KNOTS AND CONTACT GEOMETRY II: CONNECTED SUMS

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ABSTRACT. We study the behavior of Legendrian and transverse knots under the operation of connected sums. As a consequence we show that there exist Legendrian knots that are not distinguished by any known invariant. Moreover, we classify Legendrian knots in some non-Legendrian simple knot types.

1. INTRODUCTION

The last few years have brought forth several advances in our understanding of Legendrian and transverse knots. Roughly speaking, our knowledge has advanced on two fronts: via the holomorphic theory and via 3-dimensional topology. The most concrete realization of the holomorphic theory is the theory of Chekanov-Eliashberg contact homology invariants [Ch, EGH]. This theory yielded the first examples of nonisotopic Legendrian knots with the same classical invariants: the topological type, the Thurston-Bennequin invariant, and the rotation number. (There are also more computable variants derived from contact homology, such as the characteristic algebra of Ng [Ng].) The purpose of these holomorphic invariants is to *distinguish*. Their counterpart is 3-dimensional contact *topology*, which has the flavor of classical 3-dimensional cut-and-paste topology with a slight twist. The main tool here is convex surface theory, introduced by Giroux [Gi]. Using recent advances in convex surfaces, the authors completely classified Legendrian torus knots and Legendrian figure eight knots [EH]. A complete classification of Legendrian knots of a certain topological type implies the complete classification of transverse knots of the same topological type [EH] (although not vice versa); hence transverse torus knots and transverse figure eight knots are classified (the predecessor to this result is [Et1]). More recently, Menasco [Me] classified all transverse iterated torus knots by using the work of Birman-Wrinkle [BW] which rephrased the classification question into a question in braid theory.

The goal of this paper is to prove a structure theorem for Legendrian knots, namely the behavior of Legendrian and transverse knots under the connected sum operation. Our main theorem (Theorem 3.4) classifies Legendrian knots in a non-prime knot type, provided we understand the classification for the prime summands. Theorem 3.4, in essence, is the relative version of Colin's gluing theorem for connected sums of tight contact manifolds [Co]. One corollary of our main theorem is the existence of Legendrian knots which are not contact isotopic but are indistinguishable by all known invariants (including the holomorphic invariants). Moreover, for any integer m, there exist Legendrian knots with identical invariants that are non-Legendrian-isotopic even after m stabilizations. Previously it was not known whether Legendrian knots (with identical invariants)

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became isotopic after one stabilization, largely due to the fact that the Chekanov-Eliashberg invariants vanish on stabilized Legendrian knots. Theorem 3.4 also implies the following: the connected sum of transversally simple knot types is transversally simple (see Section 3 for definitions).

The plan for the paper is as follows. After reviewing some background (especially on connected sums of knots) in Section 2, we give precise statements of the main theorem in Section 3 and its applications in Section 4. The main theorem is proved in Sections 5 and 6.

2. Some background and notation

We assume familiarity with basic notions in contact geometry, such as characteristic foliations and convex surface theory. This can be found in [Et2] (see also [Ae, El, Gi]). As this paper is a sequel to [EH] we assume the reader is familiar with its contents. In particular, Sections 2 and 3 of [EH] are foundational and develop the necessary terminology and background on Legendrian knots and transversal knots.

In this paper, our ambient 3-manifolds and knots are *oriented*, and "knot types" are oriented knot types. Let M_1 and M_2 be 3-manifolds. We first describe the *connected sum* of two (topological) knots $K_1 \subset M_1$ and $K_2 \subset M_2$. Let B_i be an open ball in M_i that intersects K_i in an unknotted arc α_i. Let $f : \partial(M_1 \setminus B_1) \to \partial(M_2 \setminus B_2)$ be an orientation-reversing diffeomorphism which sends $K_1 \cap \partial(M_1 \setminus B_1)$ to $K_2 \cap \partial(M_2 \setminus B_2)$. (Here, $X \setminus Y$ denotes the metric closure of the complement of Y in X.) Now the connected sum of M_1 and M_2 is

$$M_1 \# M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$$

and the *connected sum* of K_1 and K_2 in $M_1 \# M_2$ is

$$K_1 \# K_2 = (K_1 \setminus \alpha_1) \cup (K_2 \setminus \alpha_2).$$

Note that there are two possible identifications of $K_1 \cap \partial(M_1 \setminus B_1)$ with $K_2 \cap \partial(M_2 \setminus B_2)$ — we choose the one which induces a coherent orientation on $K_1 \# K_2$. It is an easy exercise to see that $K_1 \# K_2$ is well-defined and its topological type is independent of the choices of B_i and f.

If $K_1, K_2 \subset S^3$, we can interpret the connected sum operation as happening entirely in S^3, since $S^3 \# S^3 = S^3$. In particular, fix a 2-sphere S in S^3 that splits S^3 into two balls B_1 and B_2. Then isotop K_1 so that it intersects B_2 in an unknotted arc and isotop K_2 so that it intersects B_1 in an unknotted arc. Moreover, we can arrange for K_1 and K_2 to intersect S at the same points. We then define $K_1 \# K_2 = (K_1 \cap B_1) \cup (K_2 \cap B_2)$. This clearly is an equivalent definition of the connected sum in S^3. From this definition it is easy to arrive at the familiar diagrammatic picture of a connected sum. See Figure 1.

A knot K in S^3 is *prime* if $K = K_1 \# K_2$ implies that either K_1 or K_2 is the unknot. Any knot $K \subset S^3$ admits a unique (minimal) decomposition into prime pieces, *i.e.*, $K = K_1 \# \dots \# K_n$ with $K_i, i = 1, \dots, n$, prime. This decomposition can be achieved by finding a collection $\{S_1, \dots, S_{n-1}\}$ of disjoint 2-spheres in S^3 that each intersects K in two points. Given such a *separating sphere* S_i, we may reverse the procedure described in the preceding paragraph to write the knot as the connected sum of two other knots.

