ON EMBEDDINGS OF 3-MANIFOLDS IN SYMPLECTIC 4-MANIFOLDS

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I enjoy trying to rethink basic facts that I learned some time ago, and attempting to express them in clear terms. Typically when I learned things from textbooks and classes, they seemed technical and complicated. Unless I go over and rethink them, my memory gradually decays, making unjustified simplifications and unfeasible shortcuts. I gain something by trying to rethink and express it, trying not to resort too much to citing authorities or my intangible belief system of what is well-known.

–William Thurston

to my parents and my football (soccer) family.

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SUMMARY

We proposed a conjecture that every 3-manifolds smoothly embedded in some closed symplectic 4-manifolds. This work shows that any closed oriented 3-manifold can be topologically embedded in some simply-connected closed symplectic 4-manifold, and that it can be made a smooth embedding after one stabilization. As a corollary of the proof we show that the homology cobordism group is generated by Stein fillable 3-manifolds. We also find obstructions on smooth embeddings: there exists 3-manifolds which cannot smoothly embed in a way that appropriately respect orientations in any symplectic manifold with weakly convex boundary.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Introduction

The embedding of 3-manifolds in higher dimensional space has always been a fascinating problem. Whitney's embedding theorem [1] says that every closed oriented 3-manifold smoothly embeds in $\mathbb{R}^6$. Hirsch improved this result by proving [2] that every 3-manifold can be smoothly embedded in S^5. Meanwhile, Lickorish [3] and Wallace [4] proved that every 3-manifold can be smoothly embedded in some 4-manifold, and in fact, a generalization of their arguments shows that every 3-manifold can be smoothly embedded in the connected sum of copies of $S^2 \times S^2$. Freedman proved [5] that all integer homology 3-spheres can be embedded topologically, locally flatly in S^4. On the other hand, the Rokhlin invariant μ and Donaldson's diagonalization theorem [6] show that some integer homology spheres cannot smoothly embed in S^4. Now, one can ask: Does there exists a compact 4-manifold in which all 3-manifolds embed? Shiomi [7] gave a negative answer to this question. Aceto, Golla, and Larson [8] studied the problem of embedding 3-manifolds in spin 4-manifolds. Symplectic manifolds form a very interesting class of 4-manifolds. Etnyre, Min, and the author conjectured the following in [9].

Conjecture 1. *Every closed, oriented smooth 3-manifolds smoothly embed in a symplectic 4-manifold.*

For example, notice that if Y is obtained by doing integer surgery on a knot in S^3 then Y can have a smooth oriented embedding in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ or $\mathbb{CP}^2\#_2\overline{\mathbb{CP}^2}$ depending on whether the surgery coefficient is odd or even. There does not seem to be an analogous result for links.

There is another reason why this conjecture is interesting. Heegaard Floer homology, defined by Ozsváth and Szabó, is a 3-manifold invariant with a choice of $Spin^c$ structure. One interesting question would be, how one can understand this homology from the perspective of 4-manifold theory. As we know that every closed, oriented 3-manifold Y is boundary of a 4-manifold W. Such a manifold W can be thought of as a cobordism from $S^3 \to Y$ after deleting a 4-ball from the interior of W. Ozsváth and Szabó defined a smooth cobordism invariant mix map in [10], which is a map from certain flavor of Heegaard Floer group of S^3 to a certain flavor of Heegard Floer group of Y. (Conjecturally this is same as Seiberg-Witten map.) Thus one can study Heegaard Floer homology of Y as an image of the mix map. Although, non-vanishing of the mix map is not very well understood. The following result gurantees the non-vanishing of the mixed map.

Theorem 1. *(Ozsváth, Szabó [11]) If (X, ω) is a closed, symplectic manifold with $b_2^+(X) > 1$, then for the canonical $Spin^c$ structure $\mathfrak{k}$ corresponding to the symplectic form, $F^{mix}_{X,\mathfrak{k}}$ is non-vanishing. Here we think of X as a cobordism from S^3 to S^3 by taking out two 4-balls.*

So if a 3-manifold Y embeds in a symplectic closed manifold X in a separating way, one can hope under certain conditions, the mix map factors through the Heegaard Floer homology group of Y. And thus, looking at the image of the mix map, one can study the Heegaard Floer group of 3-manifold from the perspective of 4-manifold theory.

1.1.1 Embeddings in symplectic manifolds

While we cannot resolve the above conjecture, we can show the existence of topological embeddings and of smooth embeddings after stabilization.

Theorem 2. *Given a closed, connected, oriented 3-manifold Y there exists a simply-connected symplectic closed 4-manifold X such that Y can be embedded topologically, lo-*

cally flatly in X. This embedding can be made a smooth embedding after one stabilization, that is Y can smoothly embed in $X\#(S^2 \times S^2)$.

When smooth embedding in symplectic manifolds do exist, they can imply interesting things about topology. For example, recall that a closed oriented rational homology sphere is called an *L-space* if its Heegaard Floer homology group is "as simple as possible," as specified in subsection 2.1.3. Embeddings of L-spaces in symplectic manifolds are constrained as follows.

Theorem 3. *If an L-space Y smoothly embeds in a closed symplectic 4-manifold X then it has to be separating. Moreover, if $X = X_1 \cup_Y X_2$ then one of the X_i has to be a negative-definite 4-manifold.*

Remark 4. *Ozsváth and Szabó [11] have established the above result for separating L-spaces in symplectic manifolds. So the main new content of the above theorem is to show that L-spaces cannot be embedded as non-separating hypersurfaces in symplectic manifolds. The proof we give of this was inspired by Agol and Lin's work on hyperbolic 4-manifolds [12].*

Theorem 3 gives rise to a very interesting question.

