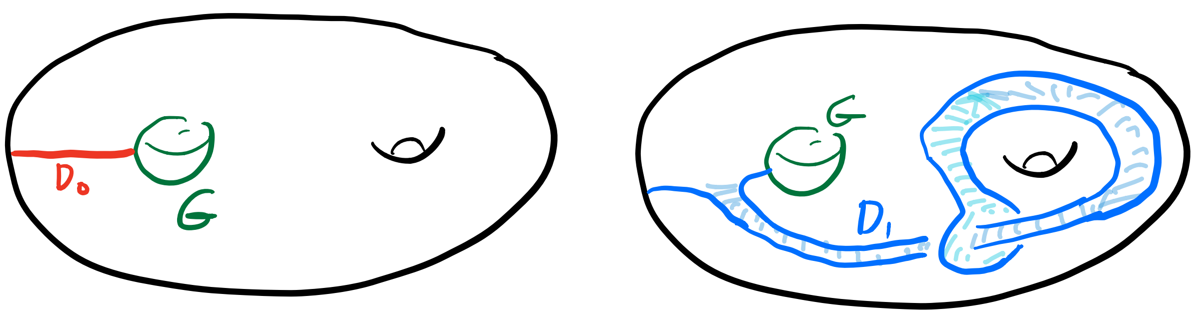



isotopy
without moving the bulb

H. Schwartz No 2-torsion is a necessary condition

R. Schneiderman - P. Teichner

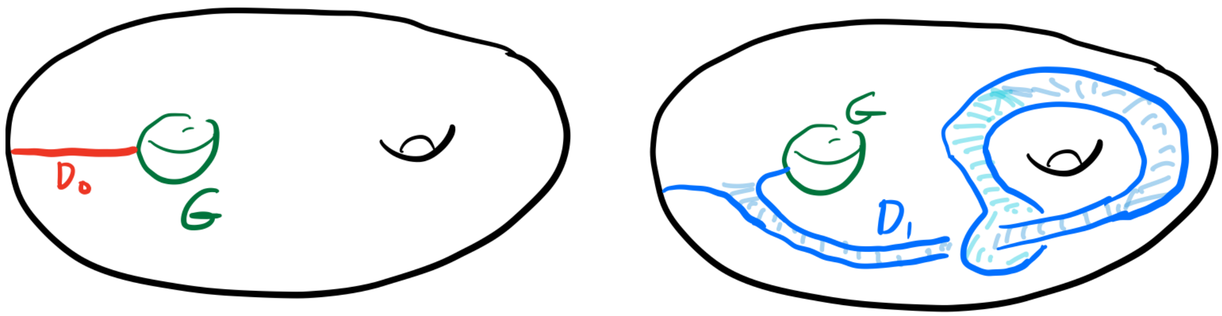
The Freedman-Quinn invariant is the exact obstruction.



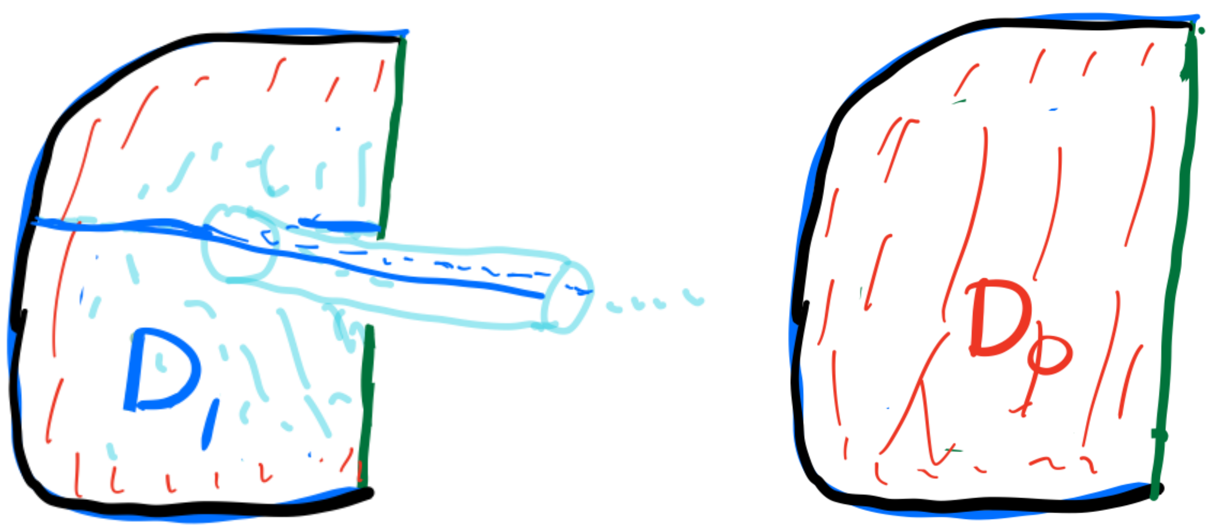
Facts 1) $FQ(D_0, D_1) = 0$ since $\pi_1(M) = \mathbb{Z}$ i.e. is torsion free