Although the collection $\{K_1, \dots, K_n\}$ is unique up to isotopy, the collection $\{S_1, \dots, S_{n-1}\}$ of separating spheres is not. To avoid confusion in what follows, whenever we decompose a knot in S^3, we will be doing so with respect to a *fixed* collection of separating spheres $\{S_1, \dots, S_{n-1}\}$. Moreover, we take $K = K_1 \# \dots \# K_n$ to mean the following: using the same notation from the

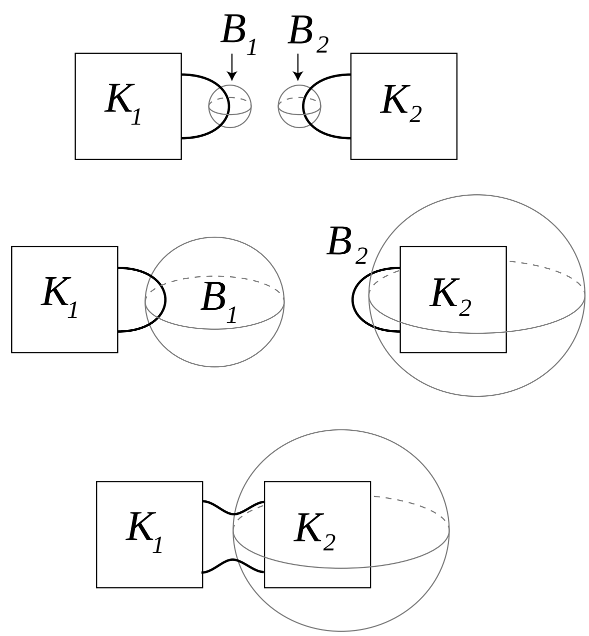

FIGURE 1. Diagrammatic connected sums.

second paragraph of this section, we glue the $K_i \setminus \alpha_i$ together so that the endpoint of $K_i \setminus \alpha_i$ connects to the initial point of $K_{i+1} \setminus \alpha_{i+1}$ modulo n. (This makes sense since the K_i are oriented.) This way, the $K_1, \dots, K_n$ are cyclically strung together in order.

Let $\mathcal{K}$ be a topological knot type in a 3-manifold M, *i.e.*, an equivalence class of (topologically) isotopic knots. Define $\mathcal{L}(\mathcal{K}, M, \xi)$ to be the set of isotopy classes of Legendrian knots in (M, ξ) of type $\mathcal{K}$. If the contact manifold (M, ξ) is implicit, then we write $\mathcal{L}(\mathcal{K})$ instead of $\mathcal{L}(\mathcal{K}, M, \xi)$.

3. THE MAIN THEOREM

We first explain Colin's gluing theorem [Co]. Denote the space of tight contact 2-plane fields on a 3-manifold M by $Tight(M)$. Then we have the following:

Theorem 3.1 (Colin)**.** *Given two 3-manifolds M_1, M_2, there is an isomorphism*

$$\pi_0(Tight(M_1)) \times \pi_0(Tight(M_2)) \xrightarrow{\sim} \pi_0(Tight(M_1 \# M_2)).$$

Let (M_i, ξ_i), $i = 1, 2$, be two tight contact manifolds. Choose $p_i \in M_i$ as well as a standard contact 3-ball B_i with coordinates (x, y, z) about p_i so that the contact structure is given by $dz + xdy = 0$. After possibly perturbing the boundary of B_i, there is an orientation-reversing map f from $S_1 = \partial(M_1 \setminus B_1)$ to $S_2 = \partial(M_2 \setminus B_2)$ that takes the characteristic foliation of S_1 to that of S_2. According to Colin's theorem, the contact structure ξ induced on

$$M = M_1 \# M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$$

is tight, and is independent of the choice of B_i, p_i, and f, up to isotopy. Moreover, every tight ξ on M arises, up to isotopy, from a unique pair (ξ_1, ξ_2) of tight contact structures.

Let us now explain the Legendrian connected sum operation, which is a relativized version of Colin's connected sum operation. In each (M_i, ξ_i), choose an oriented Legendrian knot L_i and a point $p_i \in L_i$. Normalize the standard contact 3-ball B_i so that $L = B_i \cap y$-axis, and further require that f maps $L_1 \cap S_1$ to $L_2 \cap S_2$ as oriented manifolds. Then we obtain the Legendrian knot $L = L_1 \# L_2 \subset M$, which is called the *connected sum* $L_1 \# L_2$ of L_1 and L_2.

Lemma 3.2. *The connected sum of two Legendrian knots does not depend on the points p_i, the balls B_i, or f used in the definition.*

Although this lemma is not difficult to prove, we defer the proof until Section 6. See [Ch] for a diagrammatic proof.

Given a nullhomologous Legendrian knot L, we can define its Thurston-Bennequin invariant $\mathrm{tb}(L)$ and it rotation number $r(L)$. (For more details, consult [Et2] and [EH], for example.) We then have:

Lemma 3.3. *If L_1 and L_2 are two nullhomologous Legendrian knots, then*

$$\mathrm{tb}(L_1 \# L_2) = \mathrm{tb}(L_1) + \mathrm{tb}(L_2) + 1, \tag{1}$$

and

$$r(L_1 \# L_2) = r(L_1) + r(L_2). \tag{2}$$

This lemma easily follows from the facts that tb and r can be computed from the characteristic foliation of a Seifert surface of a knot (see [EF]) and that we can control the characteristic foliation on the Seifert surface for $L_1 \# L_2$ in terms of the foliations on the surfaces for L_1 and L_2.