Question 1. *Does every L-space bound a definite 4-manifold?*

Remark 5. *Notice that if the Conjecture 1 is true then the above question has a positive answer.*

Remark 6. *We now discuss a strategy to show negative answer to Question 1 about L-spaces bounding definite 4-manifolds. Before that notice, all lens spaces bound both positive-definite and negative-definite 4-manifolds, because every lens space can be thought of as the boundary of a negative plumbed manifold and $-L(p, q) = L(p, p - q)$. One can obstruct rational homology spheres Y with $H_1(Y, \mathbb{Z}) \neq 0$ from bounding negative-definite manifolds by using the technique developed by Owens and Strle [13] where in Theorem*

2 they proved that if maximum value of d-invariant of Y is smaller than $1/4$ (with some more algebraic conditions) then Y cannot bounds a negative-definite 4 manifold. Now $d(Y, \mathfrak{s}) = -d(-Y, \mathfrak{s})$, so if we find an L-space Y with $H_1(Y, \mathbb{Z}) \neq 0$, for which the absolute differences between d-invariants for different $Spin^c$ structures are very small then that could be used to obstruct it from bounding positive-definite and negative-definite 4-manifolds (as both Y and $-Y$ cannot bound negative-definite 4-manifolds). So we can ask,

Question 2. *For every $n \in \mathbb{N}$ does there exist an L-space which is not an $\mathbb{Z}HS^3$ and whose d-invariant values are in $(-1/n, 1/n)$?*

1.1.2 Cobordisms and symplectic structures

We say that a closed oriented 3-manifold is *Stein fillable* if there is a Stein fillable contact structure on it, whose definition is deferred until subsection 2.1.2. Not all 3-manifolds are Stein fillable. Work of Eliashberg [14] and Gromov [15] proved that Stein fillable contact structures are always tight. Lisca [16] gave the first example of a non-Stein fillable manifold, and Etnyre and Honda [17] improved the result by showing the existence of a 3-manifold without a tight contact structure.

Recall that we call a integral homology cobordism from Y_0 to Y_1 a $\mathbb{Z}$*-ribbon cobordism* if this integer homology cobordism is achieved by attaching handles of index only 1 and 2 to $Y_0 \times [0, 1]$ along $Y_0 \times \{1\}$. We also indicate such a cobordism by saying Y_0 is *ribbon cobordant* to Y_1. Note that this relation is a partial ordering on 3-manifolds and not necessarily a symmetric relation. (We can similarly define $\mathbb{Q}$-ribbon homology cobordism.)

Theorem 7. *Given any closed oriented 3-manifold Y there exists a Stein fillable 3-manifold Y' and there is a $\mathbb{Z}$ ribbon homology cobordism W from Y to Y' which is obtained from $Y \times [0, 1]$ by attaching a single pair of algebraically cancelling 1- and 2-handle. Moreover,* *this is an invertible cobordism, that is there is a cobordism W' from Y' to Y such that $W \cup_{Y'} W'$ is diffeomorphic to $Y \times [0,1]$. In particular Y' embeds in $Y \times [0,1]$.*

Remark 8. *Yasui has pointed out to the author that, while they do not talk about cobordisms, this result, without the statement of only needing a single 1- and 2-handle pair, can be proven by putting together several results from [18].*

In low-dimensional topology the study of integer homology cobordism group $\Theta^3_{\mathbb{Z}}$ and rational homology cobordism group $\Theta^3_{\mathbb{Q}}$ are of special interest. The above result gives a new generating set for these groups.

Corollary 9. *The homology cobordism groups $\Theta^3_{\mathbb{Z}}$ and $\Theta^3_{\mathbb{Q}}$ are generated by Stein fillable 3-manifolds.*

Remark 10. *It is not known whether $\Theta^3_{\mathbb{Q}}$ is generated by L-spaces or not. Nozaki, Sato and Taniguchi [19] proved that $\Sigma(2,3,11)\#_2(-\Sigma(2,3,5))$ does not bound definite 4-manifold. If we can find an L-space Y which is rationally cobordant to $\Sigma(2,3,11)$, then $Y\#_2(-\Sigma(2,3,5))$ cannot bounds a definite 4-manifold. Since $Y\#_2(-\Sigma(2,3,5))$ is an L-space, 3 says this manifold cannot be smoothly embedded in any symplectic 4-manifold. Conversely, if all 3-manifolds embed in some symplectic 4-manifold, then $\Theta^3_{\mathbb{Q}}$ is not generated by L-spaces. So finally we can ask the following question.*

Question 3. *Is $\Sigma(2,3,11)$ rationally cobordant to some L-space?*

For a closed oriented 3-manifold Y, $H_3(Y;\mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. So a map $f : Y_0 \to Y_1$ induces a homomorphism on the top-dimensional homology group, $f_* : \mathbb{Z} \to \mathbb{Z}$. The degree of f is $f_*(1) \in \mathbb{Z}$.

Corollary 11. *Given any 3-manifold Y there exists a Stein fillable 3-manifold Y' and a degree one map $f : Y' \to Y$.*

Although we cannot give a complete answer about smooth embeddings of 3-manifolds in closed symplectic 4-manifolds, we can find obstructions to smooth embeddings in compact symplectic 4-manifolds with convex boundary. There is an ambiguity when we think about smooth embeddings of a 3-manifold Y in a smooth 4-manifold X. As oriented manifold Y is very different from $-Y$: for example the Poincaré homology sphere with positive orientation bounds a negative-definite 4-manifold but the Poincaré homology sphere with negative orientation does not. On the other hand, if Y smoothly embeds in X then the boundary of a small neighbourhood of Y is $-Y \sqcup Y$, so $-Y$ smoothly embeds in X as well. But we can fix this issue in terms of cobordisms.