2) $\text{Stong}(D_0, D_1) = 0$ since D_0, D_1 have dual spheres (see M. Klug - M. Miller)

D_1 is not isotopic to D_0 rel ∂



$D_1 \rightsquigarrow$ loop of embeddings of the interval which is essential.



Theorem G - "Self referential discs and the light bulb lemma"

Theorem D_0 properly embedded 2-disc $\subset M^4$ compact, with dual sphere $G \subset \partial M$. $\mathcal{D} =$ isotopy classes of embedded discs homotopic to D_0 rel ∂

Then there is a homomorphism

$$\phi_{D_0} : \mathcal{D} \longrightarrow \pi_1^D(\mathcal{E}mb(I, M), I_0)$$

$$= \mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

It maps onto subgroup generated by elements $g + g^{-1}$ and $\hat{\lambda}$ where $\hat{\lambda}^2 = 1$

<u>Example</u> when $M = S^2 \times D^2 \natural S^1 \times B^3$,

$D(I_0) = 1$ and $\vartheta \cong$

to subgroup $\{[z^n + z^{-n}] \mid n \in \mathbb{N}\}$

z generator of π_1

Extension and generalization of D. Kosanovic and P. Teichner

1) The above homomorphism is an $\cong$

2) understand the homotopy type of space of embedded spheres with dual sphere in ∂W^4

Hannah Schwartz proves a LBT for discs with dual sphere $G \subset \mathring{M}$ such that $\pi_1(M - G) \xrightarrow{\cong} \pi_1(M)$

<u>Question</u> (Schwartz) Is there a LBT when $\pi_1(M - G) \to \pi_1(M)$ not $\cong$

THÉORÈME A. — *Soient* V^n *et* M^m *deux variétés différentielles de classe* C^∞, *la variété* V^n *étant compacte sans bord.*

Soit $f : V^n \to M^m$ *une application continue. Si* $2m - 3n - 3 \geqq 0$, f *est homotope à un plongement si et seulement si* $\alpha_0(f)$ *est l'élément neutre du groupe* $\Omega_{2n-m}(\mathcal{C}_f, \partial W; \theta_f)$.

Soient k *un entier* $\geqq 1$ *et* $f_0 : V^n \to M^m$ *un plongement. L'homomorphisme (application pointée si* $k = 1$*) :* $n=1$ $m=4$ $k=2$

$$\alpha_{\cancel{k}\,2} : \quad \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^n, M^m),\ \mathrm{Pl},\ f_0) \to \Omega_{\underbrace{2n-m+k}_{0}}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

est un isomorphisme (bijection si $k = 1$*) pour* $k \leqq 2m - 3n - 3$, *un épimorphisme (surjection si* $k = 1$*) pour* $k = 2m - 3n - 2$.