We denote by $S_\pm(L)$ the $\pm$ stabilization of the Legendrian knot L. Recall that this is a procedure to reduce tb of a Legendrian knot by 1 (see [EH]) and diagrammatically corresponds to "adding kinks" to L. The following is our main theorem.

Theorem 3.4. *Let $\mathcal{K} = \mathcal{K}_1 \# \dots \# \mathcal{K}_n$ be a connected sum decomposition of a topological knot type $\mathcal{K} \subset (M, \xi)$ into prime pieces $\mathcal{K}_i \subset (M_i, \xi_i)$, where $(M, \xi) = (M_1, \xi_1) \# \dots \# (M_n, \xi_n)$ is tight. Then the map*

$$C : \left(\frac{\mathcal{L}(\mathcal{K}_1) \times \cdots \times \mathcal{L}(\mathcal{K}_n)}{\sim} \right) \longrightarrow \mathcal{L}(\mathcal{K}_1 \# \dots \# \mathcal{K}_n) \tag{3}$$

given by $(L_1, \dots, L_n) \mapsto L_1 \# \dots \# L_n$ is a bijection. Here the equivalence relation $\sim$ is of two types:

1. $(L_1, \dots, S_\pm(L_i), L_{i+1}, \dots, L_n) \sim (L_1, \dots, L_i, S_\pm(L_{i+1}), \dots, L_n)$,
2. $(L_1, \dots, L_n) \sim \sigma(L_1, \dots, L_n)$, *where σ is a permutation of the $\mathcal{K}_i \subset (M_i, \xi_i)$ such that $\sigma(M_i, \xi_i)$ is isotopic to (M_i, ξ_i) and $\sigma(\mathcal{K}_i) = \mathcal{K}_i$.*

Theorem 3.4 will be proved in Section 6. We now discuss its consequences. Let $\overline{\mathrm{tb}}(\mathcal{K})$ denote the maximal Thurston-Bennequin invariant over elements in $\mathcal{L}(\mathcal{K})$. Then,

Corollary 3.5. $\overline{\mathrm{tb}}(\mathcal{K}_1 \# \mathcal{K}_2) = \overline{\mathrm{tb}}(\mathcal{K}_1) + \overline{\mathrm{tb}}(\mathcal{K}_2) + 1$.

This corollary was independently proven by Torisu in [To].

Theorem 3.4 takes on a particularly simple form when one restricts attention to maximal Thurston-Bennequin knots.

Corollary 3.6. *If L is a Legendrian knot which is a maximal tb representative of $\mathcal{L}(\mathcal{K})$, then L admits a unique prime decomposition (modulo potential permutations).*

Recall the strategy described in [EH] for classifying Legendrian knots of a given a topological knot type. The idea was to (1) prove that Legendrian knots in a particular knot type would always destabilize to a knot with maximal tb and then (2) classify all Legendrian knots of this type with maximal tb. When executing the final stage of this strategy, Corollary 3.6 is useful because it allows us to concentrate on prime knots.

4. Applications

In discussing the applications of Theorem 3.4, we restrict our attention to Legendrian knots in the standard tight contact (S^3, ξ_{S^3}) or $(\mathbb{R}^3, \xi_{\mathbb{R}^3})$. Similar results hold in other manifolds.

First we reformulate the connected sum operation. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two topological knot types. Given $L_i \in \mathcal{L}(\mathcal{K}_i)$, $i = 1, 2$, we define their *connected sum* as follows: let B_i be a standard contact 3-ball about a point p_i on L_i. By Eliashberg's classification theorem of tight contact structures on the 3-ball [El], there is a contact isotopy $\phi_t : S^3 \to S^3, t \in [0, 1]$, from $\phi_0 = \mathrm{id}_{S^3}$ to ϕ_1 which takes B_1 to $S^3 \setminus B_2$. Moreover, it is easy to arrange L_2 and $\phi_1(L_1)$ to intersect ∂B_2 in the same two points. We may now define $L_1 \# L_2$ to be the Legendrian knot $(L_2 \cap (S^3 \setminus B_2)) \cup (\phi_1(L_1) \cap B_2)$. We leave it as a simple exercise to check that this definition of the connected sum of knots is equivalent to the one given above. It has the advantage of being done ambiently, *i.e.*, we do not take connected sums of the ambient manifolds, only of the knots.

Since any Legendrian isotopy can be assumed to miss a preassigned point, the classification of Legendrian knots in (S^3, ξ_{S^3}) and in $(\mathbb{R}^3, \xi_{\mathbb{R}^3})$ are equivalent. Moreover, there is a convenient diagrammatic description of Legendrian knots in $\mathbb{R}^3$ in terms of front projections (for example, see [EH]). Figure 2 indicates two ways in which the ambient connected sum (described in the previous

