What is rational 4-genus?

Tech Topology Conference

Katherine Raoux – University of Arkansas

12. 9. 23

We start with a knot:

$$K \subset Y^3$$

$$p[K] = 0 \in H_1(Y;\mathbb{Z})$$

for some $p \in \mathbb{Z}$

BUT!

K does not bound a Seifert surface.

$$\mathbb{Z} \cong H_2(Y,K) \longrightarrow H_1(K) \longrightarrow H_1(Y)$$

$$c \quad \overset{\text{lifts}}{\dashleftarrow} \quad p[K] \longmapsto 0$$

Represent c by surface S.

THEN

S is a rational Seifert surface for K.

S $\qquad \partial S$ p-covers K

$\mathrm{int}(S) \hookrightarrow Y \setminus K$

Rational Seifert genus

$$\|K\|_Y := \inf_{S,p} -\frac{\chi(S)}{2p}$$

min. genus Seif. surf for $K_{p,q}$ in $Y - \nu(K)$

$$= \frac{-\chi(S_{min})}{2p} = \frac{1}{2}\left(A_{max}(K) - A_{min} - 1\right)$$

HF-Alexander grading

Calegari-Gordon $\qquad$ Ni-Wu

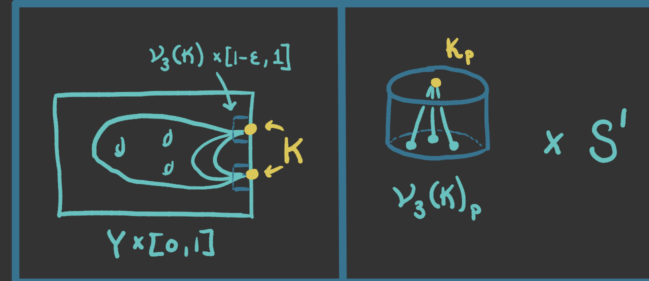
What about rational 4-genus?

A rational slicing surface:

$\partial\Sigma$ p-covers K

$\mathrm{int}(\Sigma) \hookrightarrow Y \times [0,1] \setminus K$

K has a simultaneous tubular and collar neigh.



Near the knot, $\Sigma \cap \nu_3(K) \times \{1-\varepsilon\} = P_\beta(K)$.

Rational 4-genus:

$$\|K\|_{Y\times[0,1]} := \inf_{\Sigma, p} \frac{-\chi(\Sigma)}{2p}$$

$$= \inf_{p \geq 1} \left(\inf_{\beta \in B_p} \left(\inf_{\Sigma} \left\{ \frac{-\chi(\Sigma)}{2p} \,\middle|\, \begin{array}{l} \Sigma \hookrightarrow Y \times [0,1] \\ \partial\Sigma = P_\beta(K) \end{array} \right\} \right) \right)$$

Theorem (Hedden-R.)

$$\frac{1}{2}\left(\tau_{max}(K) - \tau_{min}(K) - 1\right) \leq \|K\|_{Y\times[0,1]}$$

max/min $\tau_\alpha(K)$, $\alpha \in \widehat{HF}(Y)$

Anosov Reeb Flows, Dirichlet Optimization and Entropy

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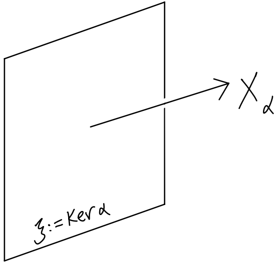
Tech Topology Conference 2024

Chern-Hamilton Question

- Suppose M is a closed oriented 3-manifold.
- Let α be a positive contact form on M and X_α the associated Reeb vector field.



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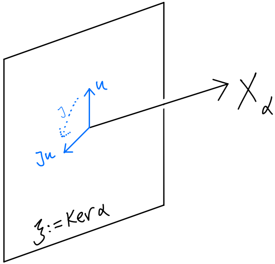

Definition

If J is an almost complex structure defined on $\xi := \ker\alpha$, we define its *Dirichlet energy* as

$$\mathcal{E}(J) := \int_M ||\mathcal{L}_{X_\alpha} J||^2 \, \alpha \wedge d\alpha.$$

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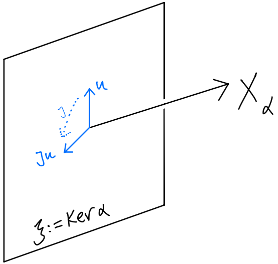

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- i.e.

$$\mathcal{E} : \mathcal{J}(\alpha) = \{\text{space of almost complex structures on } \xi =: \ker \alpha\} \to \mathbb{R}$$

is an *energy functional*.

Chern-Hamilton Question

Chern-Hamilton Question 1984

For which contact manifolds (M, α) does the Dirichlet energy functional

$$\mathcal{E} : \mathcal{J}(\alpha) \to \mathbb{R}$$

attains its minimum?

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Theorem (H. 23)

It rarely does! We can classify all such (M, α) *s.*

Using variational techniques,

Theorem (Tanno 1989)

An almost complex structure J is critical for

$$\mathcal{E} : \mathcal{J}(\alpha) \to \mathbb{R},$$

if and only if,

$$\nabla_{X_\alpha}(\mathcal{L}_{X_\alpha} J) = 2(\mathcal{L}_{X_\alpha} J)J.$$

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Theorem (Deng 1991)

Any critical J is in fact the minimizer of $\mathcal{E} : \mathcal{J}(\alpha) \to \mathbb{R}$.

If J is a *critical* almost complex structure,

- the scalar torsion $||\mathcal{L}_{X_\alpha} J||$ is constant along the Reeb flow X_α, i.e.

$$X_\alpha \cdot ||\mathcal{L}_{X_\alpha} J|| = 0.$$

Consequences of Tanno's Formula

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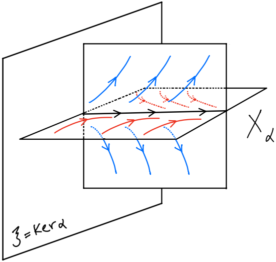

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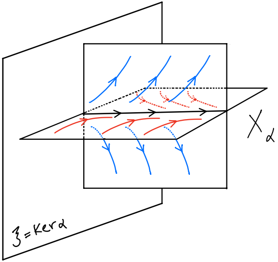

- $\implies$ The pre-image of a regular value of the scalar torsion $\Sigma_\lambda = ||\mathcal{L}_{X_\alpha} J||^{-1}(\lambda)$ is an invariant closed surface.

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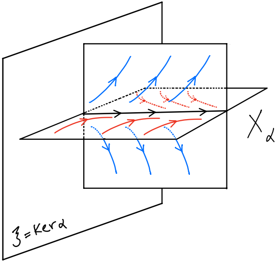

- $\Longrightarrow$ The pre-image of a regular value of the scalar torsion $\Sigma_\lambda = ||\mathcal{L}_{X_\alpha} J||^{-1}(\lambda)$ is an invariant closed surface.
- (NOT possible in hyperbolic dynamics)

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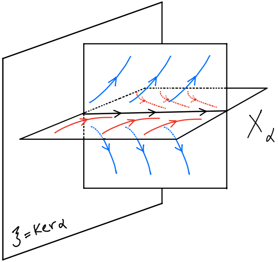

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Corollary

For critical J, we have $||\mathcal{L}_{X_\alpha} J|| \equiv C \geq 0$ for some constant C.

Case 1: $||\mathcal{L}_{X_\alpha} J|| \equiv 0$

- That is when X_α is a Killing vector field.

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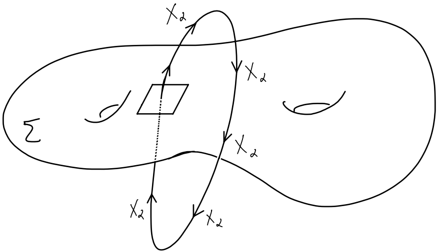
- That is when X_α is a Killing vector field.
- Virtually, X_α traces a $\mathbb{S}^1$-fibration over a surface (called *Boothby-Wang fibrations*).



Case 2: $||\mathcal{L}_{X_\alpha} J|| \equiv C > 0$

- We have *hyperbolic behavior* everywhere. Such Reeb flows are called Anosov.

[1] Image source: https://thatsmaths.com/2013/10/11/poincares-half-plane-model/

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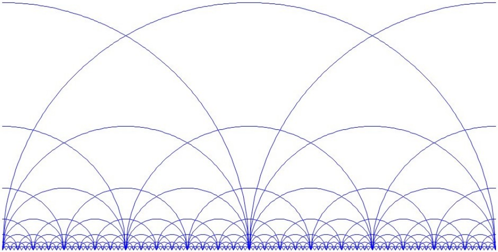
- We have *hyperbolic behavior* everywhere. Such Reeb flows are called Anosov.
- $||\mathcal{L}_{X_\alpha} J||$ being constant corresponds to *entropy rigidity*.
- (Foulon 01) $\Rightarrow$ Virtually, X_α is the geodesic flow of a hyperbolic surface Σ on $UT\Sigma$.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Theorem (H. 23)

The Dirichlet energy functional $\mathcal{E}:\mathcal{J}(\alpha)\to\mathbb{R}$ admits a minimizer, if and only if,

(1) (M,α) is virtually equivalent to a Boothb-Wang fibration.

or

(2) X_α is virtually equivalent to the geodesic flow of a hyperbolic surface Σ on $UT\Sigma$.

Infimum Dirichlet energy

What about the infimum of the Dirichlet energy for a general (M, α)?

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Theorem (H. 23)

If (M, α) is an Anosov contact manifold. Then,

$$\inf_{J \in \mathcal{J}(\alpha)} \mathcal{E}(J) = \frac{h^2_{\alpha \wedge d\alpha}(X_\alpha)}{Vol(\alpha \wedge d\alpha)},$$

where $h^2_{\alpha \wedge d\alpha}(X_\alpha)$ is the measure entropy.

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Conjecture

For an arbitrary contact manifold (M, α), we have

$$\inf_{J \in \mathcal{J}(\alpha)} \mathcal{E}(J) = \frac{h^2_{\alpha\wedge d\alpha}(X_\alpha)}{Vol(\alpha \wedge d\alpha)}.$$

Thank you!

:)

Infinite Order Knot Traces

Yikai Teng

Rutgers University-Newark

December, 2023

Cork Twists

Corks

A (loose) cork (W, f) is a contractible compact smooth 4-manifold W together with a boundary automorphism f such that f does not extend to a self-diffeomorphism of W.

Cork Twisting Theorem ([4], [8], [1])

Every pair of exotic simply-connected smooth 4-manifolds are related by a cork twist.

Examples

- $E(1)$ and $E(1)_{2,3}$ are related by the positron cork [3].
- $E(2)\#\overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ are related by the Mazur cork [2].

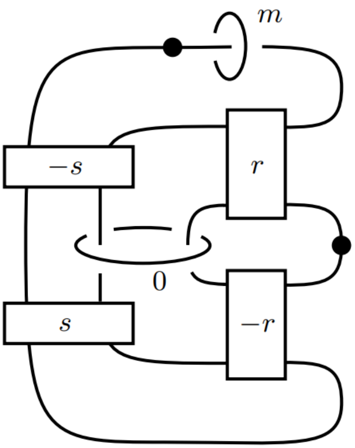
Infinite order corks

Theorem 1 ([6], [5])

There exists a cork X together with a boundary automorphism ϕ on ∂X such that the iterated automorphism ϕ^k does not extend to a self-diffeomorphism on X for any $k > 0$.

- $\forall n, X \hookrightarrow E(n)$
- Twisting via ϕ^k changes $E(n)$ to the smooth manifold obtained by doing Fintushel-Stern knot surgery using the k-twist knots.
- We can upgrade X to a family of infinite order corks by repeatedly blowing up $E(n)$ combined with orientation changes.