Definition 12. *We call a smooth embedding of Y in a cobordism W from Y_0 to Y_1 an* oriented cobordism embedding *if Y is either non-separating or Y separates W into $W_1 \sqcup W_2$ such that Y as an oriented manifold is a boundary component of W_1, and all other components of ∂W_1 (if they exist) are part of Y_0.*

Theorem 13. *If an L-space Y does not bound a negative-definite 4-manifold then Y cannot have an oriented cobordism embedding in any symplectic 4-manifold with weakly convex boundary.*

Remark 14. *There are many such L-spaces, for instance the Poincaré homology sphere with negative orientation and r-surgery for $r \in [9, 15)$ on the pretzel knot $P(-2, 3, 7)$ in S^3 [20, 21] (the latter L-spaces are in fact hyperbolic). It was already known that these manifolds are not Stein fillable [16, 20, 21]. Here, we proved that in addition to not being weakly fillable they cannot even have a smooth oriented cobordism embedding in any weak filling of any 3-manifold.*

Corollary 15. *If Y' admits a weakly fillable contact structure then any L-space Y which does not bound negative-definite 4-manifolds cannot have any smooth oriented cobordism embedding in $Y' \times I$.*

The difference between smooth and topological embeddings can be used to detect exotic structures on compact manifolds. If we find two homeomorphic 4-

manifolds such that a 3-manifold embeds smoothly in one but not the other then they are not diffeomorphic, i.e. they are an exotic pair. The next Corollary was first proved by Akbulut [22] and since then by many others, but we will give an alternative proof that follows from the study of embeddings of 3-manifolds into 4-manifolds.

Corollary 16. *There exists compact 4-manifolds with boundary X and X' such that $b_2(X) = b_2(X') = 1$ that are homeomorphic but not diffeomorphic.*

We now turn to studying when a 3-manifold has a Stein filling for some contact structure, and start by discussing obstructions. The *Rokhlin invariant* $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$ is defined as $\mu(Y) = \sigma(W)/8 \pmod 2$, where W is any compact, spin 4-manifold with boundary Y and $\sigma(W)$ is its signature. This invariant μ is an invariant under homology cobordism. The Brieskorn homology sphere $\Sigma(2,3,7)$ cannot bound a $\mathbb{Z}HB^4$ since its Rokhlin invariant μ is 1. So any 3-manifold Y that is homology cobordant to $\Sigma(2,3,7)$ cannot have an integer homology ball ($\mathbb{Z}HB^4$) as a Stein filling. But Fintushel and Stern [23] proved that $\Sigma(2,3,7)$ bounds a rational homology ball ($\mathbb{Q}HB^4$). So one can ask if $\Sigma(2,3,7)$ has a $\mathbb{Q}HB^4$ as a Stein filling. The following lemma is well-known and can be proven easily by looking at the long exact homology sequence.

Lemma 17. *If a $\mathbb{Z}HS^3$ bounds a $\mathbb{Q}HB^4$ which is not a $\mathbb{Z}HB^4$ then it must have a 3-handle.*

This implies that if $\Sigma(2,3,7)$ bounds a $\mathbb{Q}HB^4$ then it cannot be Stein as every handle decomposition has a 3-handle which contradicts a result of Eliashberg [24]. From the previous discussion we can see that the same conclusion is true for any 3-manifold Y that is integer homology cobordant to $\Sigma(2,3,7)$. So it is natural to ask if there exists a 3-manifold Y that is rationally cobordant to $\Sigma(2,3,7)$ and it bounds a rational homology Stein ball. We know that S^3 is rationally cobordant to $\Sigma(2,3,7)$

[23]. But such a cobordism must have a 3-handle. So a modified question would be: Does there exist a 3-manifold Y such that there is a rational ribbon homology cobordism from $\Sigma(2,3,7)$ to Y and Y bounds a rational Stein ball?

Theorem 18. *If X is an oriented compact 4-manifold with connected boundary $\partial X = Y$,*

(i) If $b_1(X) = 0$ then there exists a Stein 4-manifold X' with boundary $\partial X' = Y'$ such that there is a rational ribbon homology cobordism from Y to Y' and $b_2(X) = b_2(X')$.

(ii) If $\partial X = Y$ a $\mathbb{Q}HS^3$, then there exists a Stein 4-manifold X' with boundary $\partial X' = Y'$ such that the intersection form of X is isomorphic to the intersection form of X' and there is a rational ribbon homology cobordism from Y to Y'.

As discussed above, not every smooth filling X of a 3-manifold Y can be given a Stein structure, or indeed there are Y that do not even admit any Stein fillings. But we can ask if there is a ribbon rational homology cobordism from Y to a manifold Y' that has a symplectic filing X' so that X' shares some of the algebraic properties of X. For example if we let

$$b_2^F(Y) = \min\{b_2(X) \mid \partial X = Y\},$$

then we can ask the following.

Question 4. *Let Y be a 3-manifold. Is there a ribbon rational homology cobordism to Y' such that $b_2^F(Y') = b_2^F(Y)$ and Y' has a Stein filling which realized $b_2^F(Y')$?*

Remark 19. *There exists a 3-manifold Y' and a rational ribbon homology cobordism from $\Sigma(2,3,7)$ to Y' such that Y' has a rational ball Stein filling. In particular it is true if we replace $\Sigma(2,3,7)$ with any 3-manifold which bounds a rational ball. A large class of such manifolds is provided by Akbulut and Larson [25] and by Şavk [26].*

CHAPTER 2
METHODOLOGY

2.1 Background

2.1.1 Contact geometry

Recall that a *(co-orientable) contact structure* ξ on an oriented 3-manifold Y is the kernel of an 1-form $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha$ is non-degenerate. Geometrically a contact structure on a 3-manifold is a distribution of a 2-plane fields on the manifold that is not tangent to any embedded surface in the manifold. Darboux's theorem says that every contact 3-manifold (Y, ξ) is locally contactomorphic to $(\mathbb{R}^3, \xi_{std} = ker(dz - ydx))$. In this sense we always assume that the contact structures are positive, i.e. the orientation on Y coincides with the orientation given by $\alpha \wedge d\alpha$. We orient the normal direction to the contact plane by α, or equivalently the contact planes are oriented by $d\alpha$.