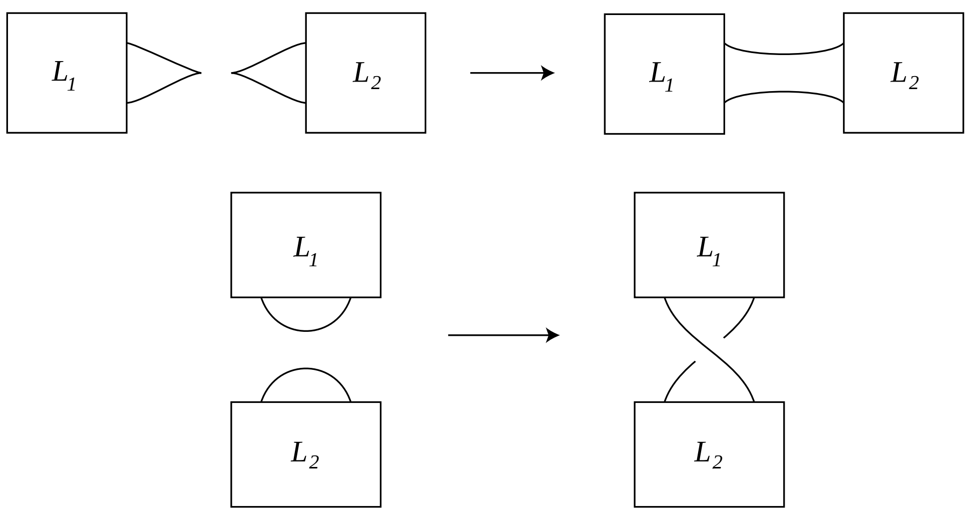

FIGURE 2. Two diagrammatic versions of Legendrian connected sum.

paragraph) can be done in terms of the front projections of Legendrian knots.

Perhaps the most interesting application of Theorem 3.4 is towards the construction of topological knot types which are not Legendrian simple. Recall that a topological knot type $\mathcal{K}$ is said to be *Legendrian simple* if Legendrian knots in $\mathcal{K}$ are classified by the Thurston-Bennequin invariant and the rotation number. The first non-Legendrian-simple knot type was discovered in [Ch] and, since then, many similar examples have been found. All examples to date have used contact homology (in one form or another) to distinguish the Legendrian knots. Although contact homology provides an intriguing way of distinguishing Legendrian knots, it currently does not provide much geometric insight into why Legendrian knots are different.

Theorem 4.1. *Given two positive integers m and n, there is a knot type $\mathcal{K}$ and distinct Legendrian knots $L_1, \dots, L_n$ in $\mathcal{L}(\mathcal{K})$ which have the same Thurston-Bennequin invariant and rotation number, and remain distinct even after m stabilizations (of any type).*

Theorem 4.1 follows from Theorem 3.4 and the classification of Legendrian torus knots (Theorem 4.2 below) from [EH]. Recall that $\mathcal{K}_{p,q}$ is a (p,q)-*torus knot* if any element of $\mathcal{K}_{p,q}$ can be isotoped to sit on a *standardly embedded* torus T in S^3 as a (p,q)-curve. Here we say a torus $T \subset S^3$ is *standardly embedded* if it is oriented and $S^3 \setminus T = N_1 \cup N_2$, where N_i, $i = 1, 2$, are solid tori with $\partial N_1 = T$ and $\partial N_2 = -T$. Now, there exists an oriented identification $T \simeq \mathbb{R}^2/\mathbb{Z}^2$ where the meridian of N_1 corresponds to $\pm(1,0)$ and the meridian of N_2 to $\pm(0,1)$.

Theorem 4.2. *Legendrian knots in $\mathcal{L}(\mathcal{K}_{p,q})$ are determined by their knot type, Thurston-Bennequin invariant and rotation number. If $p < 0$ and $-p > q > 0$, then $\overline{\mathrm{tb}}(\mathcal{K}_{p,q}) = pq$ and the corresponding values of r are*

$$r(K) \in \{\pm(|p| - |q| - 2qk) : k \in \mathbb{Z}, 0 \leq k < \frac{|p| - |q|}{|q|}\}.$$

Moreover, all other Legendrian knots in this knot type are obtained by stabilization.

If we plot the possible values of tb and r for a negative torus knot, we obtain a picture similar to that of Figure 3.

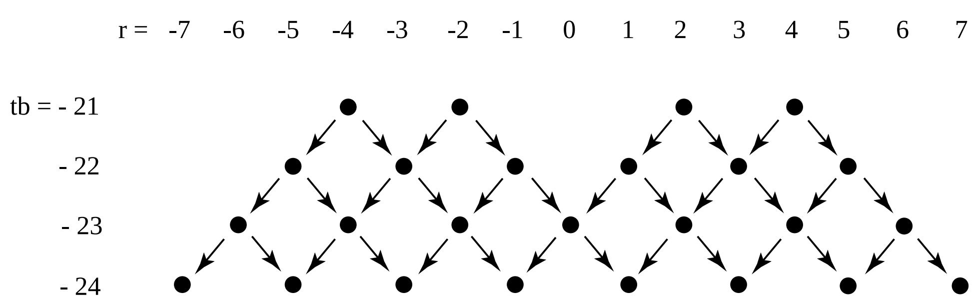

FIGURE 3. Some possible tb and r's for the $(-7,3)$ torus knot.

Proof of Theorem 4.1. Let $p = -(4n+1)s - 1$ and $q = 2s$, with s an even number greater than $m+1$. Then, according to Theorem 4.2, there are $4n$ Legendrian knots in $\mathcal{L}(\mathcal{K}_{p,q})$ with maximal $\mathrm{tb} = pq$ and distinct rotation numbers $-(4n-3)s+1, \dots, (4n-5)s+1, (4n-1)s+1$ and $-(4n-1)s-1, -(4n-5)s-1, \dots, (4n-3)s-1$. Let $L_r \in \mathcal{L}(\mathcal{K}_{p,q})$ with $\mathrm{tb} = pq$ and rotation number r. For $k = 0, \dots, 2n-1$, let $L^k = L_{(4(n-k)-1)s+1} \# L_{-(4(n-k)-1)s-1}$. Note that all the L^k are topologically isotopic, have the same $\mathrm{tb} = 2pq+1$ and $r = 0$, yet are not Legendrian isotopic by Theorem 3.4. See Figure 4. Since the spacings in r between adjacent maximal tb representatives are at least $2m$ by our choice of p and q, the L^k remain distinct even after m stabilizations. $\square$