Realising the cork $(r, s > 0 > m)$



Twisting knot traces

Knot traces

A knot n-trace $X_n(K)$ is the smooth 4-manifold with a single 2-handle glued to a 0-handle along the knot K, with framing coefficient n. Its boundary turns out to be the n-surgery of the 3-sphere along the knot K, $S^3_n(K)$.

Applications combining with trace embedding lemma

- Knot traces can be used as plugs to generate exotic $\mathbb{R}^4$'s.
- With a little modifications, knot traces can possibly generate exotic S^4's or $\#^n\mathbb{CP}^2$'s [7] [9].

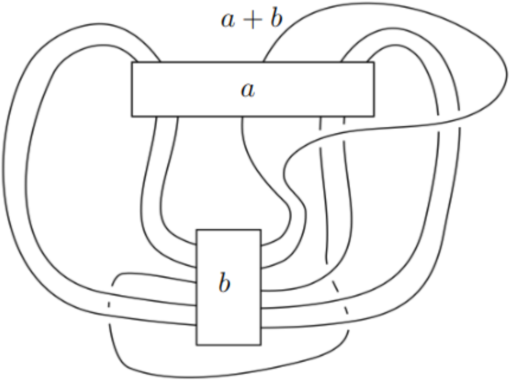
Knot trace as infinite order plugs

Theorem 3

For any integer n with $|n| \geq 2$, there exists a knot trace $X_n(K_n)$ together with a boundary automorphism ϕ_n on $S^3_n(K_n)$ such that the iterated automorphism ϕ_n^k does not extend to a trace self-diffeomorphism for any $k > 0$.

- Base case: $X_{-2}(K_{-2}) \hookrightarrow E(n)$
- Twisting via ϕ^k_{-2} changes $E(n)$ to the smooth manifold obtained by doing Fintushel-Stern knot surgery using the k-twist knots.
- We can obtain infinite order knot traces of other framings by repeatedly blowing up $E(n)$ combined with orientation changes.

Realising the knot trace $(a, b < 0)$



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Dissecting Symplectic 4-Manifolds

Randy Van Why
Northwestern University

Symplectic 4-manifolds :

- (M^4, ω) where

  (i) $d\omega = 0$

  (ii) $\omega \wedge \omega \neq 0$

- If M is closed then $[\omega] \neq 0$ in $H^2_{dR}(M)$

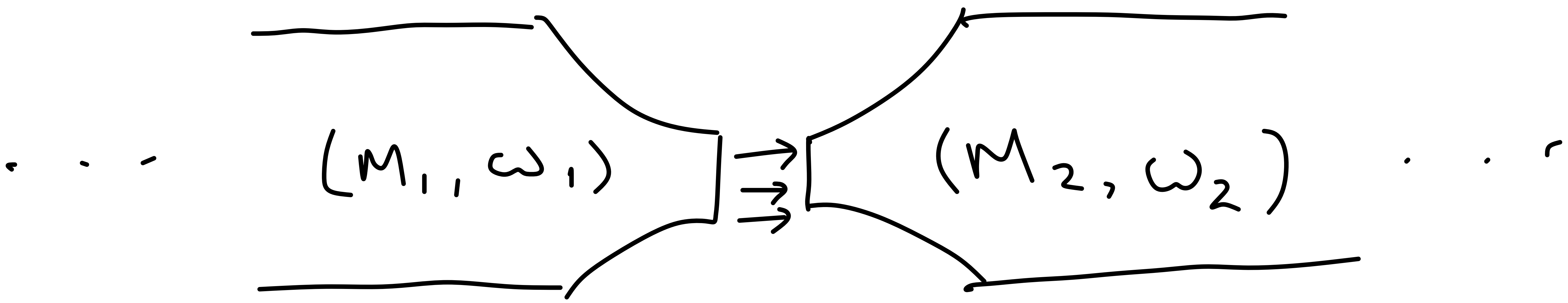
  (so S^4 is not symplectic)

Q: How to study (M^4, ω) in pieces?

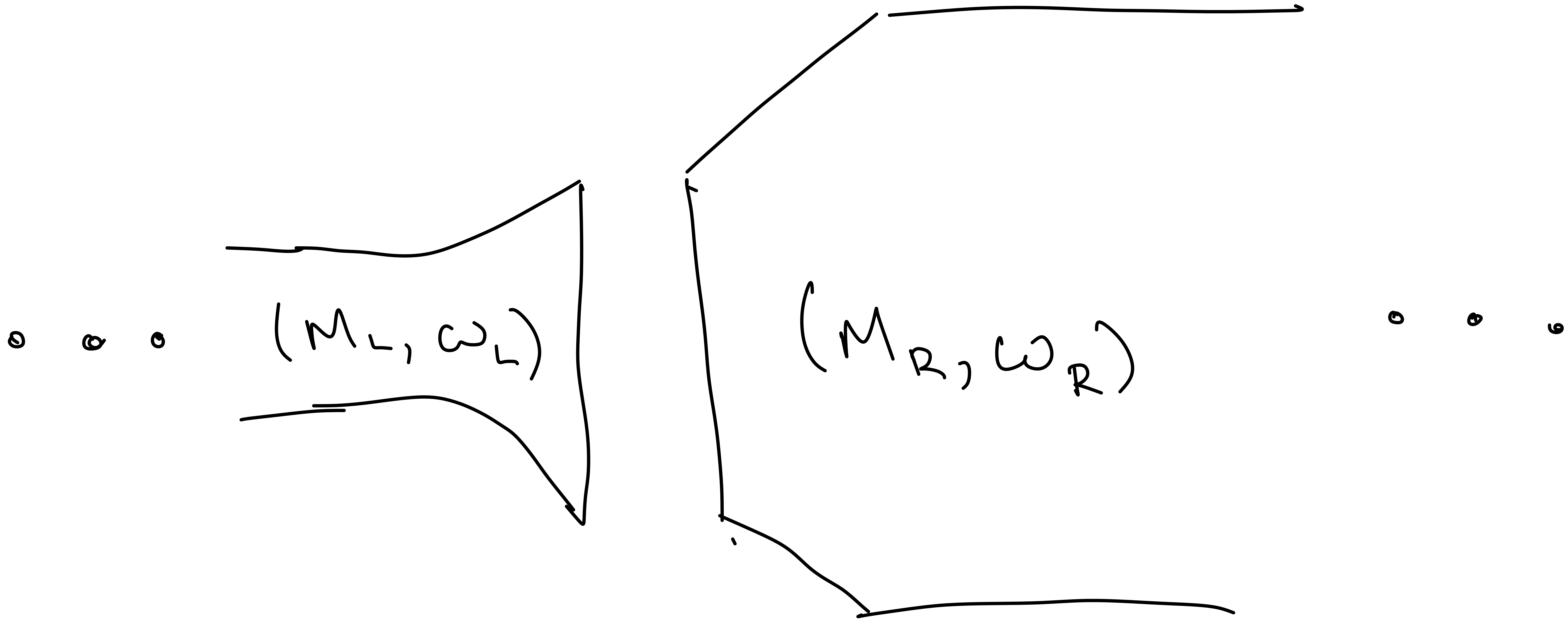
- Gluing two symplectic manifolds along a common boundary may not produce a symplectic manifold

$$S^4 = B^4 \cup_{S^3} B^4 \qquad (B^4, \omega_{std})$$

Why?: Closed condition is hard to gaurantee even if we could extend symplectic forms across.

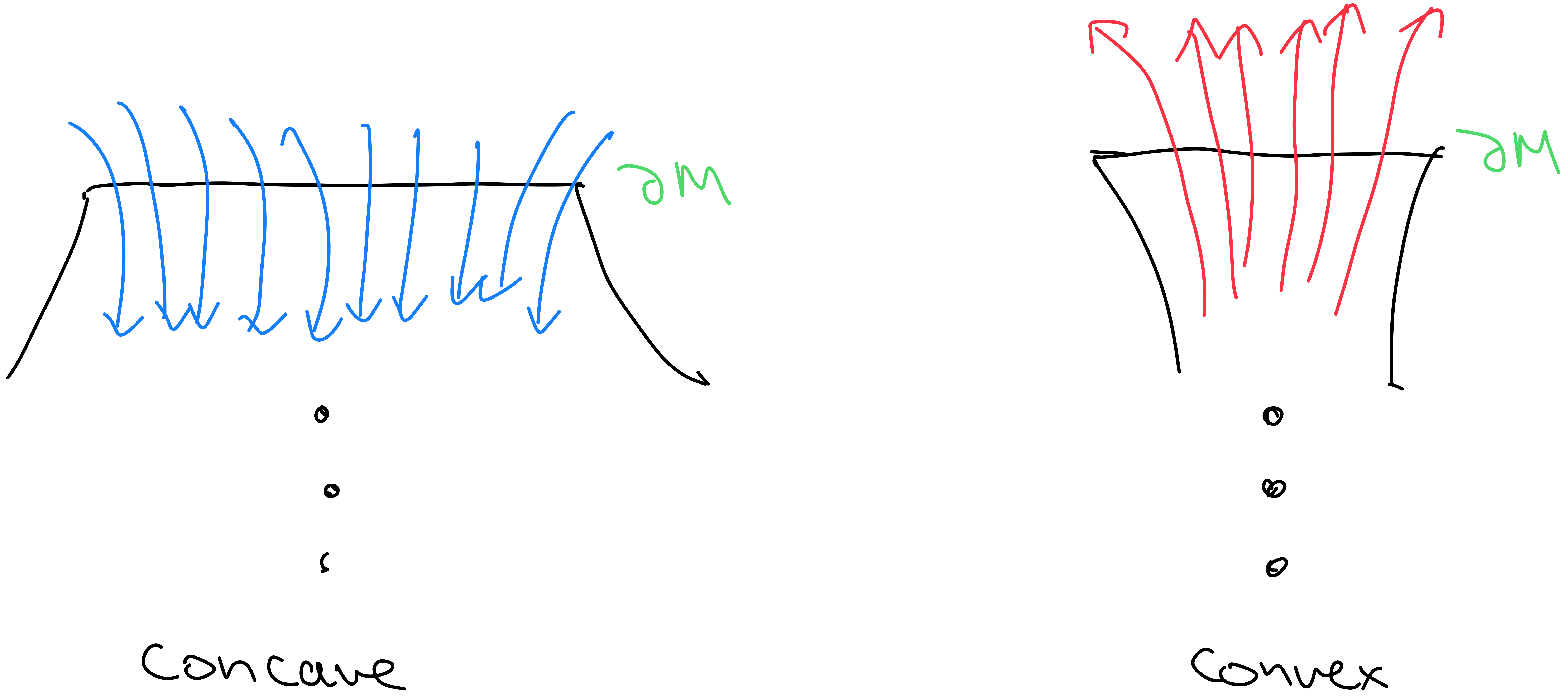


Q: When can a symplectic gluing be done?



A: Along a (symplectically) concave/convex pair of ends.