A knot $L \subset (Y, \xi)$ is called *Legendrian* if at every point of L the tangent line to L lies in the contact plane at that point, i.e. $TL \subset \xi$ or equivalently $\alpha(TL) = 0$ for the contact 1-form α defining ξ. The *contact framing* of a Legendrian knot is defined by the orthogonal of ξ along L, in other words, push L off in the normal direction to ξ. Equivalently, we can take the framing obtained by pushing L off in the direction of a nonzero vector field transverse to L which stays inside the contact planes. This framing is the *Thurston–Bennequin framing* of L. If L is null-homologus in (Y, ξ) then it admits a natural 0-framing provided by any embedded surface $\Sigma \subset Y$ with $\partial\Sigma = L$. In this case the Thurston–Bennequin framing can be converted into an integer $tb(L) \in \mathbb{Z}$ — measure the Thurston–Bennequin framing with respect to Seifert framing, i.e., the natural 0-framing. Notice that the 0-framning doesn't depend on the choice of the surface Σ, therefore the number $tb(L)$ is independent of Σ and thus it is an invariant for null-homologus Legendrian knots, namely the *Thurston–Bennequin invariant.*

In order to have a better understanding of the topological constructions we discuss a way to visualize Legndrian knots in $\mathbb{R}^3$ (or, equivalently, in S^3) equipped with the standard contact structure $\xi_{std} = ker(dz - ydx)$.

Consider a Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ and projects to a closed curve γ in the xz–plane which is also known as *front projection* of L. The curve γ uniquely determines the Legendrian knot L which can be reconstructed by setting $y(t)$ as the slope of $\gamma(t)$. Notice that the projection has no vertical tangencies since $\frac{dz}{dx} = y \neq \infty$. And for a similar reason, at a crossing of the projection, the most negatively sloped curve always stays at front. There are two types of cusps singularity possible when $\frac{dz}{dx} = 0$ whcih are called left cusps and right cusps. See Figure 2.1.

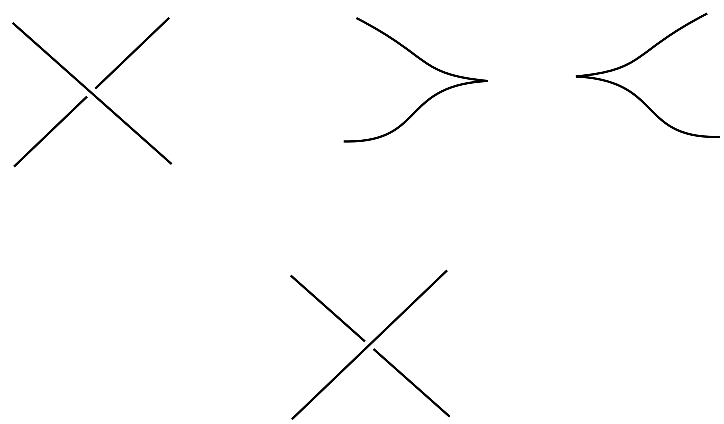

Figure 2.1: The top row indicates the correct crossing and the cusps in the front projection. And the bottom picture crossing will not occur in the front projection diagram.

Looking at an oriented front projection one can compute the Thurston–Bennequin invariant of a Legendrian knot $tb(L)$ as follows. Given L an orientaion, we can define $w(L)$ (the writhe of L) as the sum of signs of the double points.

Lemma 20. *If $c(L)$ is the number of cusps, then the Thurston–Bennequin invariant $tb(L)$*

given by the contact structure is equal to $w(L) - \frac{1}{2}c(L)$.

For more details we refer [27] and [28].

Definition 21. *A contact 3-manifold* (Y, ξ) *is called overtwisted if there exists a Legendrian unknot* K *with Thurston–Bennequin number* 0*, i.e., if the contact framing of* K *conincides with the framing given by the seifert-disk* D*. A contact 3-manifold* (Y, ξ) *is called tight if it is not overtwisted.*

If one does $fr_\xi - 1$ surgery on L by removing N and gluing back a solid torus so as to effect the desired surgery, then there is a unique way to extend $\xi|_{Y-N}$ over the surgery torus so that it is tight on the surgery torus. The resulting contact manifold is said to be obtained from (Y, ξ) by *Legendrian surgery* on L.

2.1.2 Symplectic fillings, cobordisms and caps

We recall that a compact symplectic manifold (X, ω) is a *strong symplectic filling* of (Y, ξ) if $\partial X = Y$ and there is a vector field v defined near ∂X such that the Lie derivative of ω satisfies $\mathcal{L}_v \omega = \omega$, v points out of X and $\iota_v \omega$ is a contact form for ξ. Moreover, (X, ω) is a strong symplectic cap for (Y, ξ) if it satisfies all the properties above, except $\partial X = -Y$ and v points into X. We also say (X, ω) is a *weak filling* of (Y, ξ) if $\partial X = Y$ and $\omega|_\xi > 0$ (here all our contact structures are co-oriented). Similarly, (X, ω) is a weak cap of (Y, ξ) if $\partial X = -Y$ and $\omega|_\xi > 0$. We shall say that (Y, ξ) is (strongly or weakly) *semi-fillable* if there is a connected (strong or weak) filling (X, ω) whose boundary is disjoint union of (Y, ξ) with an arbitrary non-empty contact manifold.

A *symplectic cobordism* from the contact manifold (Y_-, ξ_-) to (Y_+, ξ_+) is a compact symplectic manifold (W, ω) with boundary $-Y_- \cup Y_+$ where Y_- is a *concave* boundary component and Y_+ is *convex*, this means that there is a vector field v near ∂W which points transversally inwards at Y_- and transversally outwards at

Y_+, and $\mathcal{L}_v\omega = \omega$. The first result we will need concerns when symplectic cobordisms can be glued together.