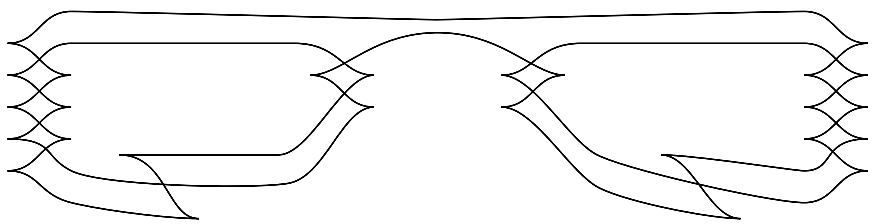

FIGURE 4. When $m = 0$ in Theorem 4.1, one can use $(p, q) = (-(2n+1), 2)$-torus knots. Here is one of those knots when $p = -7$.

Remark 4.3. The Legendrian knots L^k appearing in the proof of Theorem 4.1 remain distinct after m stabilizations. However, it is well-known that the Chekanov-Eliashberg contact homology invariants are unable to distinguish stabilized knots (because the invariants vanish). Thus these are the first examples of Legendrian knots which are not distinguished by the known holomorphic invariants. We also note that the examples in Theorem 4.1 have nontrivial contact homology. We are unable to determine if these invariants are the same or not, but all easily computable invariants derived from contact homology are the same for these examples.

Remark 4.4. Using Theorems 3.4 and 4.2, we can classify Legendrian knots isotopic to (multiple) connected sums of torus knots. This is the first classification of Legendrian knots in a non-Legendrian-simple knot type.

Remark 4.5. Observe that the connected sums of torus knots are fibered knots. Thus we have examples of non-Legendrian-simple fibered knots, contrary to a (perhaps overly optimistic) conjecture that fibered knots are Legendrian simple.

We end by observing that, while the connected sum of Legendrian simple knot types need not be Legendrian simple, the connected sum of transversally simple knot types is always transversally simple. Here a knot type $\mathcal{K}$ is *transversally simple* if transversal knots in $\mathcal{K}$ are determined by their self-linking number.

Theorem 4.6. *The connected sum of transversally simple knot types is transversally simple.*

Proof. Recall that, according to Theorem 2.10 of [EH], a knot type is transversally simple if and only if it is stably simple. A knot type $\mathcal{K}$ is *stably simple* if any two knots in $\mathcal{L}(\mathcal{K})$ for which $s = \mathrm{tb} - r$ agree are Legendrian isotopic after some number of negative stabilizations.

Now assume that $\mathcal{K}_1$ and $\mathcal{K}_2$ are stably simple knot types. Let $L_1, L_1' \in \mathcal{L}(\mathcal{K}_1)$ and $L_2, L_2' \in \mathcal{L}(\mathcal{K}_2)$ such that $s(L_1 \# L_2) = s(L_1' \# L_2')$. It follows that $s(L_1) = s(L_1') + 2n$ and $s(L_2) = s(L_2') - 2n$ for some integer n, which we may take to be ≥ 0. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are stably simple, there exist m_1 and m_2 such that $S_-^{m_1} \circ S_+^n(L_1)$ is Legendrian isotopic to $S_-^{m_1}(L_1')$ and $S_-^{m_2} \circ S_+^n(L_2')$ is Legendrian isotopic to $S_-^{m_2}(L_2)$. Thus

$$\begin{aligned} S_-^{m_1+m_2}(L_1 \# L_2) &= (S_-^{m_1}(L_1)) \# (S_-^{m_2}(L_2)) = (S_-^{m_1}(L_1)) \# (S_-^{m_2} \circ S_+^n(L_2')) \\ &= (S_-^{m_1} \circ S_+^n(L_1)) \# (S_-^{m_2}(L_2')) = (S_-^{m_1}(L_1')) \# (S_-^{m_2}(L_2')) \\ &= S_-^{m_1+m_2}(L_1' \# L_2'). \end{aligned}$$

This proves that $\mathcal{K}_1 \# \mathcal{K}_2$ is stably simple. $\square$

Remark 4.7. In contrast to the situation for Legendrain knots discussed in Remark 4.5, it still does not seems unreasonable to believe that fibered knots are transversely simple. See also [BW, Me].

5. THE MAIN TECHNICAL RESULT

Given a Legendrian knot L in a tight contact manifold (M, ξ), we may always find a sufficiently small tubular neighborhood N of L such that $T = \partial N$ is a convex torus with dividing set Γ_T consisting of two parallel, homotopically nontrivial dividing curves. We make an oriented identification $T \simeq \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (μ, λ), so that the μ-direction is the meridional direction and λ-direction is the longitudinal direction given by a Seifert surface. (Note that this convention is different from the usual Dehn surgery convention.) The slope of a homotopically nontrivial closed curve on T will be given in the $\mu\lambda$-coordinates. With respect to these coordinates, the slope of Γ_T is $\frac{1}{\mathrm{tb}(L)}$. Using the Legendrian Realization Principle we may arrange, and shall always assume, that T is in *standard form* and the ruling slope on T is 0.

An embedded sphere S in M that intersects L transversely in exactly two points and separates M will be called a *separating sphere* for (M, L). Given a separating sphere S, let $M \setminus S = M_1^o \sqcup M_2^o$ and $L_i^o = (L \setminus S) \cap M_i^o$, $i = 1, 2$. We call S a *trivial* separating sphere if one of the M_i^o is a 3-ball and L_i^o is an unknotted arc in $M_i^o = B^3$. The separating sphere S can be isotoped so that $S \cap T$ ($T = \partial N$) consists of two ruling curves. We may further isotop S, relative to $S \cap T$, so that S becomes convex and the annular component of $S \setminus T$ admits a ruling by closed Legendrian curves parallel to the boundary of the annulus. Such an S will be called a *standard convex separating sphere*.