Jean-Pierre Dax 1972

Étude homotopique des espace de plongements

Dax Isomorphism Theorem:

$$\alpha_{\cancel{k}\,2} : \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^{\cancel{n}\,1}, M^{\cancel{m}\,4}), \mathrm{Pl}, f_0) \xrightarrow{\simeq} \Omega_{\cancel{2n-m+k}\,0}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

Let I_0 be a properly embedded closed interval in the oriented M^4.

i) $\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is generated by $\{ \mathcal{J}_g \mid g \neq 1, g \in \pi_1(M) \}$ and is canonically $\cong$ to

$$\mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

ii) There is a homomorphism

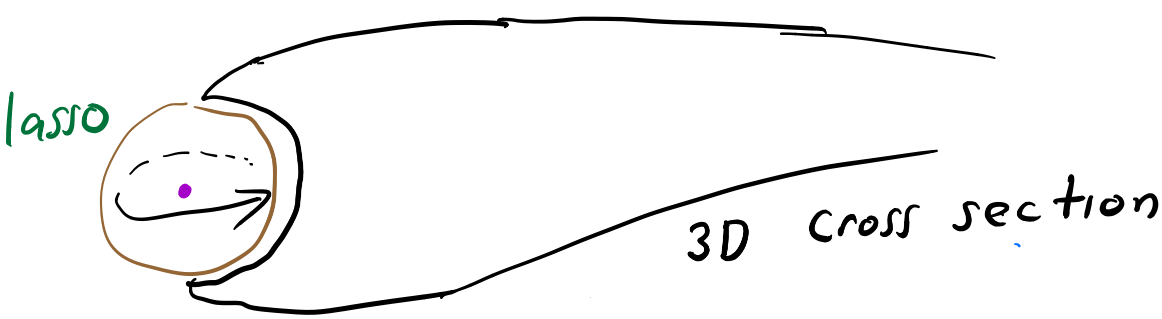
$$d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$$

with image $D(I_0)$ - the Dax kernal

$\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is the subgroup of loops that are $\simeq *$ in the space of maps - The Dax group

Reference "Self-Referential discs and the light bulb lemma"

Spinning (Definition of $\mathcal{J}_g$)



- conventions
  1) base of band below lasso on I_0
  2) an orientation rule determines sign

Fact: Spinnings commute

Dax's key idea:

Let $\alpha_t : I \longrightarrow M$

with $\alpha_0 = \alpha_1 = 1_{I_0}$ (id map to I_0)

$\alpha_t \in \pi_1^D(\mathrm{Emb}(I,M), I_0) \Longrightarrow$

$\exists\, \alpha_{t,u} \in \mathrm{Maps}(I, M; I_0)$ with

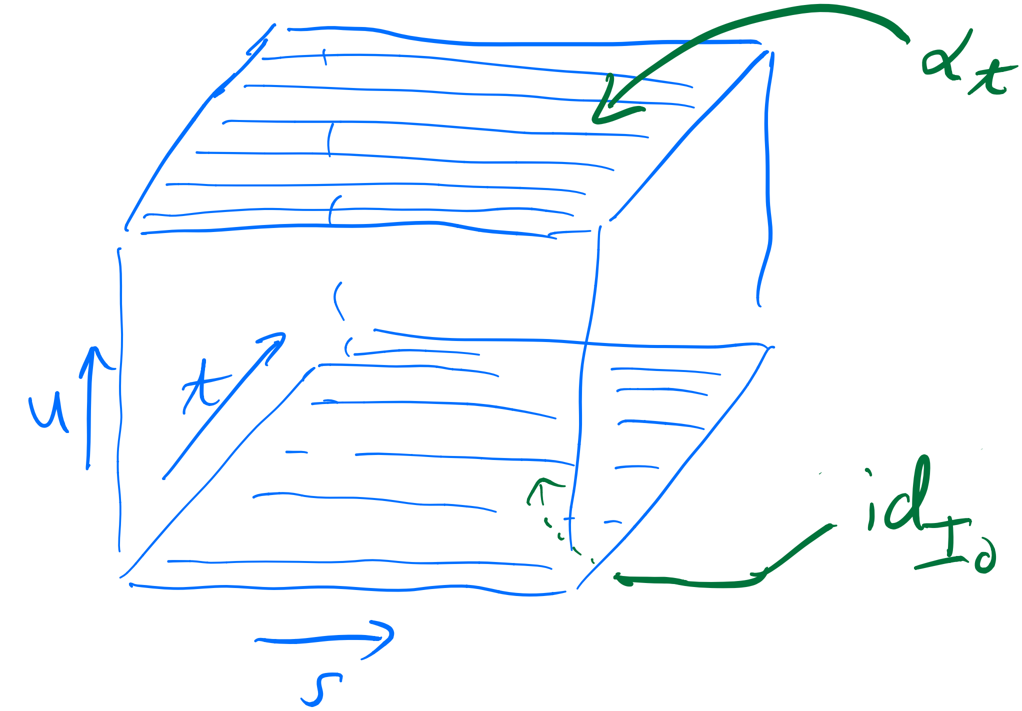
$\alpha_{t,u} = 1_{I_0}$ u near 0, $t \in I$; $= \alpha_t$ u near 1