Lemma 22. *[29] Let (X_i, ω_i) be a symplectic cobordism from (Y_i^-, ξ_i^-) to (Y_i^+, ξ_i^+), for $i = 1, 2$, and (Y_1^+, ξ_1^+) is contactomorphic to (Y_2^-, ξ_2^-). Then we can construct a symplectic cobordism (X, ω) from (Y_1^-, ξ_1^-) to (Y_2^+, ξ_2^+) such that X is diffeomorphic to $X_1 \cup_{Y_1^+} X_2$.*

Definition 23. *A smooth function $\phi : X \to \mathbb{R}$ on a complex manifold X is (strictly) plurisubharmonic if ϕ is (strictly) subharmonic on every holomorphic curve $C \subset X$. Recall that ϕ is subharmonic if for r small enough $\phi(x_0) \leq \frac{1}{2\pi r}\int_{B(x_0,r)} \phi(x)dx$. A function $\phi : X \to \mathbb{R}$ is an exhausting function if $\{x \in X | \phi(x) < c\}$ is relatively compact in X for all $c \in \mathbb{R}$. Recall that a map $\phi : X \to \mathbb{R}$ is proper if the image of a compact set is compact.*

Definition 24. *If a complex manifold X admits a proper plurisubharmonic function $\phi : X \to [0, \infty)$ then it is called Stein.*

A compact manifold W with boundary will be called *Stein domain* if there is a Stein manifold $X$and $\psi : X \to \mathbb{R}$ is a proper plurisubharmonic functionsuch that $W = \phi^{-1}((-\infty, a])$ for some regular value a. So a compact manifold with boundary (and complex structure on its interior) is a Stein domain if it admits a proper plurisubharmonic function which is constant on the boundary. More genrally, a cobordism W with boundary $-Y_1 \cup Y_2$ is a *Stein cobordism* if W is a complex cobordism with a plurisubharmonic function $f : W \to \mathbb{R}$ such that $f^-(t_i) = Y_i, t_1 < t_2$.

A closed contact 3-manifold (Y, ξ) is called Stein fillable if there exists a Stein manifold (X, J, ψ) such that ψ is bounded from below, M is an inverse image of a regular value of ψ and $\xi = ker(-d\psi \circ J)$. In fact we have the following charecterization of Stein 4-manifolds.

Theorem 25. *(Elaishberg [24]; Gompf [28]) A 4-manifold is a Stein domain if and only if it has a handle decomposition with 0-handles, 1-handles, and 2-handles and the 2-handles* *are attached along Legendrian knots in $\#S^1 \times S^2$ with framing one less than the contact framing.*

Another way to build cobordisms is by Weinstein handle attachment, [**Weinstein91**]. One may attach a 0, 1, or 2-handle to the convex end of a symplectic cobordism to get a new symplectic cobordism with the new convex end described as follows. For a 0-handle attachment, one merely forms the disjoint union with a standard 4–ball and so the new convex boundary will be the old boundary disjoint union with the standard contact structure on S^3. For a 1-handle attachment, the convex boundary undergoes a, possibly internal, connected sum. A 2-handle is attached along a Legendrian knot L with framing one less that the contact framing, and the convex boundary undergoes a Legendrian surgery.

Theorem 26. *Given a contact 3-manifold (Y, ξ) let W be a part of its symplectization, that is $(W = Y \times [0, 1], \omega = d(\alpha e^t))$. Let L be a Legendrian knot in (Y, ξ) where we think of Y as $Y \times \{1\}$. If W' is obtained from W by attaching a 2-handle along L with framing one less than the contact framing, then the upper boundary (Y', ξ') is still a convex boundary. Moreover, if the 2-handle is attached to a Stein filling (respectively strong, weak filling) of (Y, ξ) then the resultant manifold would be a Stein filling (respectively strong, weak filling) of $(Y'\xi')$.*

The theorem for Stein fillings was proven by Eliashberg [24], for strong fillings by Weinstein [30], and was first stated for weak fillings by Etnyre and Honda [31].

Starting with a Stein filling (respectively strong, weak filling) of (Y, ξ) one can construct a symplectic closed manifold by capping it off. Various people have studied concave caps on contact manifold but for our purpose we need the result of Etnyre, Min and the author [9].

Theorem 27. *If (W, ω) is weak filling of (Y, ξ) then there exists a closed symplectic 4-manifold (X, ω') in which (W, ω) symplectically embeds such that the complement of W*

in X is simply-connected and has $b_2^+ > 0$.

2.1.3 Heegard Floer homology

Recall that Heegaard Floer homology is an Abelian group associated to a 3-manifold Y, equipped with a $Spin^c$ structure $\mathfrak{t} \in Spin^c(Y)$. These homology groups are invariant of the pair $(Y, \mathfrak{t})$ and are denoted by $HF^\infty(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U, U^{-1}]$ module; $HF^+(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U^{-1}]$ module; $HF^-(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U]$ module. These invariants fit into a long exact sequence

$$\cdots \longrightarrow HF^-(Y, \mathfrak{t}) \xrightarrow{\iota} HF^\infty(Y, \mathfrak{t}) \xrightarrow{\pi} HF^+(Y, \mathfrak{t}) \xrightarrow{\delta} \cdots$$

Recall that associated to this long exact sequence there is another 3-manifold invariant

$$HF^+_{red}(Y, \mathfrak{t}) = Coker(\pi) \cong Ker(\iota) = HF^-_{red}(Y, \mathfrak{t})$$

The isomorphism in the middle is induced by the co-boundary map. Recall that $d(Y, \mathfrak{t})$ is the minimum grading of the torsion-free elements in the image$\{\pi : HF^\infty(Y, \mathfrak{t}) \rightarrow HF^+(Y, \mathfrak{t})\}$. For more details, readers are referred to [32, 33].

Now recall that an *L-space* Y is a rational homology 3-sphere whose Heegard Floer homology is as simple as possible, that is $HF^+_{red}(Y, \mathfrak{t}) = 0$ for all $Spin^c$ structures $\mathfrak{t} \in Spin^c(Y)$.

A cobordism between two 3-manifold induces a map on Heegad Floer homology. More precisely if W is a cobordism from Y_0 to Y_1 and $\mathfrak{s}$ is a $Spin^c$ structure in W whose restriction on Y_i is denoted as $\mathfrak{s}_i$ for $i = 0, 1$, then there is a map $F^\circ_{W,\mathfrak{s}} : HF^\circ(Y_0, \mathfrak{s}_0) \rightarrow HF^\circ(Y_1, \mathfrak{s}_1)$, where $\circ = +, -$ *or* ∞.