We now introduce a standard object to cap off our cut-open manifold/knot pairs (M_i^o, L_i^o). To this end, let N_s be a convex tubular neighborhood of the y-axis in the standard tight contact $(\mathbb{R}^3, \xi_{std})$ given by the 1-form $dz + x dy$. We can assume the dividing curves on N_s consist of two lines parallel to the y-axis and arrange the ruling curves to be all meridional. Now let B_s be a 3-ball about the origin with convex ∂B_s, such that $\partial B_s \cap \partial N_s$ consists of two ruling curves. Finally, let $L_s = y\text{-axis} \cap B_s$. We call the pair $((B_s, \xi_{std}|_{B_s}), L_s)$, consisting of the tight contact manifold $(B_s, \xi_{std}|_{B_s})$ and the Legendrian arc L_s, the *standard trivial pair*.

Given a convex separating sphere S as above, we can apply the Giroux Flexibility Theorem so that the characteristic foliations on S and ∂B_s agree. For each $i = 1, 2$, we then glue (B_s, L_s) to (M_i^o, L_i^o) to get a closed contact manifold (M_i, ξ_i) and a Legendrian knot $L_i \subset M_i$. The following is a consequence of Theorem 3.1.

Corollary 5.1. *(M_i, ξ_i) is tight, and, up to isotopy, does not depend on the choice of convex separating sphere S (provided the topological type is preserved) or on the gluing map.*

We now consider the relative version of the corollary which takes into account the splitting of the Legendrian knot $L \subset M$. We then have:

Theorem 5.2. *Let $((M, \xi), L)$ be a tight contact manifold together with a Legendrian knot $L \subset M$, and let S, S' be (smoothly) isotopic standard convex separating spheres. Let $((M_i, \xi_i), L_i)$ (resp. $((M_i, \xi_i), L_i'))$, $i = 1, 2$, be the glued-up manifolds together with Legendrian knots, obtained by cutting M along S (resp. S') and gluing in copies of the standard trivial pair. Then there exists a sequence $(L_1^0, L_2^0) = (L_1, L_2), (L_1^1, L_2^1), \dots, (L_1^k, L_2^k) = (L_1', L_2')$, where:*

(1) *L_i^j is a Legendrian knot in (M_i, ξ_i) isotopic to, but not necessarily contact isotopic to, L_i, $i = 1, 2$,*
(2) *(L_1^{j+1}, L_2^{j+1}) is obtained from (L_1^j, L_2^j) by performing one of the following:*
 (i) *Legendrian isotopy,*
 (ii) *$L_1^{j+1} = S_\pm(L_1^j)$ and $L_2^{j+1} = (S_\pm)^{-1}(L_2^j)$, or*
 (iii) *$L_1^{j+1} = (S_\pm)^{-1}(L_1^j)$ and $L_2^{j+1} = S_\pm(L_2^j)$.*
 Here, $(S_\pm)^{-1}$ indicates destabilization.

In other words, the Legendrian knots L_i and L_i' which arise from isotopic but not contact isotopic separating spheres differ only by successively shifting Legendrian stabilizations from one side to the other.

The remainder of this section is devoted to the proof of Theorem 5.2. The proof is essentially a concrete application of the state traversal technique.

Step 1. Let ξ be a $[0, 1]$-invariant tight contact structure on $A = S^2 \times [0, 1]$, viewed as a neighborhood of ∂B_s sitting in $(\mathbb{R}^3, \xi_{std})$. It follows from the proof of Eliashberg's classification of tight contact structures on the 3-ball [El] that ξ is the unique (up to isotopy rel boundary) tight contact structure on A, with the given characteristic foliation on the boundary.

The intersection of A with the y-axis has two components L_+ and L_-; these are Legendrian arcs running between the two boundary components of A. Let $L_\pm^{m,n} = S_+^m \circ S_-^n(L_\pm)$.

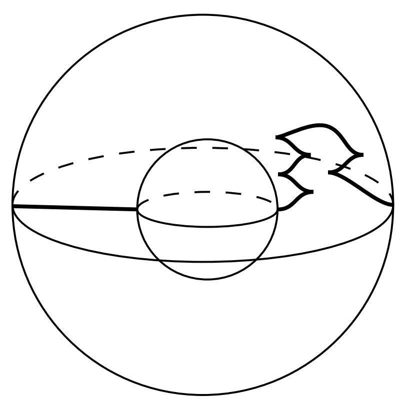

FIGURE 5. The arcs $(L_-, L_+^{2,1})$ in A.

Lemma 5.3. *Let L_+' and L_-' be Legendrian arcs in A which have the same endpoints as L_+ and L_-, respectively, and such that $L_+' \sqcup L_-'$ is (smoothly) isotopic to $L_+ \sqcup L_-$ inside A, rel ∂A. Then, after applying a contactomorphism which is isotopic to the identity through an isotopy which fixes only one of the boundary components, $L_-' \sqcup L_+'$ is Legendrian isotopic to $L_- \sqcup L_+^{m,n}$ for some uniquely determined m and n.*

Proof of Lemma 5.3. We define the *twisting number* (or the *relative Thurston-Bennequin invariant*) $tw(L_\pm')$ to be the difference between the contact framings of $L_\pm'$ and $L_\pm$. Note that the well-definition of $tw(L_\pm')$ follows from the fact that $L_\pm'$ and $L_\pm$ have the same endpoints.