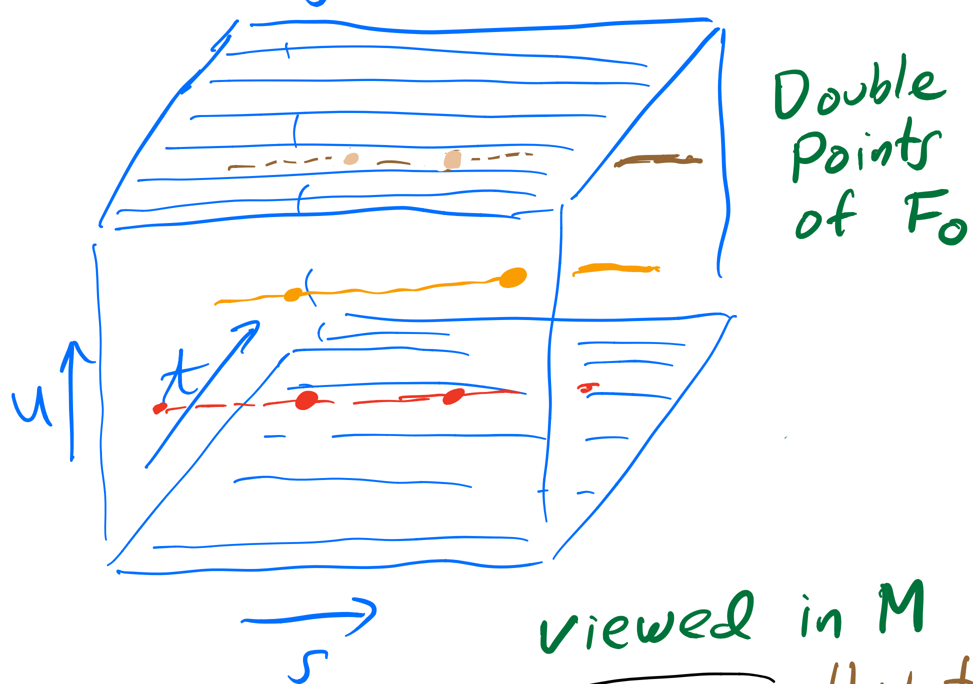
Define $F_0 : I \times I^2 \longrightarrow M \times I^2$

$F_0(s,t,u) = \left(\alpha_{t,u}(s), t, u\right)$

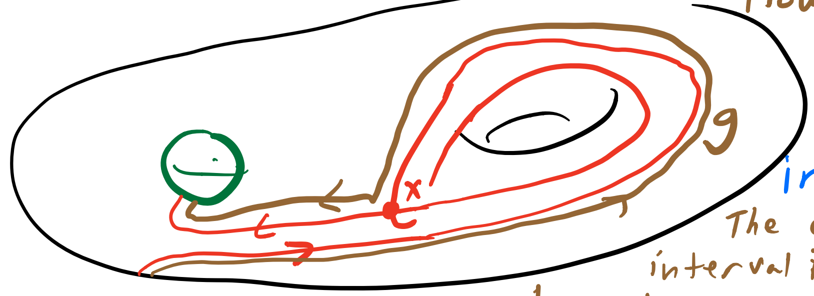
we can assume F is an immersion with finitely many double points, no triple points.