Theorem 28. *(Ozsváth-Szabó [10]) If W is a cobordism between Y_0 to Y_1 and $\mathfrak{s}$ is a $Spin^c$ structure on W whose restriction on Y_i is denoted as $\mathfrak{s}_i$ for $i = 0, 1$ then we have the following,*

$$
\begin{tikzcd}
\cdots \arrow[r] & HF^-(Y_0,\mathfrak{s}_0) \arrow[r, "\iota_0"] \arrow[d, "F^-_{W,\mathfrak{s}}"] & HF^\infty(Y_0,\mathfrak{s}_0) \arrow[r, "\pi_0"] \arrow[d, "F^\infty_{W,\mathfrak{s}}"] & HF^+(Y_0,\mathfrak{s}_0) \arrow[r, "\delta_0"] \arrow[d, "F^+_{W,\mathfrak{s}}"] & \cdots \\
\cdots \arrow[r] & HF^-(Y_1,\mathfrak{s}_1) \arrow[r, "\iota_1"] & HF^\infty(Y_1,\mathfrak{s}_1) \arrow[r, "\pi_1"] & HF^+(Y_1,\mathfrak{s}_1) \arrow[r, "\delta_1"] & \cdots
\end{tikzcd}
$$

where the vertical maps are uniquely determined up to an overall sign, and all the squares are commutative.

The *composition law* states that if W_0 is a cobordism from Y_0 to Y_1 and W_1 is a cobordism from Y_1 to Y_2, and let $\mathfrak{s}_i$ be the $Spin^c$ structure on W_i for $i = 0, 1$, then the relationship between composition of $F_{W_0,\mathfrak{s}_0}$ with $F_{W_1,\mathfrak{s}_1}$, and the maps induced by the composite cobordism $W = W_0 \cup_{Y_1} W_1$ is

$$F^\circ_{W_1,\mathfrak{s}_1} \circ F^\circ_{W_0,\mathfrak{s}_0} = \sum_{\{\mathfrak{s}\in Spin^c(W)|\ \mathfrak{s}|W_i=\mathfrak{s}_i, i=0,1\}} \pm F^\circ_{W,\mathfrak{s}}.$$

2.1.4 Closed 4-manifold invariants

There is a variant of the cobordism invariant which is defined for cobordism with $b_2^+(W) > 1$. This following lemma is proven by Ozsváth and Szabó [10]

Lemma 29. *Let W be a cobordism between Y_0 and Y_1 with $b_2^+(W) > 0$. Then the induced cobordism map $F^\infty_{W,\mathfrak{s}}$ vanishes for all $Spin^c$ structures on W.*

If we have a cobordism W with $b_2^+(W) > 1$, then we can cut W along a 3–manifold N, which divides W into two cobordism W_0 and W_1, both of which have $b_2^+(W_i) > 0$, in such a way that the map induced by the restriction

$$Spin^c(W) \to Spin^c(W_0) \times Spin^c(W_1)$$

is injective. Such a cut N is called $admissible\ cut$.

Remark 30. *Notice that if in a cobordism W with $b_2^+(W) > 1$ we find a separating rational homology 3–sphere N such that both the pieces have $b_2^+ > 0$, then N is an admissible cut.*

If $\mathfrak{s}$ is a $Spin^c$ structure on W whose restriction to W_i is $\mathfrak{s}_i$ and the induced $Spin^c$ structures on 3-manifolds Y_0, Y_1 and N is $\mathfrak{t}_0, \mathfrak{t}_1$ and $\mathfrak{t}$, then

$$F^-_{W_0,\mathfrak{s}_0} : HF^-(Y_0, \mathfrak{t}_0) \to HF^-(N, \mathfrak{t})$$

factors through the inclusion $HF^-_{red}(N, \mathfrak{t}) \to HF^-(N, \mathfrak{t})$, and

$$F^+_{W_1,\mathfrak{s}_1} : HF^+(N, \mathfrak{t}) \to HF^+(Y_1, \mathfrak{t}_1)$$

factors through the projection $HF^+(N, \mathfrak{t}) \to HF^+_{red}(N, \mathfrak{t})$. And thus by using the identification of $HF^+_{red}(N, \mathfrak{t}) \cong HF^-_{red}(N, \mathfrak{t})$ in the middle, we can define the *mixed invariant* as a map

$$F^{mix}_{W,\mathfrak{s}} : HF^-(Y_0, \mathfrak{t}_0) \to HF^+(Y_1, \mathfrak{t}_1).$$

Remark 31. *It is also proven in [10] that F^{mix} does not depend on the choice of the admissible cut.*

From the discussion above one immediately sees the following result.

Lemma 32. *If an admissible cut N of W is an L-space then $F^{mix}_{W,\mathfrak{s}}$ vanishes.*

Theorem 33. *(Ozsváth, Szabó [11]) If (X, ω) is a closed, symplectic manifold with $b_2^+(X) > 1$, then for the canonical $Spin^c$ structure $\mathfrak{k}$ corresponding to the symplectic form, $F^{mix}_{X,\mathfrak{k}}$ is non-vanishing. Here we think of X as a cobordism from S^3 to S^3 by taking out two 4-balls.*

Remark 34. *The above discussion implies that an L-space cannot be an admissible cut for a closed symplectic 4-manifold with $b_2^+ > 1$.*

CHAPTER 3

RESULTS

3.1 Topological embedding of 3-manifolds in symplectic 4-manifolds

Now we will begin the proof of topological embedding of 3-maifolds into symplectic 4-manifolds.

Proof of Theorem 2. We will topologically embed a 3-manifold Y into a symplectic manifold X in three steps. In the fourth step we will show that the embedding is smooth after a single stabilization with $S^2 \times S^2$. We start with a Kirby picture, consisting of only a 0-handle and 2-handles, for a 4–manifold whose boundary is Y.