Concavity/convexity:



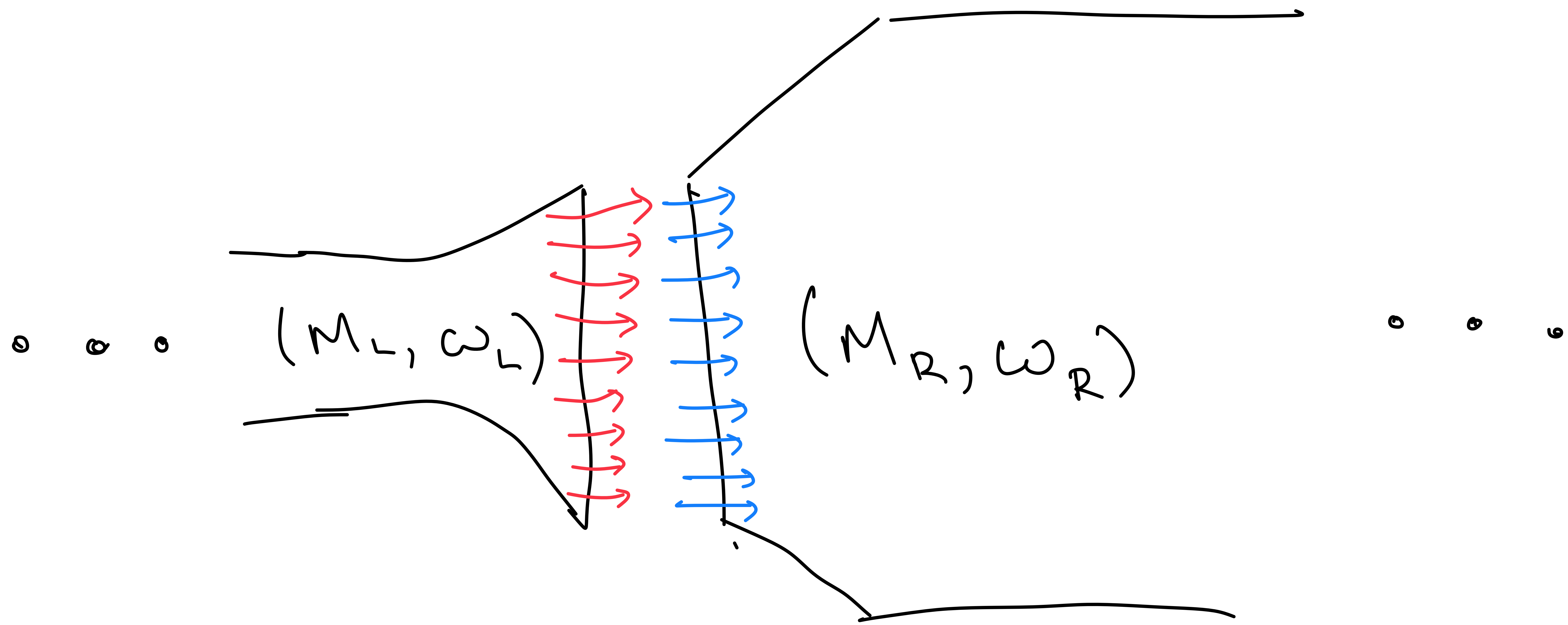
Liouville vector fields: V: vector fields s.t. $i_V \omega = \lambda$ has $d\lambda = \omega$, $V \pitchfork \partial M$.

Liouville fields induce contact structures

- $\lambda := i_V \omega$ has $\alpha = \lambda|_{\partial M}$ contact if $V \pitchfork \partial M$

- $\xi = \ker(\alpha) \subseteq T\partial M$ is a maximally non-integrable 2-plane field

$$(\partial M, \xi) \text{ contact.}$$



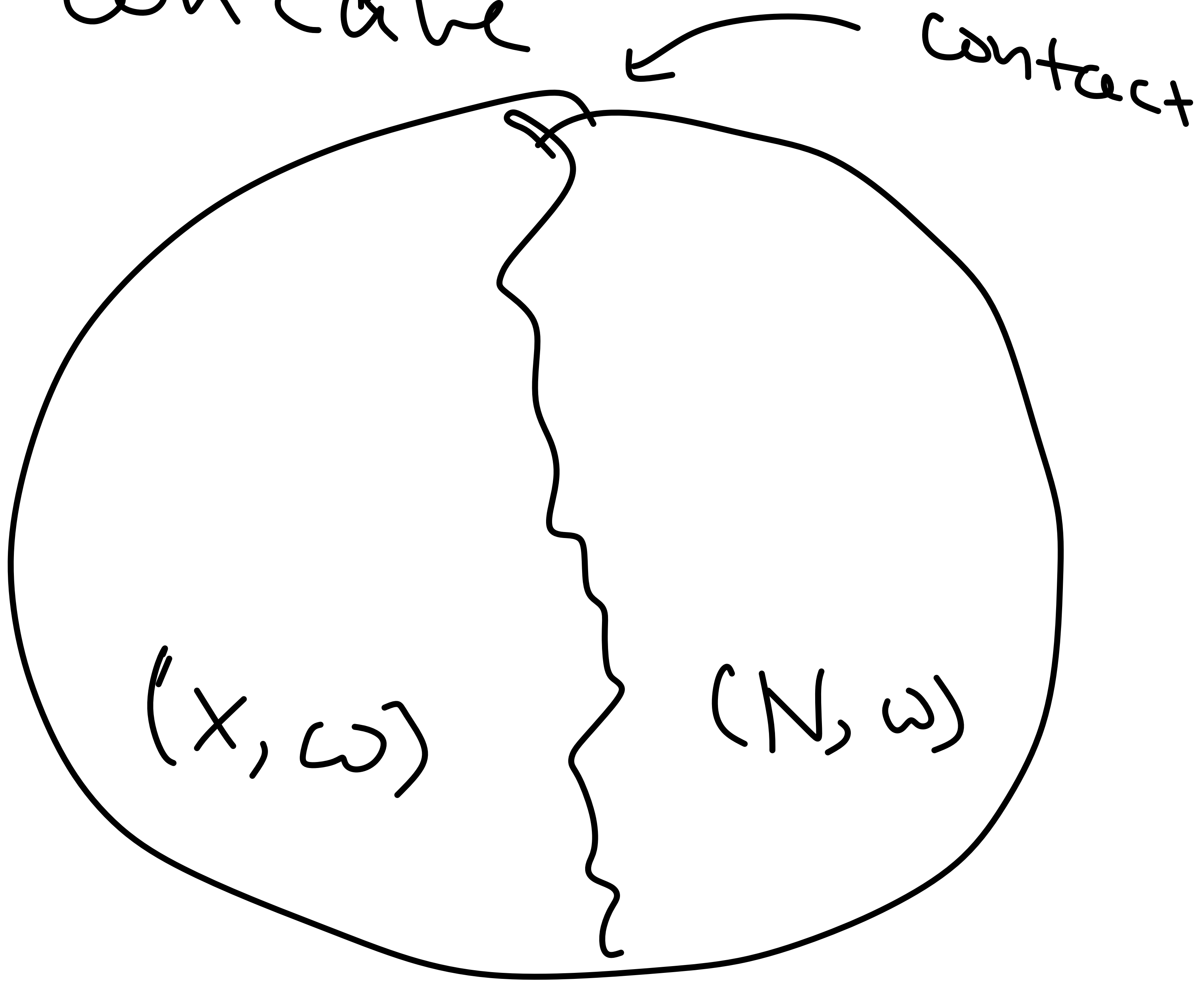
If the gluing is done along a <u>contactomorphism</u>

$$\phi : (\partial M_L, \xi_L) \longrightarrow (M_R, \xi_R) \quad [\phi_* \xi_L = \xi_R]$$

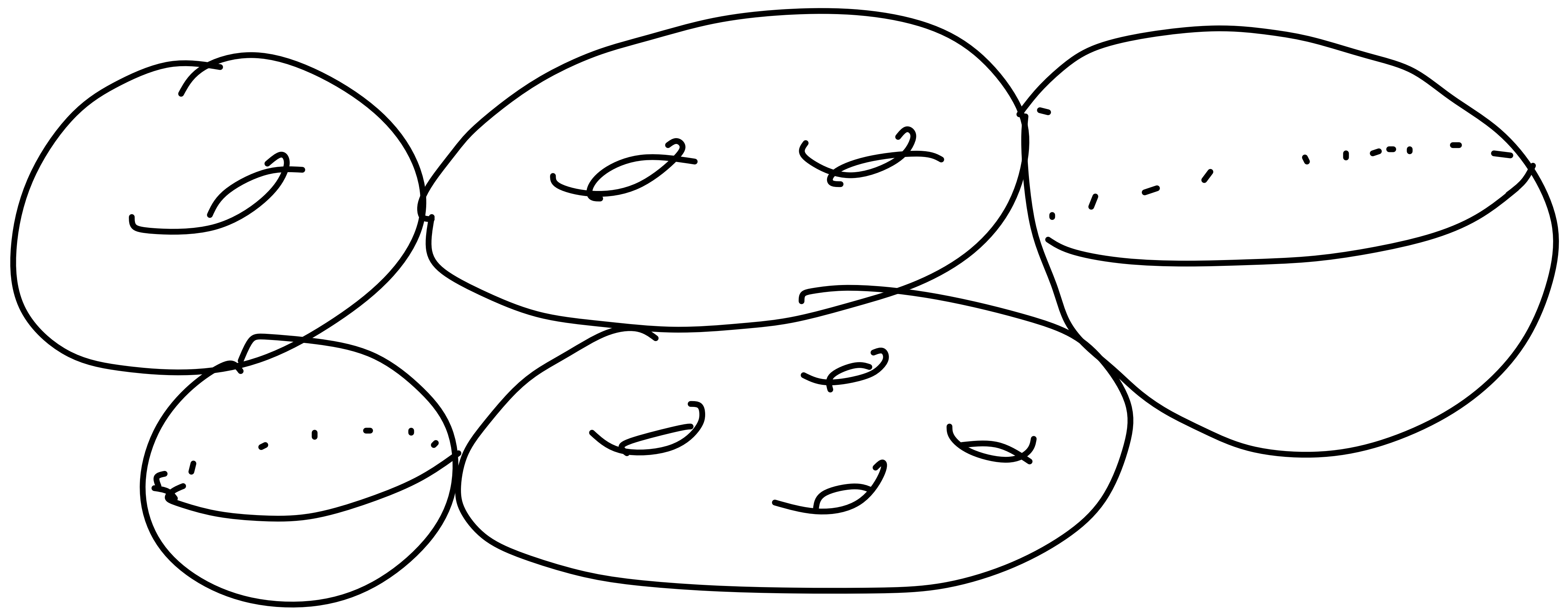
then the result is <u>symplectic</u>

Cutting open closed symplectic manifolds:

- (M^4, ω) : closed
  - ↳ $X \subseteq M$ symplectic + convex
  - ↳ $N \subseteq M$ symplectic + concave
- How do we find these?