self $\pitchfork$ at double pts





viewed in M



How to Compute g_x
sign $= \sigma_x$
oriented self intersection #

The orientation of the interval informs which tangent space comes first.

$$F_0 \rightsquigarrow d(\alpha_{t,u}) = \sum_{i=1}^{n} \sigma_{x_i} g_{x_i} \in \mathbb{Z}[\pi_1(M) \setminus 1]$$

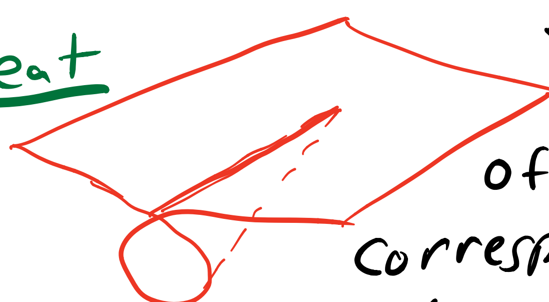
Summed over double points with

$g_{x_i} \neq 1$

This is well defined

If $\alpha^0_{t,u}$, $\alpha^1_{t,u}$ two homotopies in $\mathrm{Maps}(I, M; I_0)$ and $\alpha^0_{t,u} \simeq \alpha^1_{t,u}$ fix ∂ then the usual intersection theory argument – considering double curves of interpolating homotopy – shows $d(\alpha^0_{t,u}) = d(\alpha^1_{t,u})$

Caveat



Some double curves cone off – but this corresponds to loops $g_x = 1$.

If $\alpha^0_{t,u} \not\simeq \alpha^1_{t,u}$ then they differ by an element of π_3.

Define $d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$

where $a \in \pi_3$ is represented by $\alpha_{t,u}(s)$ where $\alpha_{t,0}, \alpha_{t,1} = 1_{I_0}$

Define $D(I_0) = d_3(\pi_3(M, x_0))$

Then

$$d : \pi_1^D(\mathrm{Emb}(I, M; I_0)) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

is a homomorphism.

Looking closely at

$$F_0 \rightsquigarrow \sum_{i=1}^{n} \sigma_{x_i} g_{x_i} \in \mathbb{Z}[\pi_1(M)]$$

where the sum is without cancellation and g_{x_i} possibly $\underline{1}$ then one sees that α_t is a concatenation of the spin maps $\sigma_{x_i} g_{x_i}$.

I.e. α_t differs from 1_{I_0} by this concatenation of spinnings.

Since spin maps $\in \pi_1^D(\mathrm{Emb}\, I, M; I_0)$

It follows that

$$d : \pi_1^D(\mathrm{Emb}(I, M; I_0)) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

is surjective. Since spinnings commute and spinning around $\underline{1}$ is $\simeq *$, d is injective.

<u>Technical pt</u> This avoids a double point s-cobordism like elimination argument of Dax

Example

$$\pi_1^D(\mathrm{Emb}(I, S^1 \times B^3; I_0)) =$$

$$\pi_1(\mathrm{Emb}(I, S^1 \times B^3; I_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0]$$

Proof $\pi_2(S^1 \times B^3) = 0$

$\pi_3(S^1 \times B^3) = 0$

Example

$$\pi_1^D(\mathrm{Emb}(I, S^2 \times D^2 \natural S^1 \times B^3; I_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0]$$

Proof Same generators - less space to kill them.