Step 1. *Stein modification of the Kirby picture.*

Let $K_1, \ldots, K_m$ be the attaching spheres for the 2-handles. We can Legendrian realize the K_i so that each K_i intersects a fixed Darboux ball B in a horizontal arc. We can blow up meridians to each K_i so that the framing on K_i is less than $tb(K_i) - 2$. All the blown-up unknots can be gathered in the Darboux ball as shown in the top left of Figure 3.1. Blow up one more unknot as indicated in the upper right of Figure 3.1 and notice that the resulting link L can be Legendrian realized as in the bottom diagram of Figure 3.1. Let K be the unknot with framing -1 in the figure. We now stabilize the components of $L - K$ so that the Thurson-Bennequin invariant of each component is one larger than its surgery coefficients. Legendrian surgery on $L - K$ together with -1 surgery on K gives our manifold Y. (Notice that to realize L each K_i might need to be stabilized one extra time as shown in the figure. This is why we arranged the surgery coefficients to be less than $tb(K_i) - 2$.) So the manifold W_0 obtained by attaching Stein 2-handles to $L - K$

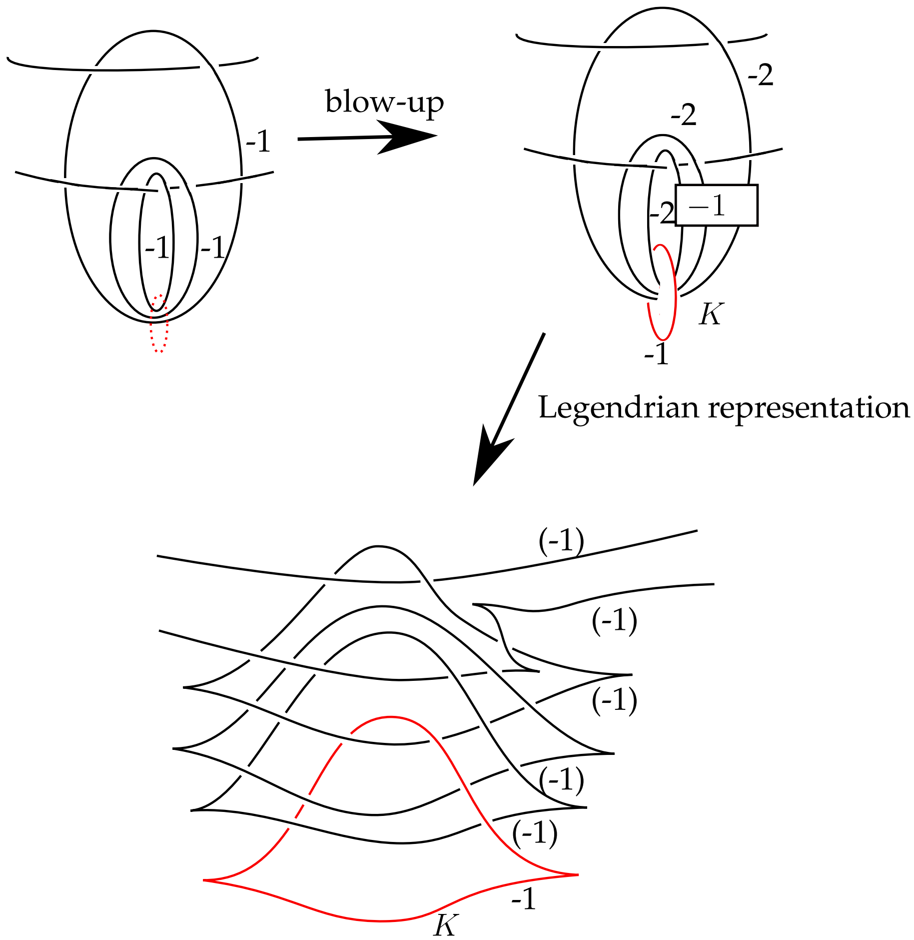

Figure 3.1: Converting a Kirby picture of the 3-manifold by blowing up such that the complement of the red knot is Stein. Here (-1) is measured relative to the contact framing.

is a Stein manifold and we denote the boundary by Y_0. Now attaching a 2-handle to W_0 along K with framing -1 gives a 4-manifold W with boundary Y.

Step 2. *Attach a cork and apply cork-twist.*

We begin by constructing a manifold W_2 by modifying the surgery presentation that is add the 1- and 2-handle shown in Figure 3.2, where the 2-handle links K as indicated and is otherwise disjoint from L (by abusing of language we will call this operation attaching a Mazur cork). Said another way, we can build a cobordism W_1 by attaching the 1- and 2-handle to $Y \times [0,1]$ along $Y \times \{1\}$. The manifold W_2 is now simply $W \cup W_1$ with ∂W glued to $-Y \subset W_1$ and W_1 is a cobordism from Y to some manifold Y'.

We can apply a cork-twist by interchanging the 1-handle and the 0-framed 2-handle. The cork twist does not change the boundary 3-manifold Y'. After the cork twist the knot K is passing over the 1-handle geometrically once and thus they cancel each others. After this handle cancellation the knot K in the original picture Figure 3.1 is replaced by -1 framed knot in the third picture of Figure 3.2. Notice that this new knot can be realized by a Legendrian knot that has Thurston-Bennequin invariant $+1$ and thus -1 smooth surgery on this knot can be realized as Legendrian surgery on a stabilization of the knot. So we get a Stein filling of the boundary.

Step 3. *Construct a simply connected a closed symplectic 4-manifold.*

Also notice that the W_1 deformation retract onto Y and W is a 2-handlebody, so in particular W_2' is simply connected. So in Step 2 when we do the cork-twist the new manifold W_2' is still homeomorphic to the compact manifold W_2 by a result of Freedman [5] and thus the original 3-manifold has a topological, locally flat embedding in W_2'. Now we use the simply connected cap constructed in Theorem 27 to cap off the upper boundary W_2' to get a closed simply connected symplectic

