Poincaré (1904): is every homotopy $S^3$ homeomorphic to $S^3$?

Generalized $^{CAT}$ Poincaré Conj: Is every homotopy $S^n$ $\underset{CAT}{\text{equivalent}}$ to $S^n$?

$$CAT = \begin{cases} TOP \\ PL \\ DIFF \end{cases}$$

$\boxed{n=3}$

TOP = PL = DIFF } yes! Perelman (2002)

$\boxed{n \geq 5}$

TOP $\neq$ PL } Yes! Smale, Stallings, Zeeman, Newman (60's)

DIFF } Not always. Kervare-Milnor (60's)

$\boxed{n=4}$

TOP } Yes! Freedman (80's)

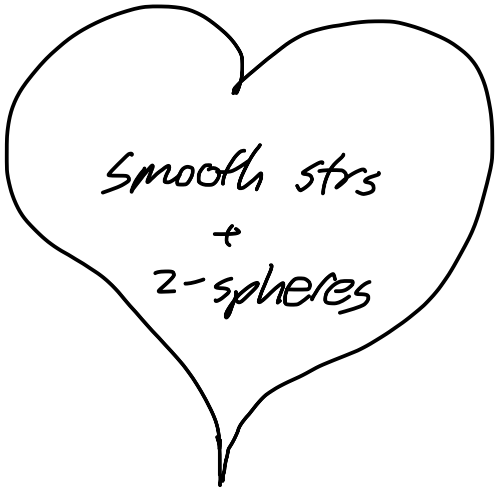
$\neq$

PL = DIFF } Unknown!

PL $\neq$ TOP: $\exists$ 4-manifolds $X_1, X_2, X_3, \ldots$ (smooth) homeomorphic but not diffeomorphic } Donaldson (80's)

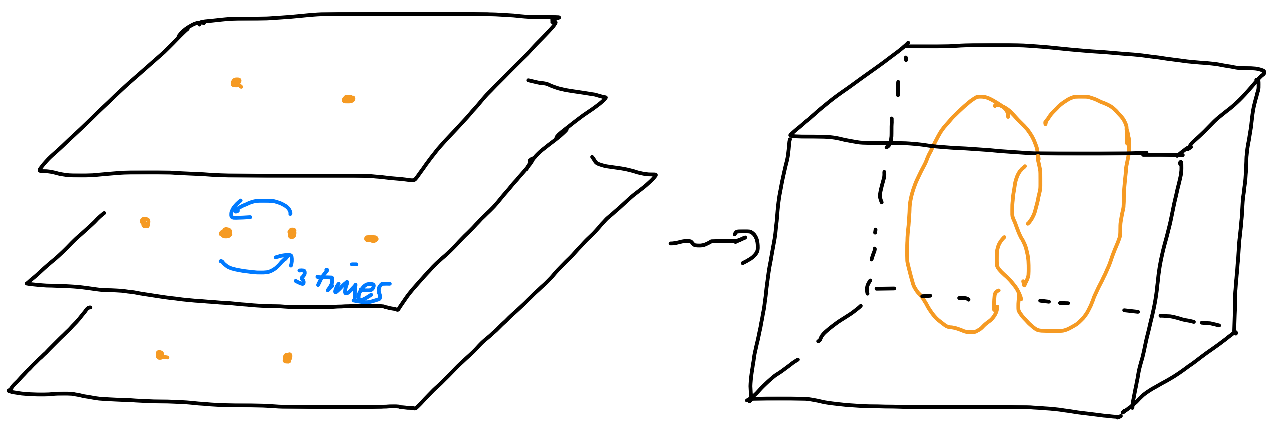
$\exists$ smooth 2-spheres $S_1, S_2, \ldots \subset X^4$

topologically isotopic but not smoothly isotopic

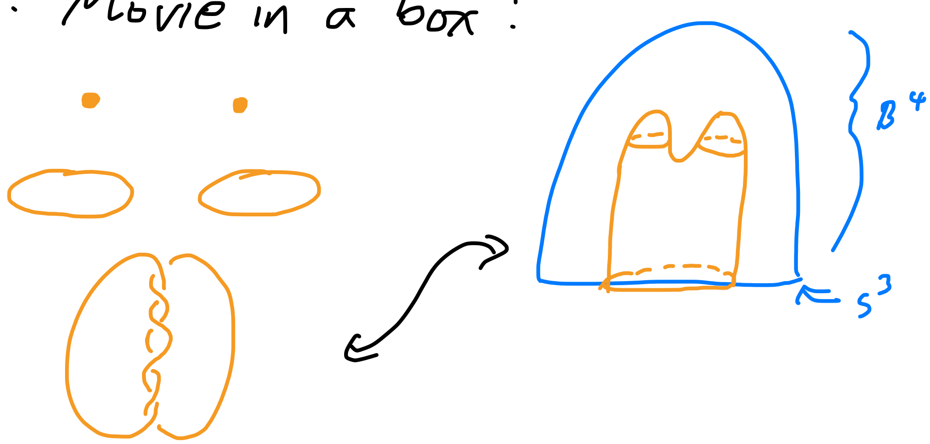



Understand 2-spheres in 4-manifolds

TAKE 1 (Warm-up) Movie in a plane

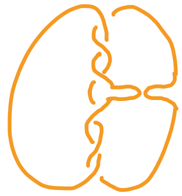


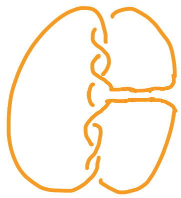

TAKE 2 : Movie in a box!