Dax Isomorphism Theorem:

$$\alpha_{\cancel{k}\,2} : \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^{\cancel{n}\,1}, M^{\cancel{m}\,4}), \mathrm{Pl}, f_0) \xrightarrow{\cong} \Omega_{\cancel{2n-m+k}\,0}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

Let I_0 be a properly embedded closed interval in the oriented M^4.

i) $\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is generated by $\{\mathfrak{z}_g \mid g \neq 1, g \in \pi_1(M)\}$ and is canonically $\cong$ to

$$\mathbb{Z}[\pi_1(M) \setminus 1]/D(I_0)$$

ii) There is a homomorphism

$$d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$$

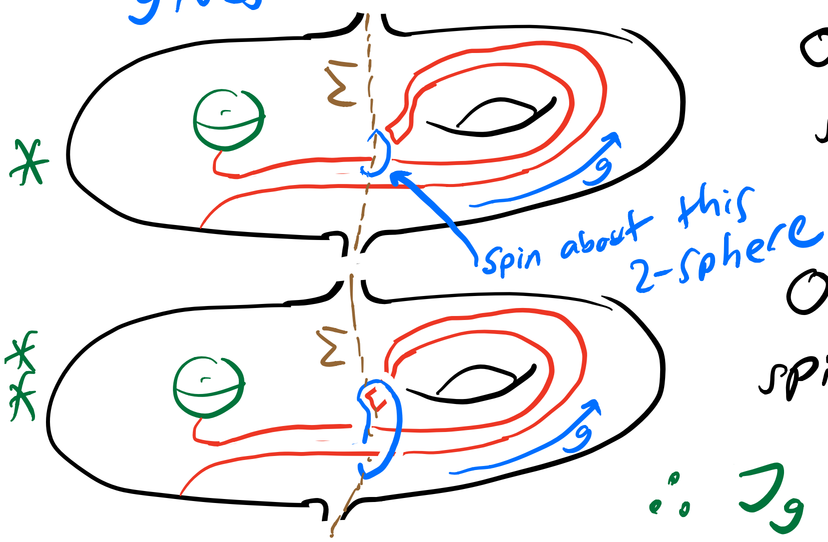
with image $D(I_0)$ – <u>the Dax Kernal</u>

Differences between two Dax $\cong$ Theorems

1) Working in different spaces
2) Part ii) is not part of his theory
3) We identify the generators geometrically

$$\pi_1^D(\mathrm{Emb}(I, S^2\times D^2 \# S^1\times B^3; I_0)) \cong \mathbb{Z}[\mathbb{N}]$$

__Idea of Proof__ The separating $S^3 := \Sigma$ gives relations. A 2-sphere $\subset \Sigma$ bounds B^3's



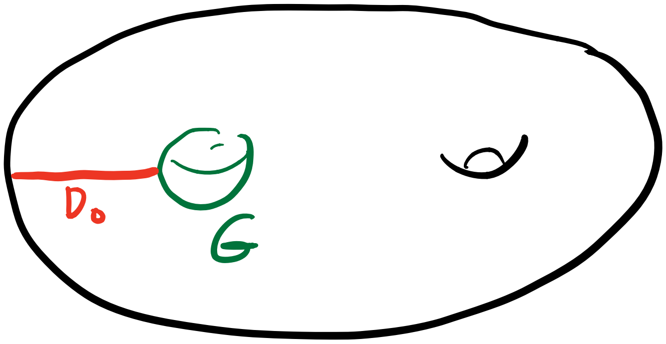
One gives spinning J_g

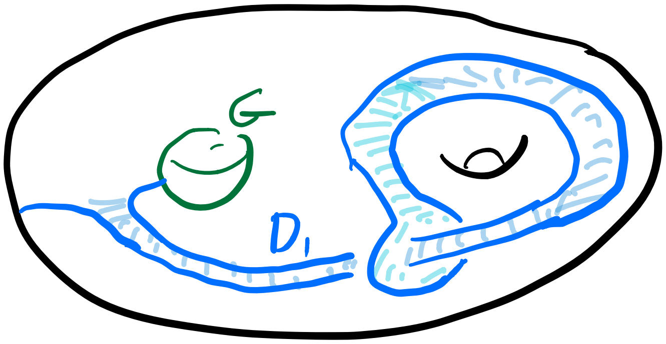
Other gives spinning J_{g-1}

$\therefore J_g \sim J_{g-1}$ up to sign

$$\alpha_{\cancel{k}\,2} : \quad \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^{\cancel{n}\,1}, M^{\cancel{m}\,4}),\ \mathrm{Pl},\ f_0) \to \Omega_{\cancel{2n-m+k}\,0}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

These "homotopies" $*$, $**$ represent different elements of the source of α_2, but represent the same elt. in π_1^D.