Now, let g be the diffeomorphism of $A = S^2 \times [0, 1]$ which rotates the sphere $S^2 \times \{t\}$ around the axis provided by $L_+ \sqcup L_-$ (*i.e.*, the y-axis) by $2\pi kt$, where k is chosen so that $tw(g(L_-')) = 0$ with respect to $g_*\xi$. Here $g_*\xi$ is isotopic to ξ rel boundary by the uniqueness of tight contact structures on A with fixed boundary.

Observe that $tw(L'_+) \leq 0$, since otherwise we could use L'_-, L'_+ and arcs on ∂A to construct a Legendrian unknot in A with nonnegative Thurston-Bennequin invariant, which would contradict tightness. Let m and n be nonnegative integers which satisfy $m+n=-tw(L'_+)$. (The precise values of m and n are to be determined later.) Hence $tw(L_+^{m,n}) = tw(L'_+)$. Next, there exists an isotopy f of A rel ∂A which takes $L'_- \sqcup L'_+$ to $L_- \sqcup L_+^{m,n}$. Since Legendrian curves and their standard tubular neighborhoods are interchangeable for all practical purposes, we may assume that f is a contactomorphism from the neighborhood U of $L'_- \sqcup L'_+$ onto its image.

It remains to extend f to a contactomorphism on all of A or, equivalently, match up two tight contact structures on the solid torus $A \setminus U$. For this, we apply the classification of tight contact structures on solid tori [Gi2, H]. The boundary slope for both tight contact structures on the solid torus $A \setminus U$ is $-(m+n)-1$. By the classification, there exists a bijection between nonnegative integer pairs (m,n) with $m+n=-tw(L'_+)$ and tight contact structures on $A \setminus U$ with slope $tw(L'_+)-1$ and two dividing curves on the boundary. Hence there is a unique choice of m, n so that the two tight contact structures on $A \setminus U$ are contact isotopic rel boundary. □

Step 2. We now establish Theorem 5.2 under an extra hypothesis on the spheres S and S'.

Claim 5.4. *Theorem 5.2 holds if S and S' are disjoint and cobound a region diffeomorphic to $S^2 \times [0,1]$.*

Proof of Claim 5.4. Let $A' \subset M$ be the region between S and S', and let M_1^c and M_2^c be components of $M \setminus A'$ so that:

$$\begin{aligned} M_1 &= M_1^c \cup B_s, \\ M_2 &= M_2^c \cup A' \cup B_s, \\ M_1' &= M_1^c \cup A' \cup B_s, \text{ and} \\ M_2' &= M_2^c \cup B_s, \end{aligned}$$

where B_s is the standard contact 3-ball.

Let $L_i^c = L \cap M_c^i$, $i=1,2$ and $L' = L \cap A'$. Thus the Legendrian arcs under consideration are:

$$\begin{aligned} L_1 &= L_1^c \cup L_s, \\ L_2 &= L_2^c \cup L' \cup L_s, \\ L_1' &= L_1^c \cup L' \cup L_s, \text{ and} \\ L_2' &= L_2^c \cup L_s, \end{aligned}$$

where L_s is the standard Legendrian arc in B_s.

Observe that B_s is contactomorphic to $B_s \cup A$, where $A = S^2 \times [0,1]$ with the $[0,1]$-invariant tight contact structure. Therefore we may think of M_1 and M_2' as composed of the appropriate M_i^c, B_s and A. Let $f : M_1 \xrightarrow{\sim} M_1'$ be the diffeomorphism which sends $M_1^c \subset M_1$ to $M_1^c \subset M_1'$ by the identity, $B_s \subset M_1$ to $B_s \subset M_1'$ by the identity, and A to A' by a diffeomorphism preserving the characteristic foliation on the boundary. It is easy to arrange for the diffeomorphism from A to A' to take the endpoints of the standard arcs $L_+ \sqcup L_-$ (described in Step 1) to the endpoints of

L'. Now, by Lemma 5.3, f is isotopic to a contactomorphism. The isotopy is fixed on M_1^c and might move one of the boundary components of A, but this can be extended over B_s. Thus we can identify the tight contact manifolds M_1 and M_1'. Moreover, according to Lemma 5.3, $S_+^m \circ S_-^n(L_1)$ is Legendrian isotopic to L_1' and $S_+^{-m} \circ S_-^{-n}(L_2)$ is Legendrian isotopic to L_2', where m, n are nonnegative integers. ☐

Step 3. We now finish the proof of Theorem 5.1 by using the following Lemma 5.5 to reduce to the previous step.

Lemma 5.5. *Let S, S' be (smoothly) isotopic standard convex separating spheres for (M, L), with $S \cap N = S' \cap N$. Then there exists a finite sequence $S_0 = S, S_1, \dots, S_l = S'$ of standard convex separating spheres where, for $i = 0, \dots, k-1$, the pair (S_i, S_{i+1}) cobounds a region diffeomorphic to $S^2 \times [0, 1]$.*