$B^3 \times I = B^4$

$\partial B^4 = S^3$





$-K \# K$

now glue mirror image

$\pi_1(S^4 - N(S_K))$

$\cong \pi_1(S^3 - N(K))$

so nontrivial

$K \subset S^3 \rightsquigarrow S_K \subset S^4$

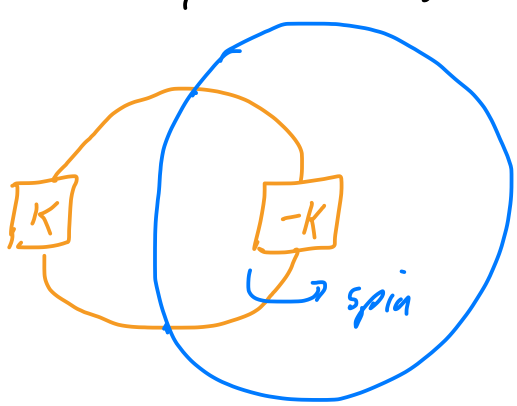
spun knots

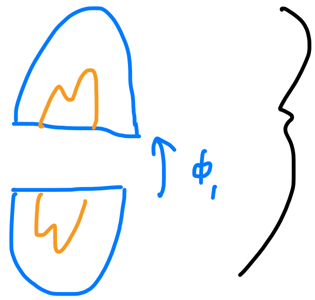
(Artin 20's)

Other ways to glue:

ex $\phi_t : S^3 \to S^3$

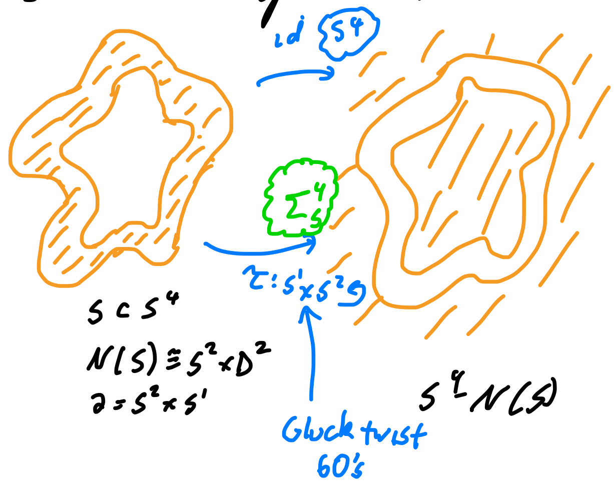
$\phi_0 = id \quad \phi_1(K \# -K) = K \# -K$ setwise






deform spin knots (Litherland 70's)
twist spun (Zeeman 60's)
roll spun (Litherland)

## What can you do with a sphere?




$\Sigma^4_S$ := Gluck twist on $S \subset S^4$
(smooth) homotopy 4-sphere

$\Sigma^4_S \cong S^4$
when $S$ is
- spun knot
- twist spun
- ribbon knot

$\Downarrow$

"simple"-ish spheres

Unknown for many spheres!
- roll spun knots

## Notions of complexity

I. Stabilization number

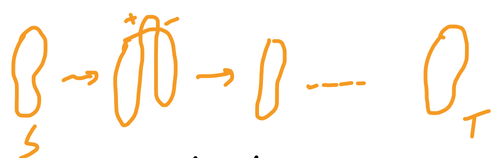
$\mu_{st}(S) :=$ min # of stabilizations needed to produce an unknotted surface


bounds a handlebody in $S^4$

II Casson-Whitney number

$\mu_{cw}(S) :=$ min # of +/− point pairs in a regular homotopy to unknotted sphere

$S, T \subset S^4$ homotopic




regular homotopy

(Klug - Joseph - Ruppik - S 2021)

Th$^{\underline{m}}$ A: For $S \subset S^4$, $\mu_{st}(S) \leq \mu_{cw}(S) + 1$

Th$^{\underline{m}}$ B: $\exists$ $\infty$'ly many $S_i$ with $\mu_{st}(S_i) = 2$ and $\mu_{cw}(S) = 1$

Th$^{\underline{m}}$ C: $\mu_{cw}(S) = 1 \rightsquigarrow \mu_{st}(S) = 1$

<u>Application of Th$^{\underline{m}}$ C</u> (Naylor - S 2022)

$\mathcal{R}_K :=$ roll spin of $K \subset S^3$

<u>Prop:</u> If $K$ has unknotting # $= 1$, then $\mu_{cw}(\mathcal{R}_K) = 1$

FACT ($\underbrace{\text{Montesinos}}_{\text{Kyle Larson}}$, $\underbrace{\text{Iwase}}_{\text{N-S}}$ 80's)

if $\mu_{st}(S)=1$, then $\Sigma_S$ is standard

$\Downarrow$ Thm C

**Thm**: The Gluck twist $\mathcal{R}_K$, $K$ has unknotting # 1 is standard!