Symplectic surface configurations:



- $D = \bigcup_i D_i \subseteq M$
- (D_i^2, ω) closed symplectic
- $D_i \pitchfork D_j$ is ω-orthogonal for $i \neq j$

- Thm: (Gay-Stipsicz, Gay-Mark, Li-Min, Li-Mak)

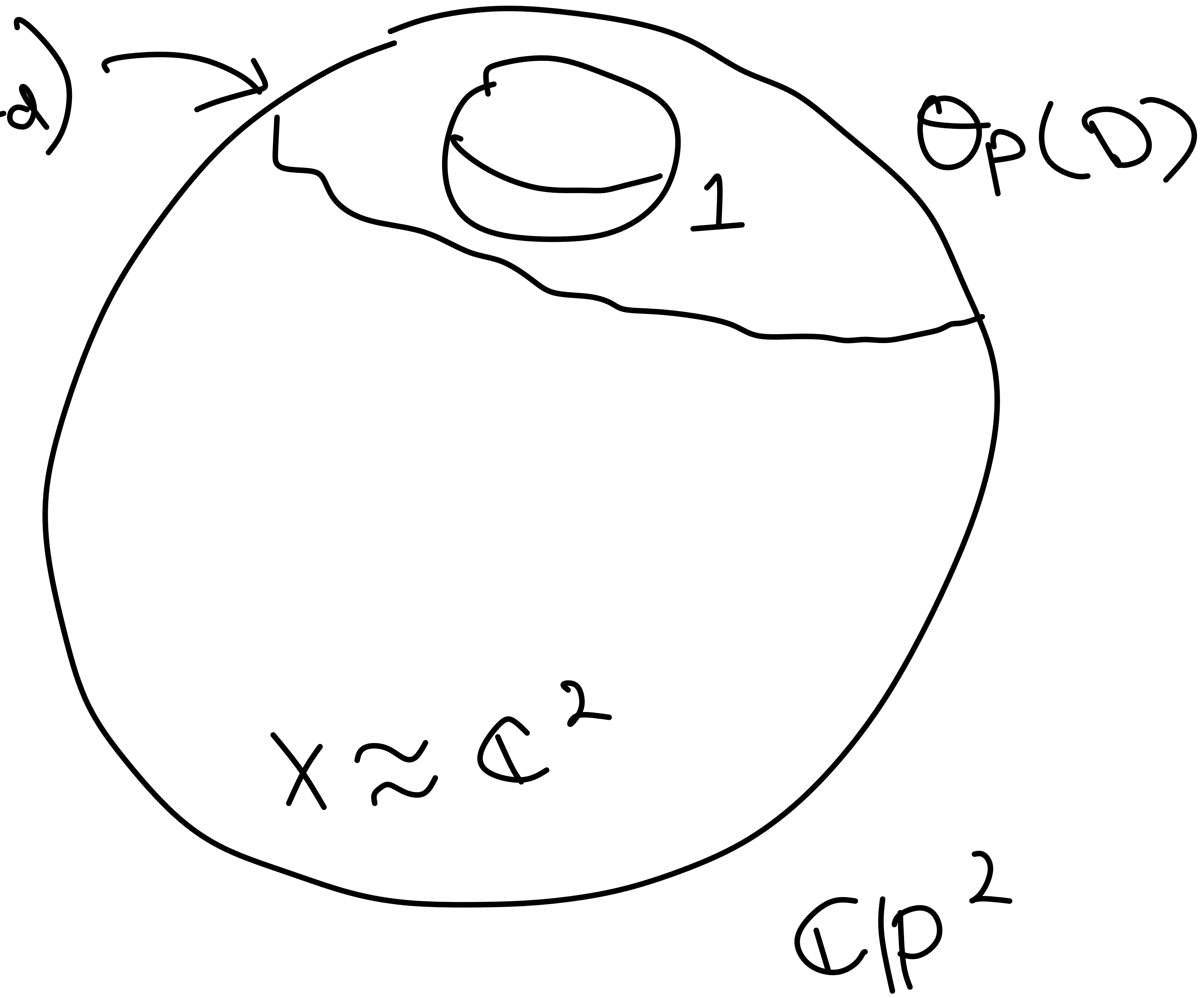
  Condition determines when $Op(D)$ is
  - Concave
  - Convex

Dissecting $\mathbb{CP}^2$:

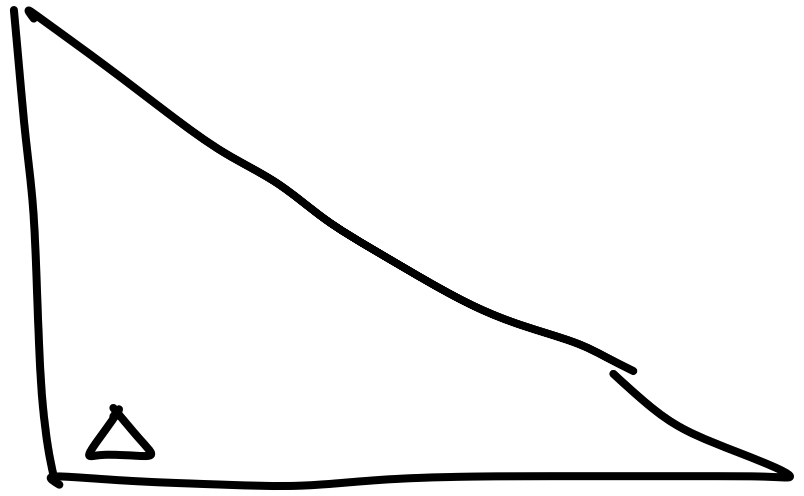
- $D = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$ at ∞
- $\mathbb{CP}^2 - D \cong \mathbb{C}^2$ so

(S^3, ξ_{std})

Contact S^3

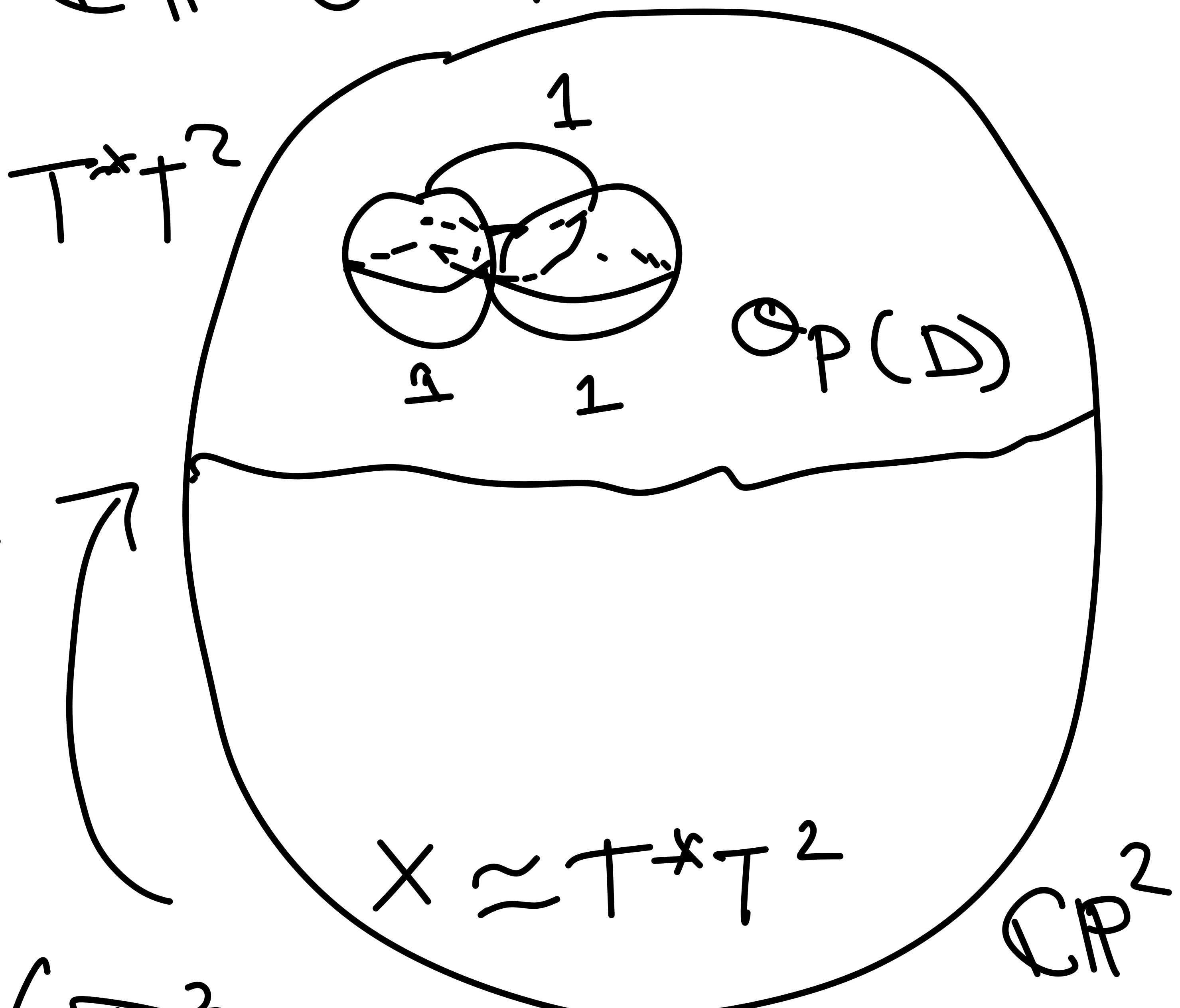


Or via moment polytope



- $\partial\Delta \sim D \cong \mathbb{CP}^1 \cup \mathbb{CP}^1 \cup \mathbb{CP}^1$
- $\mathbb{CP}^2 - D \cong T^*T^2$

open nbhd of T^2 fiber



Contact (T^3, ξ_1)

Heegaard diagrams for 5-manifolds

Geunyoung Kim
University of Georgia

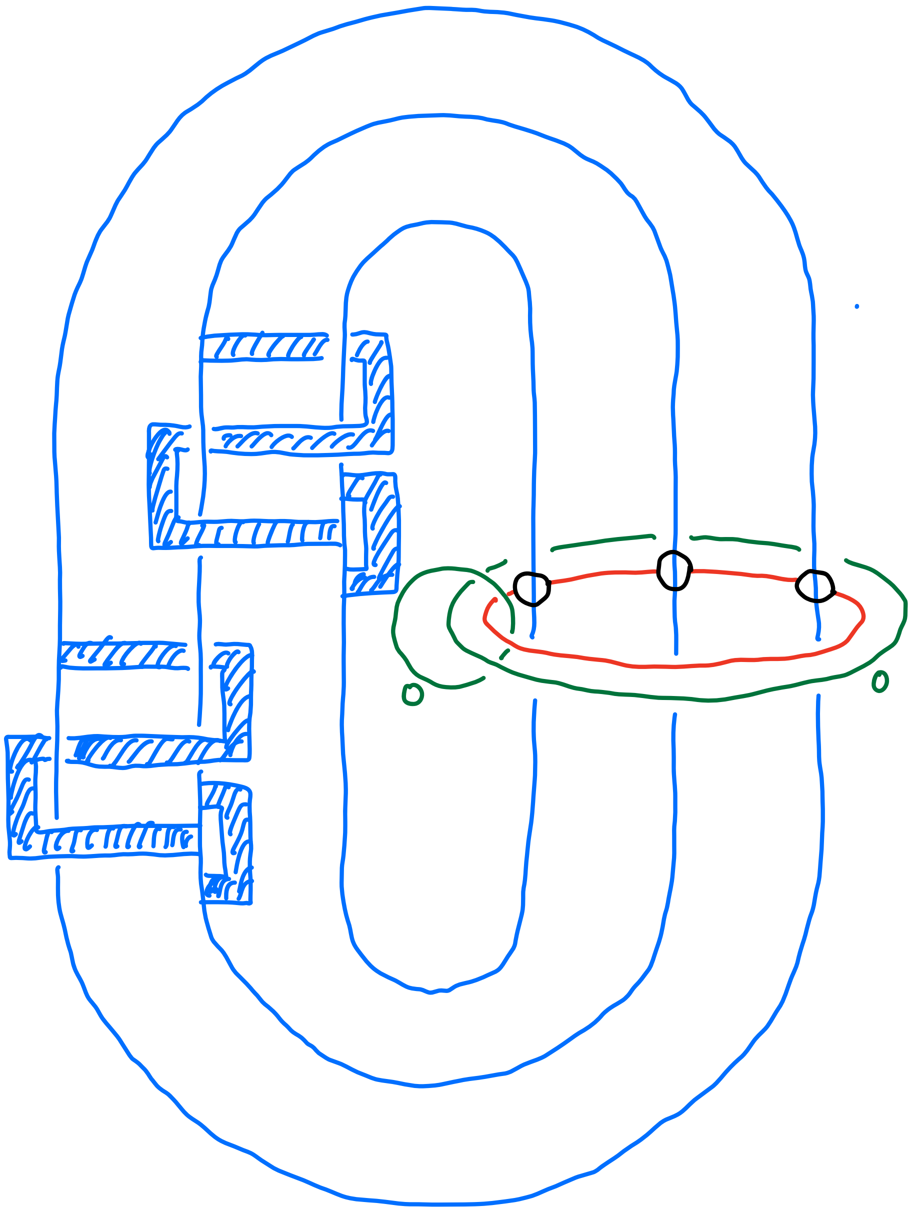
Definition

A <u>(5-dimensional) Heegaard diagram</u> is a triple (Σ, α, β) such that

- Σ is a closed, orientable, smooth 4-manifold,
- $\alpha = \alpha_1 \cup \cdots \cup \alpha_n \subset \Sigma$ is a 2-link with $\nu(\alpha) \cong \coprod_n S^2 \times B^2$,
- $\beta = \beta_1 \cup \cdots \cup \beta_m \subset \Sigma$ is a 2-link with $\nu(\alpha) \cong \coprod_m S^2 \times B^2$.