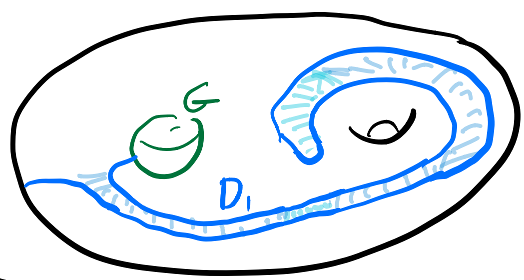


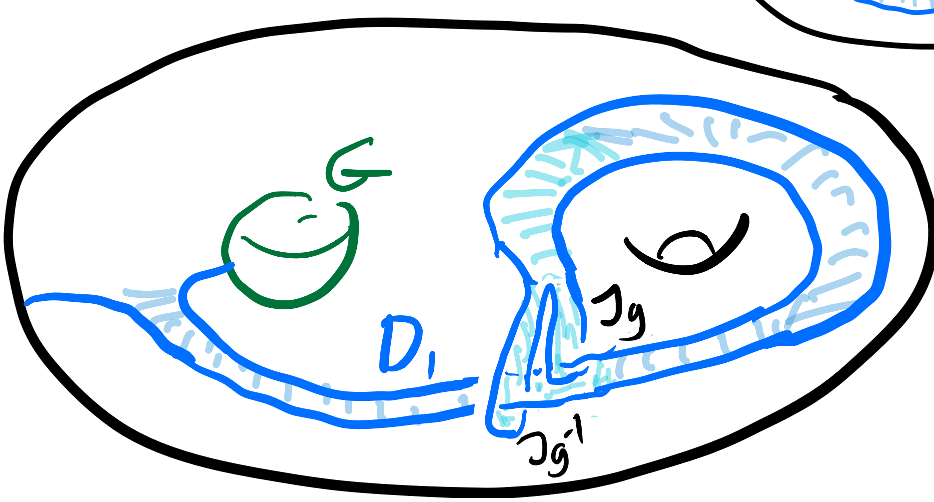
View $D_1 \in \pi_1^D(\mathrm{Emb}(I, M); I_0)$.

$d(D_1) = g + g^{-1}$

(using correct choice of sweeping across D_1)

<u>**Proof**</u> Viewed as a loop of embedded intervals D_1 is the concatenation of J_g, $J_{g^{-1}}$





□

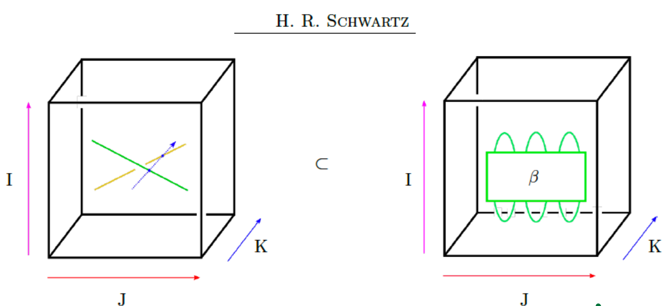
How to compute $d(D_1)$ up to $I(D_0)$ (after H. Schwartz)

Reference (Schwartz) "A LBT for discs"

Step 1 Consider a regular homotopy of D_0 to D_1 $\rightsquigarrow$ $I \times D^2 \to I \times M^4$

Step 2

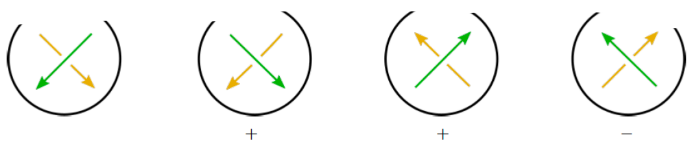
Consider the self intersection locus viewed as a plat.



(Here K is the "interval direction")

Step 3 Add up crossings (projection into I, J plane) corresponding to identified arcs.

- each crossing comes with a sign and group element.



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Ignore crossings with $g_x = 1$