Proof of Lemma 5.5. We use Colin's *isotopy discretization* technique [Co]. Let Σ_t, $t \in [0, 1]$, be the images of a smooth isotopy which takes $\Sigma_0 = S$ to $\Sigma_1 = S'$. We may additionally assume that each Σ_t intersects the standard neighborhood N of L in meridional ruling curves and that each $\Sigma_t \cap N$ is a standard convex meridional disk. For each t, there exists a tubular neighborhood $N(\Sigma_t)$ of Σ_t and an interval $[t - \varepsilon, t + \varepsilon]$ such that $\Sigma_s \subset N(\Sigma_t)$ for all $s \in [t - \varepsilon, t + \varepsilon]$. By the compactness of $[0, 1]$, there exist $t_0 = 0 < t_1 < \cdots < t_k = 1$ such that each $\Sigma_{t_{i+1}}$ is contained in a tubular neighborhood $N(\Sigma_{t_i})$ of Σ_{t_i}. Since convex surfaces are C^∞-dense in the space of closed embedded surfaces [Gi], we may assume that the Σ_{t_i} and $\partial(N(\Sigma_{t_i})) = \Sigma'_{t_i} - \Sigma''_{t_i}$ are convex. Now simply take the sequence

$$\Sigma_{t_0}, \Sigma'_{t_0}, \Sigma_{t_1}, \Sigma'_{t_1}, \dots .$$

It is easily verified that this sequence satisfies the cobounding condition. ☐

6. Proof of Theorem 3.4

In this section we complete the proof of Theorem 3.4. For simplicity we assume that $\mathcal{K} = \mathcal{K}_1 \# \mathcal{K}_2$ and there are no equivalence relations of type 2 in Theorem 3.4, *i.e.*, there are no extra symmetries. We show that

$$C : \left(\frac{\mathcal{L}(\mathcal{K}_1) \times \mathcal{L}(\mathcal{K}_2)}{\sim} \right) \to \mathcal{L}(\mathcal{K}_1 \# \mathcal{K}_2) \tag{4}$$

given by $(L_1, L_2) \mapsto L_1 \# L_2$ is a bijection. The proof is broken down into the following three claims.

Claim 6.1. *The connect sum operation is well-defined.*

Proof. Theorem 5.2 indicates the ambiguity in splitting a manifold along different standard convex separating spheres. Since we are always removing a standard trivial pair (B_s, L_s) from a manifold/knot pair when taking a connected sum, and L_s is not stabilized, the only state transition that actually occurs (among the possibilities listed in Theorem 5.2) is Legendrian isotopy. Hence there is no ambiguity in the connected sum operation. ☐

It is clear that $C(S_\pm(L_1), L_2)$ and $C(L_1, S_\pm(L_2))$ are isotopic Legendrian knots, since stabilizations of a knot can be transfered from one side of the separating sphere to the other. Therefore the map C is well-defined.

Claim 6.2. *The map C is surjective.*

Proof. Let L be a Legendrian knot in $\mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$ and S be a 2-sphere in M^3 that intersects L transversely in exactly two points and topologically divides the knot into the appropriate knot types. Also let N be a standard convex neighborhood of L, where we arrange the ruling curves on ∂N to be meridional. First isotop S so that $S\cap\partial N$ consists of precisely two ruling curves, and then apply a further isotopy of S rel $S\cap\partial N$ so that S becomes convex. Denote by (M_1^o, L_1^o) and (M_2^o, L_2^o) the components of the cut-open manifold $M\setminus S$ together with the cut-open Legendrian knot $L\setminus S$. Let L_s be a trivial Legendrian arc in B_s. We now glue the standard contact 3-ball B_s (with convex boundary) onto M_i^o, $i=1,2$, to form a closed tight contact manifold M_i; at the same time we glue L_s and L_i^o into a Legendrian knot $L_i\subset M_i$ with $L_i\in\mathcal{L}(\mathcal{K}_i)$. Moreover, since we formed the connected sum of L_1 and L_2 by removing B_s from each of M_1 and M_2 and gluing the resulting boundaries together, it is also clear that $L=L_1\#L_2$. □

Claim 6.3. *If $C(L_1,L_2)=C(L_1',L_2')$, then $(L_1,L_2)\sim(L_1',L_2')$.*

Proof. Assume that $L_1\#L_2 = L_1'\#L_2'$, and let S (resp. S') be a standard convex separating sphere for $L_1\#L_2$ (resp. $L_1'\#L_2'$.) Since S and S' are smoothly isotopic, Theorem 5.2 implies that $(L_1,L_2)\sim(L_1',L_2')$. □

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REFERENCES

[Ae] B. Aebischer, et. al., *Symplectic Geometry. An introduction based on the seminar in Bern, 1992*, Progress in Mathematics **124**, Birkhäuser Verlag, Basel, 1994.

[BW] J. Birman and N. Wrinkle, *On transversally simple knots*, J. Differential Geom. **55** (2000), 325–354.

[Ch] Y. Chekanov, *Differential algebras of Legendrian links*, preprint 1997.

[Co] V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 659–663.

[El] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), 165–192.

[EF] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, in *Geometry, topology, and dynamics* (Montreal, PQ, 1995), 17–51, CRM Proc. Lecture Notes **15**, Amer. Math. Soc., Providence, RI, 1998.

[EGH] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. **2000**, Special Volume, Part II, 560–673.

[Et1] J. Etnyre, *Transversal torus knots*, Geom. Topol. **3** (1999), 253–268 (electronic).

[Et2] J. Etnyre, *Introductory lectures on contact geometry*, to appear in Proceedings of the 2001 Georgia International Topology and Geometry Conference.

[EH] J. Etnyre and K. Honda, *Knots and contact geometry I: torus knots and the figure eight knot*, to appear in J. Symplectic Geom.

[FT] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.

[Gi] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991), 637–677.

[Gi2] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), 615–689.

[H] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368 (electronic).

[Me] W. Menasco, *On iterated torus knots and transversal knots*, Geom. Topol. **5** (2001), 651–682 (electronic).

[Ng] L. Ng, *Computable Legendrian invariants*, preprint 2000, `ArXiv:math.GT/0011265`.

[To] I. Torisu, *On the additivity of the Thurston-Bennequin invariant of Legendrian knots*, preprint 2001.

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