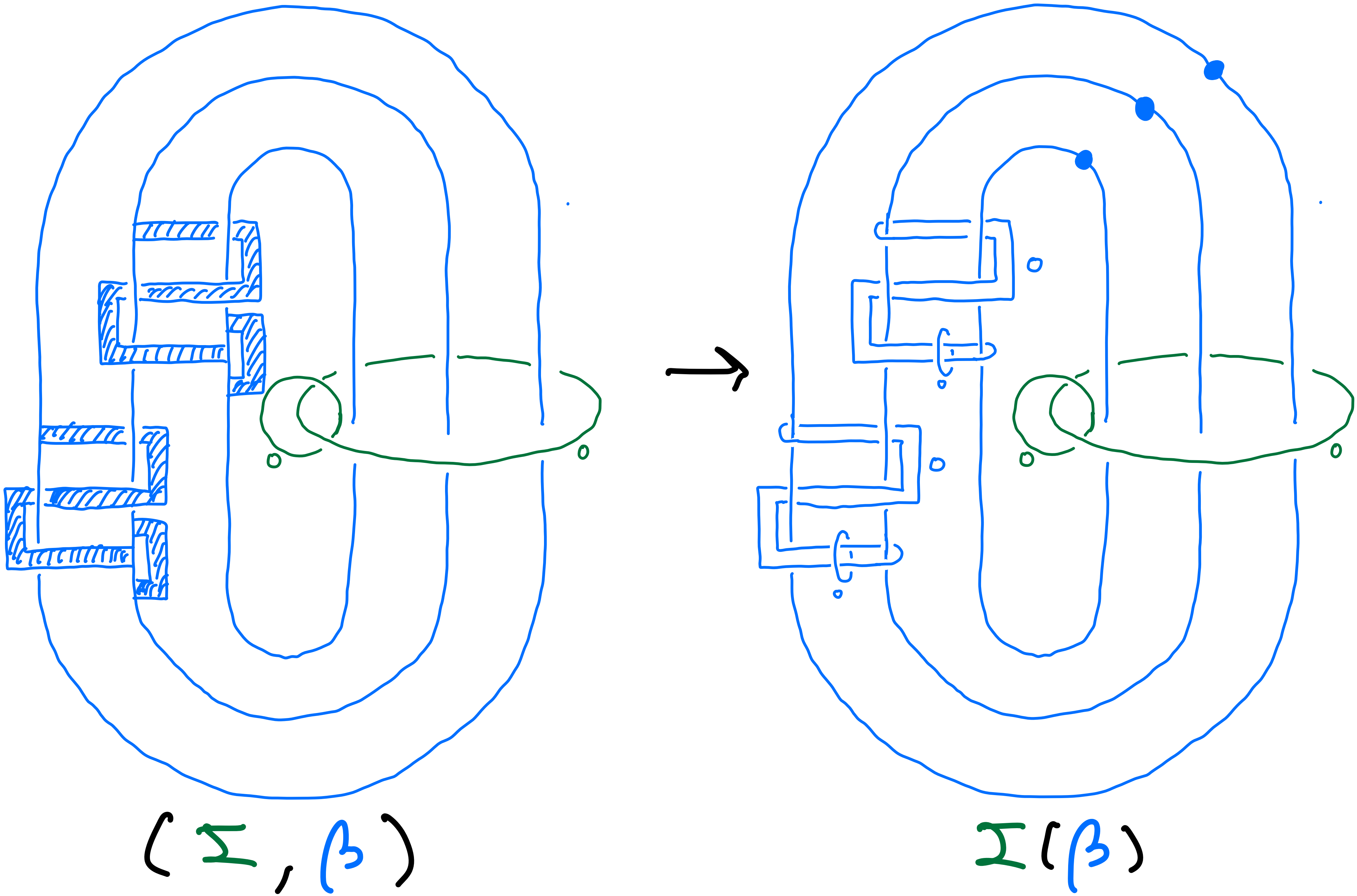
Example

$(\Sigma, \alpha, \beta) = (S^2 \times S^2, pt \times S^2, \beta)$



- $|\alpha \pitchfork \beta| = 3$ and $\alpha \cdot \beta = 1$
- $M_\alpha \cup_\Sigma M_\beta$: a cobordism from $\Sigma(\alpha)$ to $\Sigma(\beta)$
  - $\Sigma(\alpha)$: surgery of Σ along α; $\Sigma(\alpha) = S^4$
  - $\Sigma(\beta)$ = homology 4-sphere with $\pi_1 \neq 0$
- $\widehat{M_\alpha \cup_\Sigma M_\beta}$: contractible 5-manifold not homeomorphic to B^5

How to read off $\Sigma(\beta)$ from (Σ, α, β)?



Theorem

Every 5-manifold admits a (5-dimensional) Heegaard diagram.

Theorem

Two (5-dimensional) Heegaard diagrams represent diffeomorphic 5-manifolds if and only if they are related by isotopies, handle slides, stabilizations, and diffeomorphisms.

Theorem Let S^4_K be the Gluck twist of S^4 along a 2-knot K in S^4.

Let $(\Sigma, \alpha, \beta) = (S^2 \tilde{\times} S^2, F, K \# F)$ be a Heegaard diagram, where F is a fiber of $S^2 \tilde{\times} S^2$.

Then $M_\alpha \cup_\Sigma M_\beta$ is a cobordism from $\Sigma(\alpha) = S^4$ to $\Sigma(\beta) = S^4_K$.

Furthermore, the following are equivalent:

- $S^4_K \underset{\text{diffeo}}{\cong} S^4$,

- $M_\alpha \cup_\Sigma M_\beta \underset{\text{diffeo}}{\cong}$ twice punctured $S^2 \tilde{\times} S^3$,

- $(S^2 \tilde{\times} S^2, F, K \# F)$ and $(S^2 \tilde{\times} S^2, F, F)$ are related by isotopies, handle slides, stabilizations, and diffeomorphisms,

- $(S^2 \tilde{\times} S^2, K \# F) \underset{\text{diffeo}}{\cong} (S^2 \tilde{\times} S^2, F)$.

Thank you!

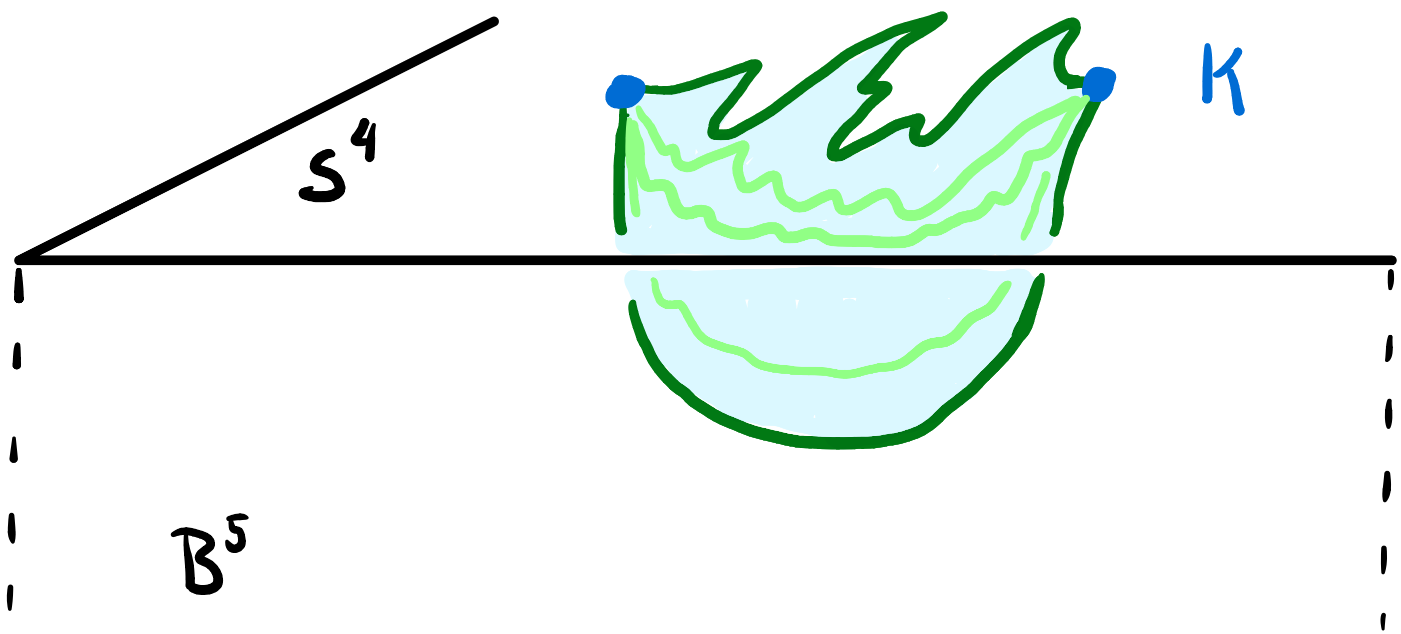
Smoothing 3-balls in the 5-ball

Daniel Hartman

Tech Topology Conference 2023

Main Theorem [H]

Let K be a smooth 2-knot bounding a locally flat top. 3-ball in S^4. Then the 3-ball is isotopic to a smooth 3-ball in the 5-ball rel boundary iff the Rochlin invariant of K vanishes.



Background

Rmk: All isotopies are rel boundary

Thm [Budney-Gabai] There exist infinitly many (Top) isotopy classes of properly embedded 3-balls bounded by the unknot in S^4.

Thm [Hughes-Kim-Miller] For every $g \geq 2$, there exists pairs of embedded Handle bodies of genus g which are not isotopic in S^4 or B^5.

Thm [H] $\pi_0 \, Emb_{\partial P}(B^3, B^5) = \{*\}$, ∂P = Boundary Parallel.

Definitions

Def Two manifolds M_0^n, M_1^n are h-cobordant if $\exists\, W^{n+1}$ such that $\partial W = -M_0^n \sqcup M_1$ and $i: M_i \to W$ is a homotopy equivalence. An S-cobordism is an h-cob where the inclusions are simple H.E.

Fact: Any h-cobordism is an S-cobordism if $Wh(\pi_1) = 0$, and this is true for $\pi_1 = \mathbb{Z}$.

Def: Let M^3 be a smooth 3-mfld with spin structure s, and let X^4 be any spin 4-manifold with $\partial X^4 = M^3$ with a compatible spin structure. Then the Rochlin invariant of M^3 is

$$\rho(M^3, s) = \sigma(X^4) \bmod 16$$

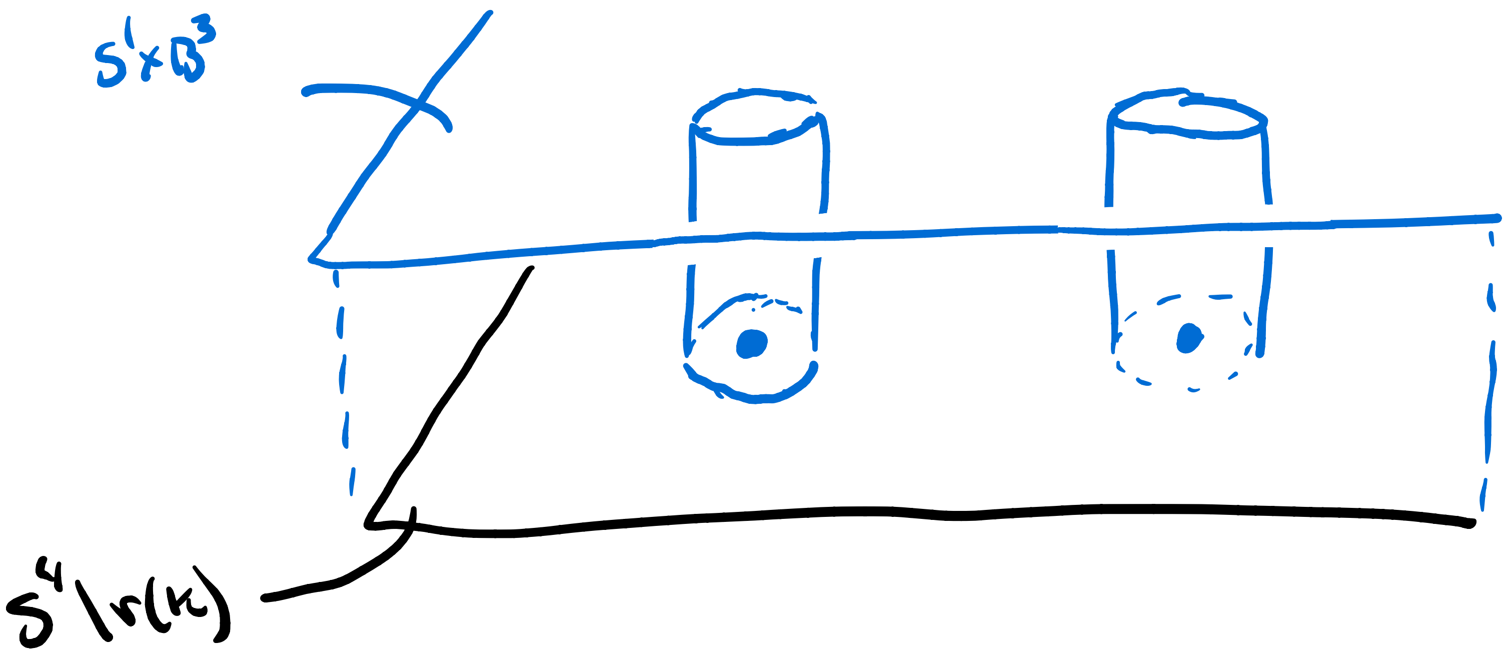
Let K be a 2-knot with seifert surface M^3 and spin structure induced by S^4. Then Rochlin inv of K is

$$\rho(K) = \frac{\rho(M^3, s)}{8}$$

Pf

Step 1: • $S^4 \setminus r(K)$ is smoothly s-cobordant to $S^1 \times B^3$ iff $\rho(K) = 0$ [Wall, Shaneson, Ruberman-Saveliev]

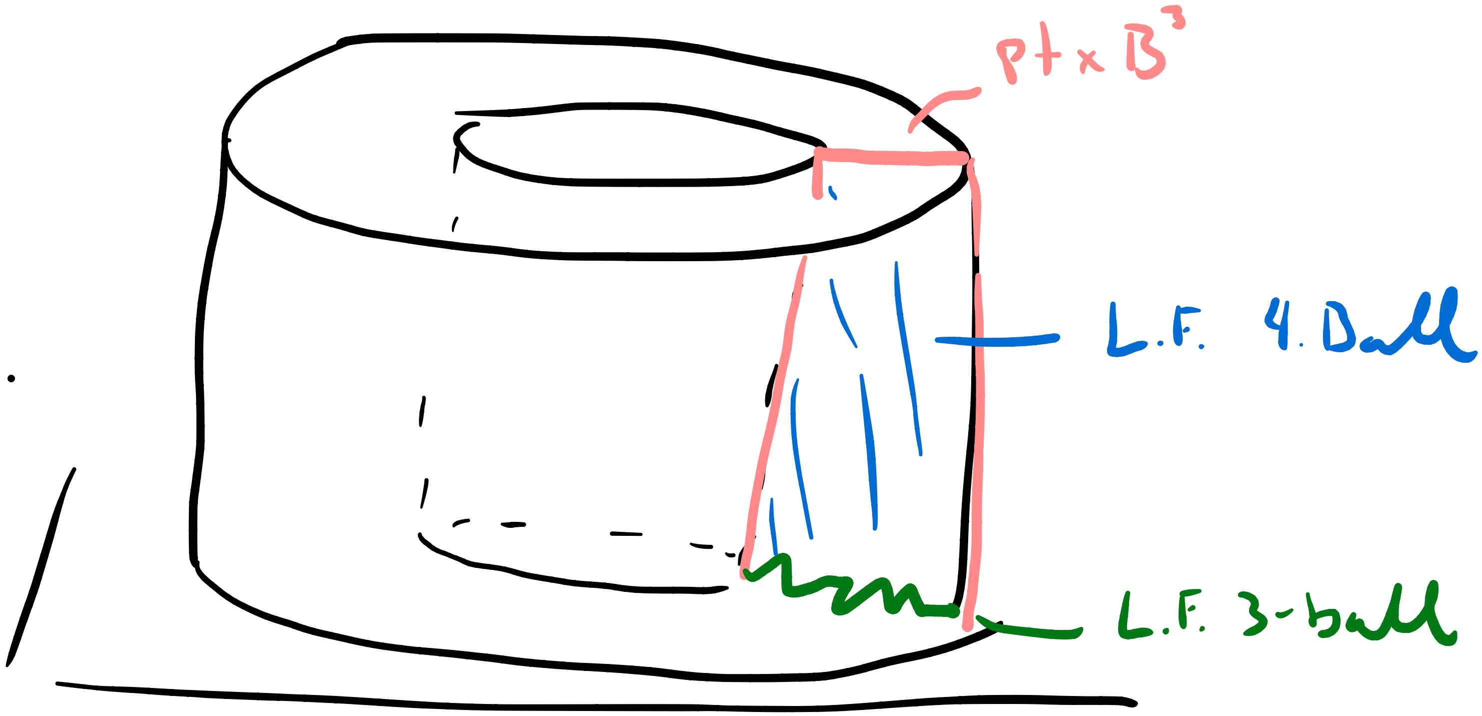
Stack the cobordism on S^4 (Top hat construction)



Step 2:
- Every s-cobordism with π_1-good is homeo to a product [Freedman-Quinn].
- Every L.F. 3-sphere in $S^1 \times S^3$ extends to a L.F. 4-ball in $S^1 \times B^4$.

Use these two facts to build a L.F. 4-ball embedded in the s-cobordism.

Note that Half the ∂ is smooth.



step 3 • There is a unique smooth structure on B^5

We use this to embed our entire construction into B^5 smoothly. $\blacksquare$

Questions

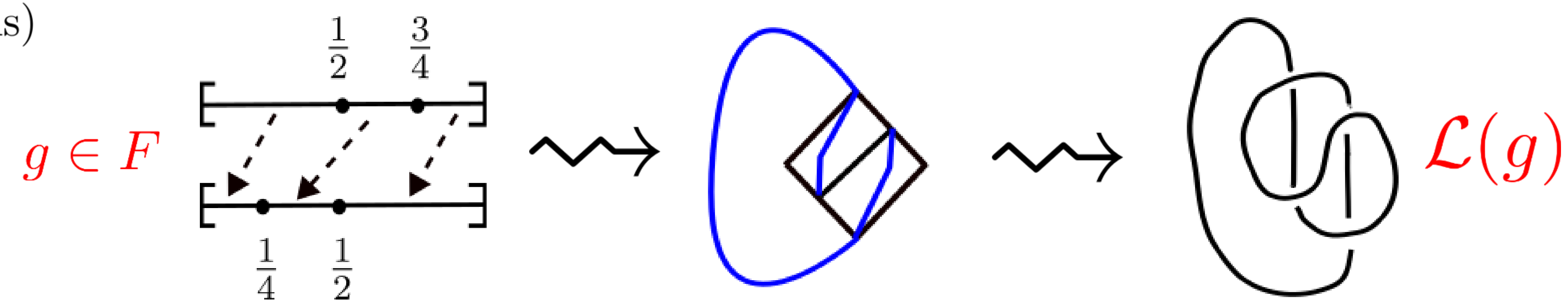
1) Can you have a Rochlin invariant 1 $\mathbb{Z}$-knot?

2) Is codim 1 top transversality an open condition in ambient dim ≥ 5?

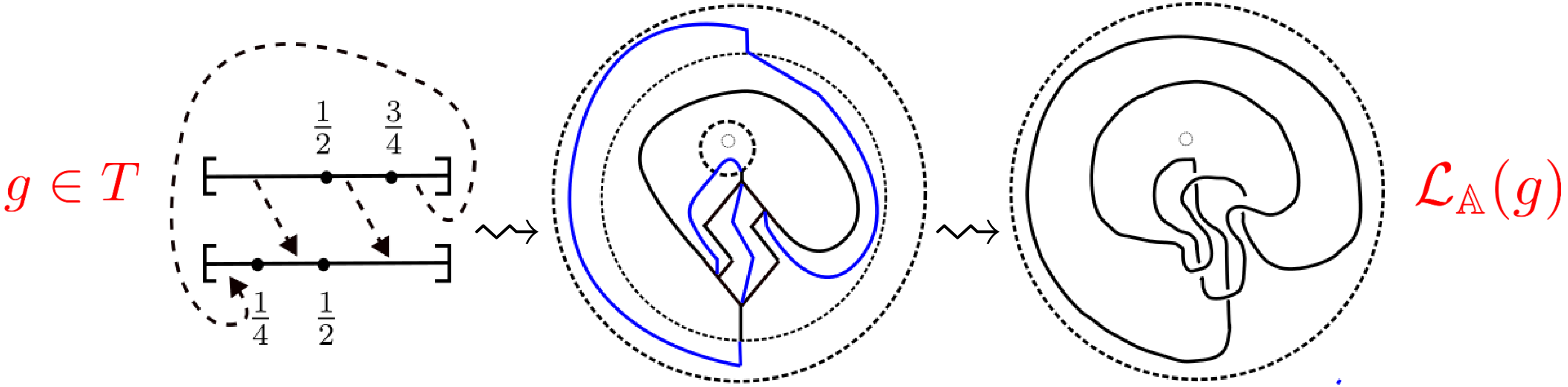
THANK YOU!

Constructing Annular Links from Thompson's group T

Jones '14: Links in S^3 from Thompson's group F (= PL functions $[0,1] \to [0,1]$ + some extra conditions)

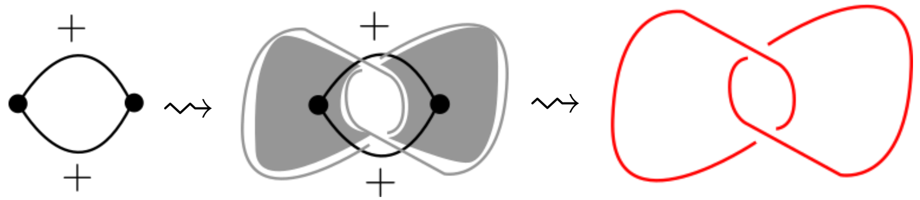


LL '23: Links in $\mathbb{A} \times I$ from Thompson's group T (= PL functions $S^1 \to S^1$ + some extra conditions)

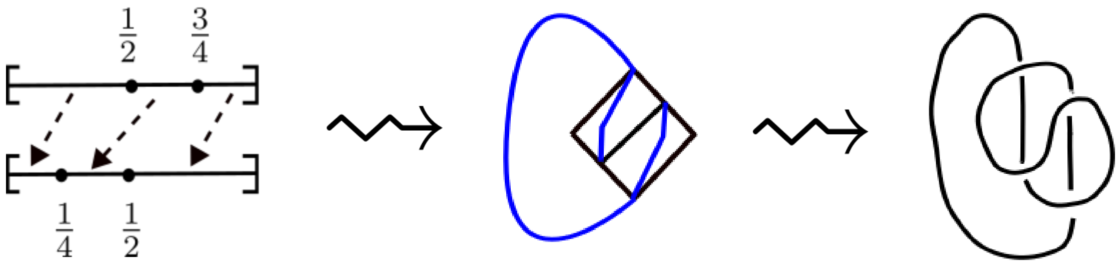


Known correspondence: graphs and links

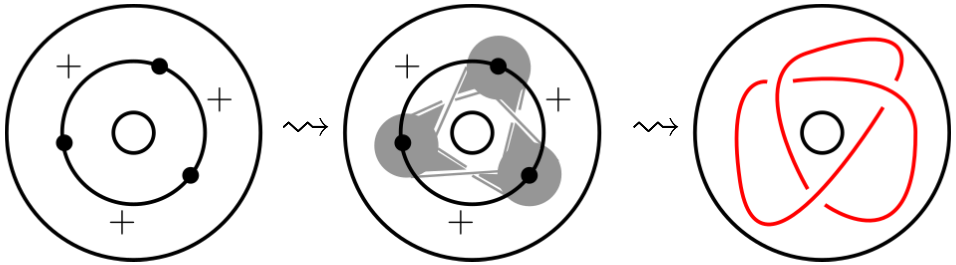
Tait graphs: $\Gamma \hookrightarrow \mathbb{R}^2 \rightsquigarrow L(\Gamma)$



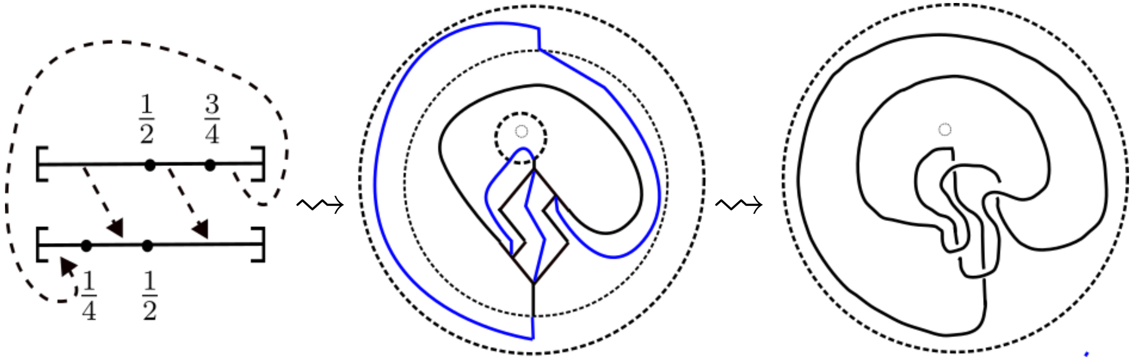
Thm (Jones '14): Given any edge-signed planar graph Γ, $\exists$ $g \in F$ such that $\mathcal{L}(g) = L(\Gamma)$.



Extension to $\mathbb{A}$: $\Gamma \hookrightarrow \mathbb{A} \rightsquigarrow L_{\mathbb{A}}(\Gamma)$



Thm (LL '23): Given any edge-signed graph $\Gamma \hookrightarrow \mathbb{A}$, $\exists g \in T$ such that $\mathcal{L}_{\mathbb{A}}(g) = L_{\mathbb{A}}(\Gamma)$.



Connections to Representation Theory

$g \in F \rightsquigarrow \mathcal{L}(g) \rightsquigarrow$ Jones polynomial $V_{\mathcal{L}(g)}(t)$

Thm (Aiello-Conti-Jones '18): For certain values of t, Jones polynomial defines a function of positive type on F.

$h \in T \rightsquigarrow \mathcal{L}_{\mathbb{A}}(h) \rightsquigarrow$ Jones polynomial $V_{\mathcal{L}_{\mathbb{A}}(g)}(t)$

Thm (LL '23): For certain values of t, Jones polynomial of annular links defines a function of positive type on T.

Fact: given a group g, $\{$functions of positive type from $g \to \mathbb{C}\} \longleftrightarrow \{$unitary representations of $g\}$.

Moral: The Jones polynomial of $\mathcal{L}(g)$ can arise as the coefficient of a unitary representation of F. Similarly, the Jones polynomial of the annular link $\mathcal{L}_{\mathbb{A}}(g)$ can arise as a unitary representation of T.

Heegaard Floer symplectic homology and generalized Viterbo's isomorphism theorem

Roman Krutowski

University of California, Los Angeles

based on joint work with Tianyu Yuan

Tech Topology conference
8-10 December, 2023

Symplectic cohomology

- Let (M^{2n}, λ) be a Liouville domain with λ a primitive of a symplectic form $\omega = d\lambda$ and Z be an outward-pointing Liouville vector field. Let

$$\hat{M} = M \cup_{\partial M} [0; +\infty) \times \partial M$$

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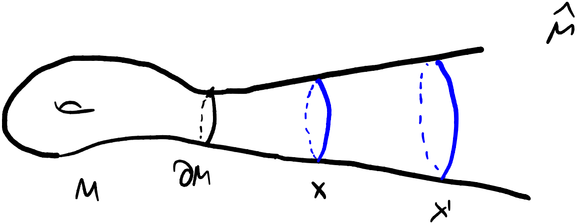

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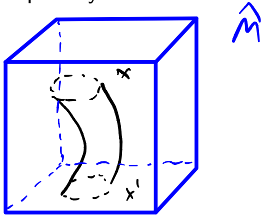

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Theorem (Viterbo, '99; Abbondandolo-Schwarz '06, Abouzaid '10)

For an oriented, closed smooth manifold Q there is an isomorphism

$$SH^*_b(T^*Q) \cong H_{n-*}(\Lambda Q).$$

Multiple particles

- Let us instead consider a Hamiltonian motion of $\kappa \geqslant 1$ identical particles in $\hat{M}$. Is there a Floer-theoretic invariant of M associated with closed orbits of such motion?

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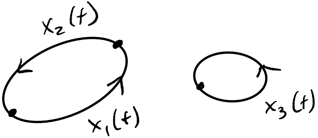
- The cochains correspond to tuples of closed Hamiltonian orbits of cumulative time κ.

$\kappa = 3$



$\mathbf{x} = (x_1, x_2, x_3)$

$\sigma(\mathbf{x}) = (12) \in S_3$

Theorem (K.-Yuan)

The Heegaard Floer symplectic cohomology groups $SH^*_\kappa(M)$ are well defined and are invariants of the Liouville domain M, independent of all intrinsic choices of the Floer data required for its setup.

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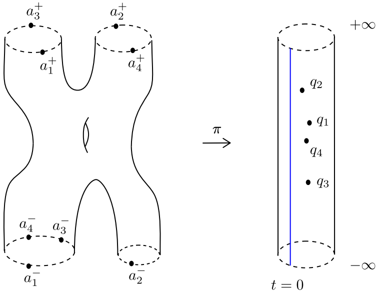
- The differential d_{SH_κ} is given by counting curves $u = (\pi, v)\colon S \to \mathbb{R} \times S^1 \times \hat{M}$ connecting two orbit tuples $\boldsymbol{x}$ and $\boldsymbol{x}'$ with $(S, \pi) \in \mathcal{H}^{\boldsymbol{\sigma},\boldsymbol{\sigma}'}_{\kappa,\chi}$, where σ and σ' are permutations in $\mathfrak{S}_\kappa$ associated with $\boldsymbol{x}$ and $\boldsymbol{x}'$.

$\sigma(x) = (13)(24)$

$\sigma(x') = (143)$



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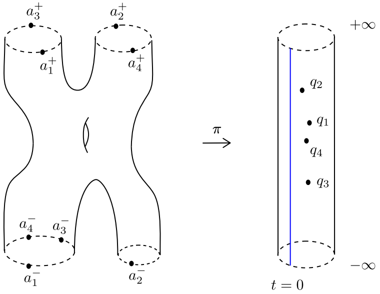
- **Key features:**



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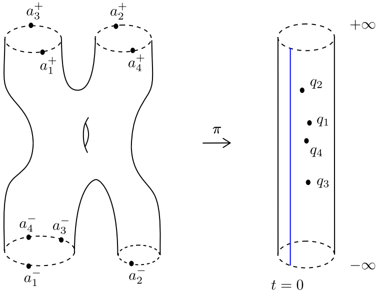

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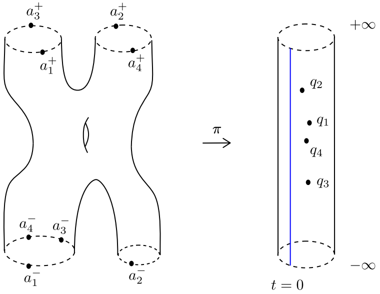
- Floer data on branched manifolds associated with Hurwitz spaces.



Morse multiloop complex

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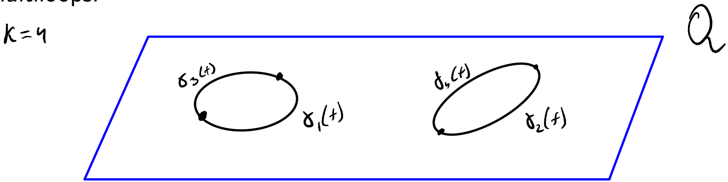
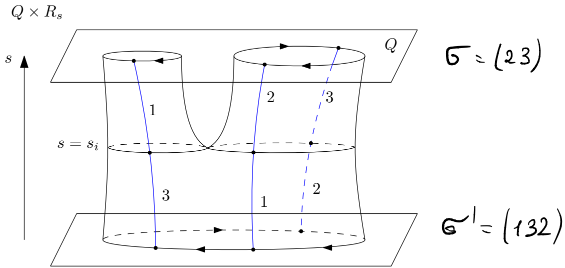
- We denote via $\Lambda^1_\kappa(Q)$ the space of free κ-multiloops of class $W^{1,2}$.
  The chain complex $CM_*(\Lambda^1_\kappa(Q))$ has generators associated with geodesic κ-multiloops.



Morse multiloop complex

- Differential ∂_M counts piecewise (pseudo)-gradient trajectories connecting two geodesic multiloops $\gamma, \gamma' \in \Lambda^1_\kappa(Q)$.



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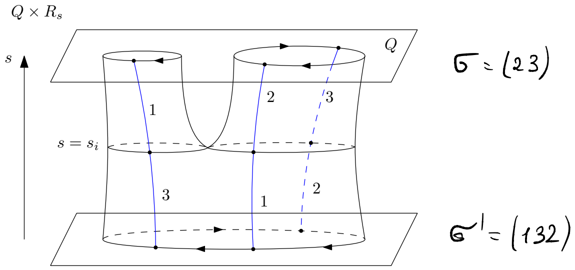


Theorem (K.-Yuan)

$(CM_*(\Lambda^1_\kappa(Q)), \partial_M)$ is a chain complex and its homology groups $HM_*(\Lambda^1_\kappa(Q))$ are independent of all auxiliary choices.

Generalized Viterbo's isomorphism theorem

Theorem (K.-Yuan)

For Q an orientable manifold with vanishing second Stiefel-Whitney class $w_2(TQ) \in H^2(T^*Q; \mathbb{Z}/2)$ there is a chain map

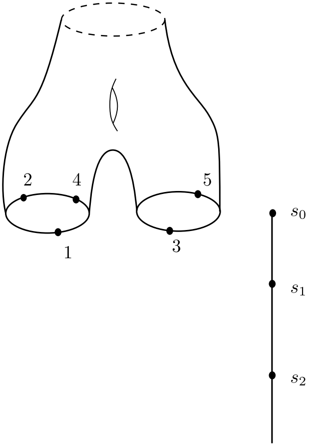
$$\mathcal{F}\colon SC^*_{\kappa,unsym}(T^*Q) \to CM_{\kappa n-*}(\Lambda^1_\kappa(Q)),$$

and it induces an isomorphism on the homology

$$SH^*_{\kappa,unsym}(T^*Q) \cong HM_{\kappa n-*}(\Lambda^1_\kappa(Q)).$$

Generalized Viterbo's isomorphism

- The chain map $\mathcal{F}$ is given by counting elements of mixed moduli spaces as in the figure and it coincides with Abouzaid's map for $\kappa = 1$.



Thank you for your attention!

CORKS FOR DIFFEOMORPHISMS

TERRIN WARREN (UGA)

TECH TOPOLOGY 2023

joint work with Slava Krushkal, Anubhav Mukherjee, and Mark Powell

MOTIVATION

EXOTIC MFLDS

<u>Thm</u>: (Freedman, Donaldson 80s)

$\exists$ X_0, X_1 : smooth simply connected 4mflds such that

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A: [KMPW]
sometimes

PSEUDOISOTOPY

(smooth)

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[Wall, Freedman]

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[Kreck, Perron, Quinn]

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∃ corks for mflds proved using h-cobordisms

PSEUDOISOTOPY

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X smooth simply connected 4 mfld

$\varphi: X \to X$ diffeo

[Kreck, Perron, Quinn] ← correction: Gabai, Gay, Hartman, Krushkal, Powell

$\varphi \underset{\text{top}}{\simeq} \text{id} \iff \varphi \underset{\text{P.I.}}{\sim} \text{id}$

IDEA: use pseudoisotopies to prove ∃ corks for diffeos

PSEUDOISOTOPY

DEF:

A diffeo $\varphi : X \to X$ is smoothly pseudoisotopic to the id if $\exists$ diffeo $\Phi : X \times I \to X \times I$ such that

$$\Phi|_{\partial X \times I} = id, \quad \Phi|_{X \times \{0\}} = id, \text{ and } \Phi|_{X \times \{1\}} = \varphi$$

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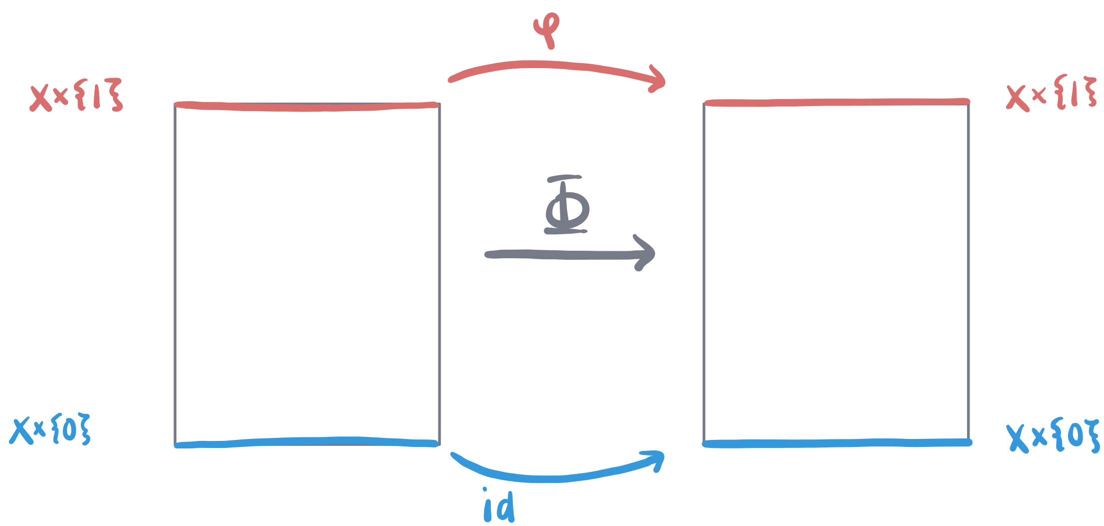

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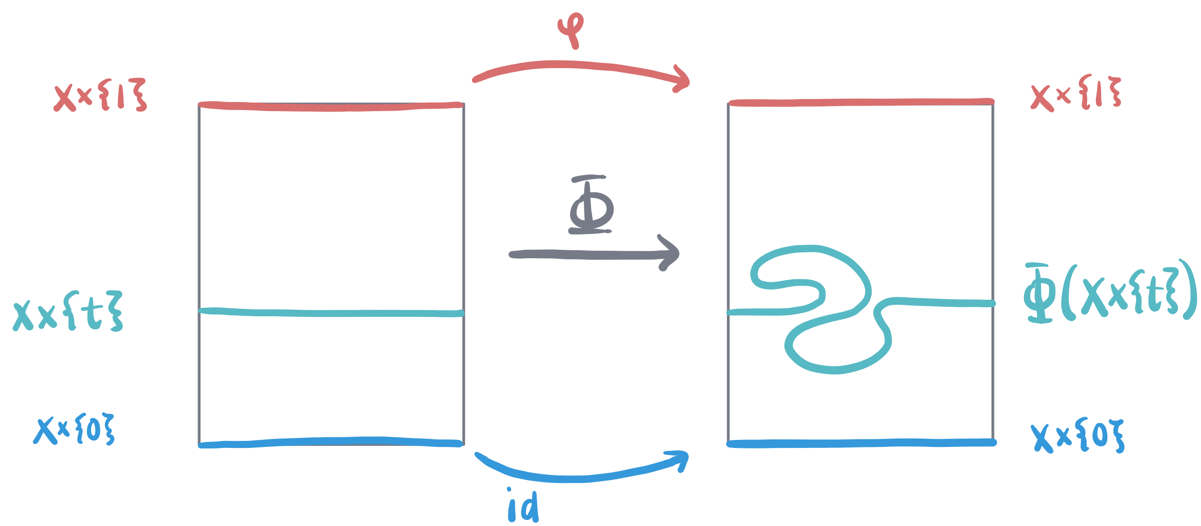

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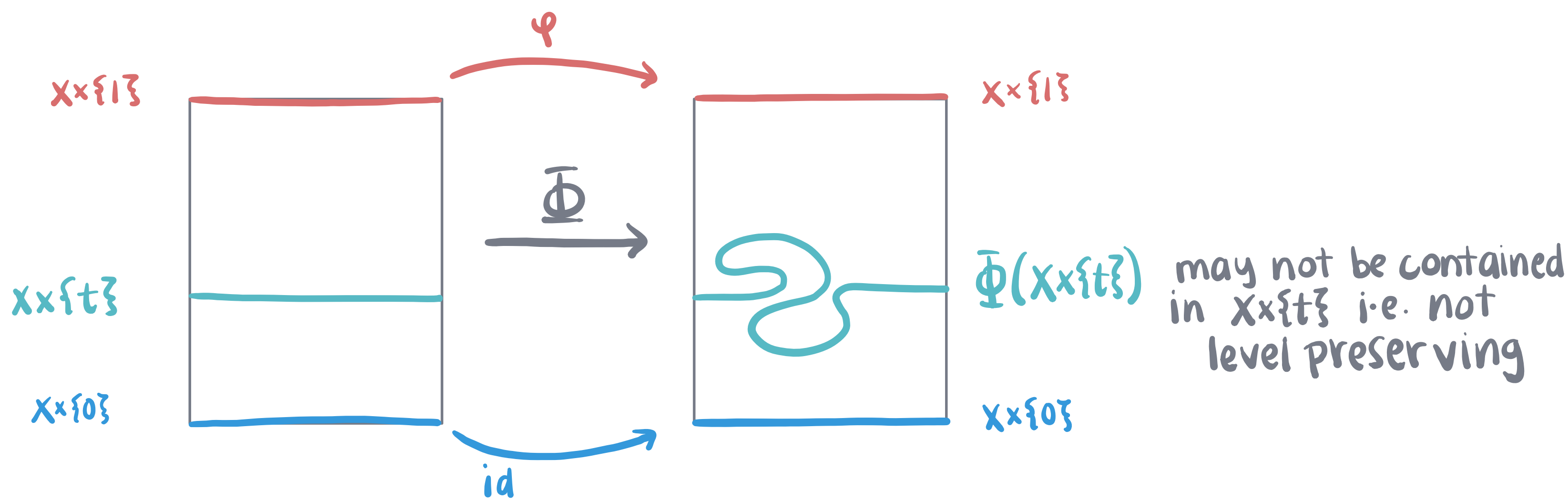

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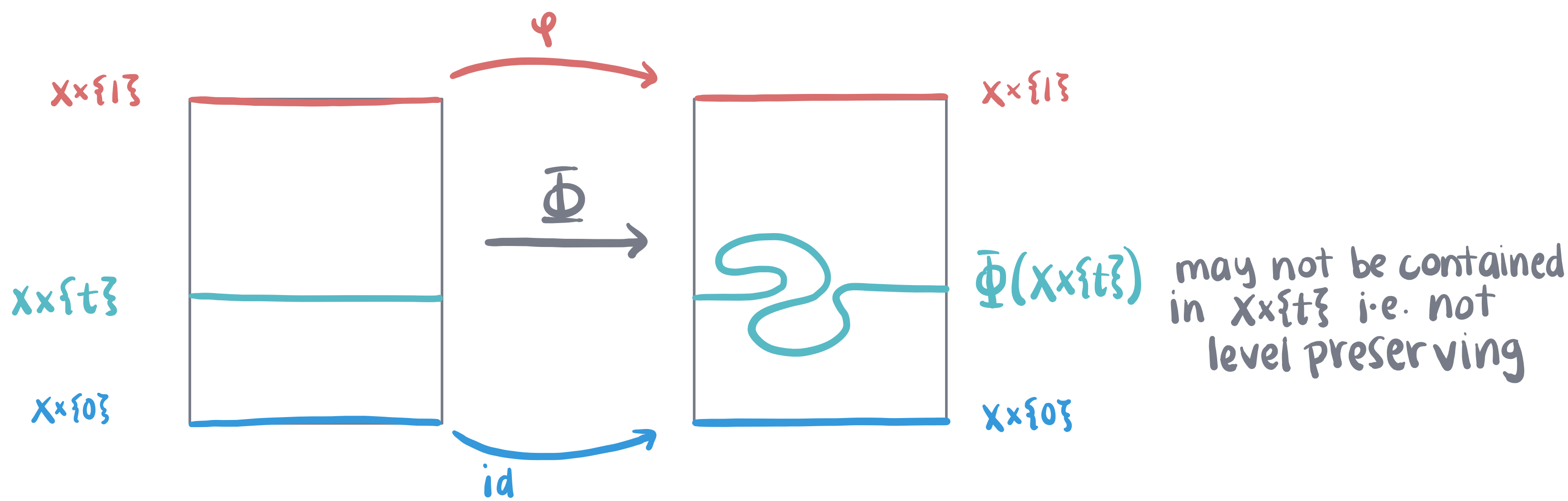

Rmk: if Φ is level preserving then Φ is a smooth isotopy

$$\Phi|_{X \times \{t\}} = \varphi_t : X \to X$$

SOME RESULTS

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φ : diffeomorphism of X

Φ : pseudoisotopy between φ and id

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Thm: [Krushkal, Mukherjee, Powell, W.]
If Φ has "one eye", then $\exists$ compact contractible $C \times I \subset X \times I$ and a smooth isotopy of $\Phi \simeq \Phi'$ such that

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Cor:

If φ is a diffeo that becomes smoothly isotopic to the identity after a single stabilization then $\exists$ compact, contractible $C \subset X$ and a smooth isotopy $\varphi \simeq \varphi'$ s.t.

$$\varphi'|_{X \setminus \mathring{C}} = id|_{X \setminus \mathring{C}}$$

THANKS FOR
LISTENING!