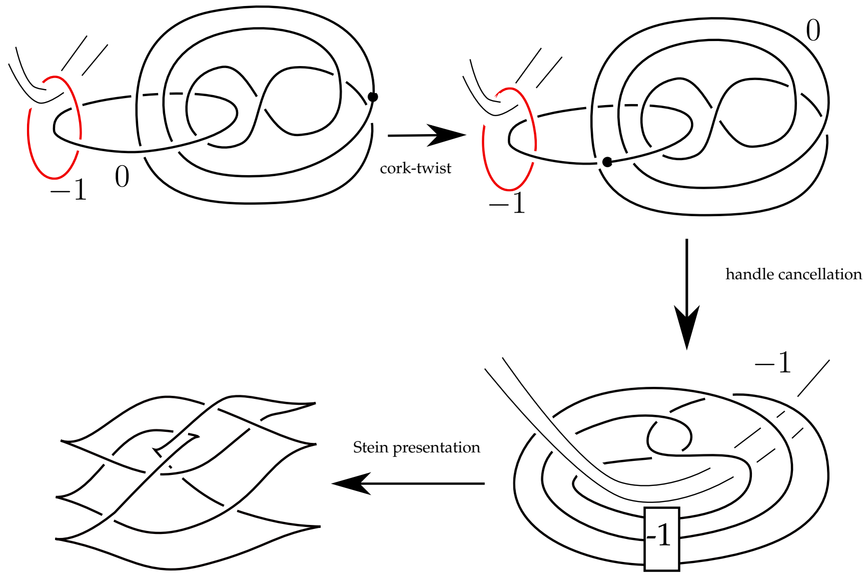

Figure 3.2: The maximal Thurston-Benequin number of the black knot in the bottom picture is $+1$. So it is a Stein $2-$handle attachment.

4-manifold X into which the 3-manifold Y topologically, locally flatly embeds.

Step 4. *Smooth embedding after one stabilization.*

We can stabilize X by adding a Hopf link to W_2' and using the same cap (of course this stabilized 4-manifold is no longer symplectic). We can now handle slide one of the components C of the Hopf link over the 0-framed knot in the Mazur cork as indicated at the top of Figure 3.3. Using the 0-framed meridian to C we can untangle C from the 2-handle in the Mazur cork as shown in the middle of Figure 3.3. We can further slide C over the 0-framed meridian to turn C into a meridian of the 1-handle. Thus the 1-handle can be cancelled with C, leaving the bottom picture in Figure 3.3. Thus we see a smooth embedding of Y into $W_2'\#S^2 \times S^2$ and thus into $X\#S^2 \times S^2$. □

We now turn to Theroem 7 that says given any 3-manifold Y there is a simple invertible $\mathbb{Z}$ ribbon homology cobordism to a Stein fillable manifold Y'.

Proof of Theorem 7. The cobordism W_1 from Step 2 is the desired ribbon $\mathbb{Z}$-homology

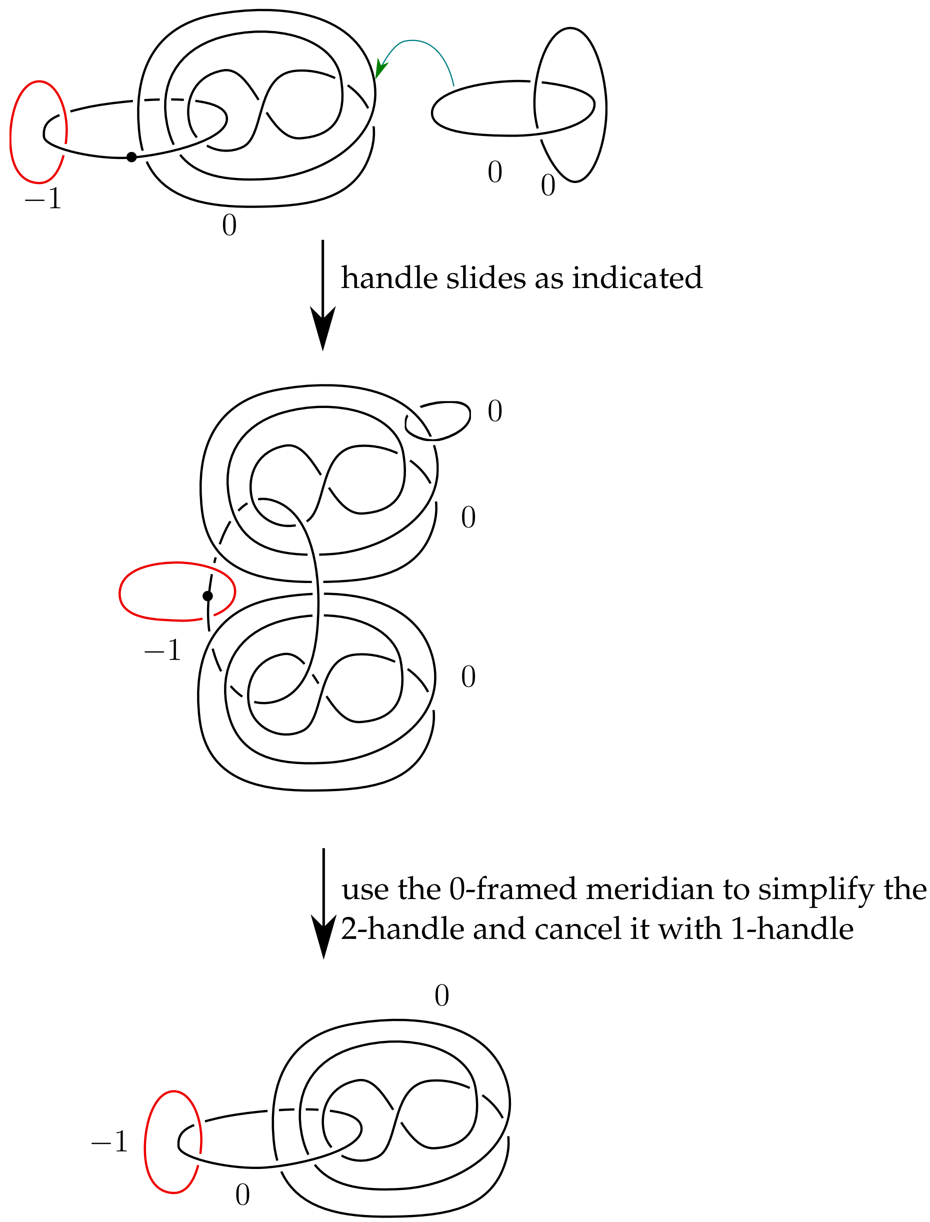

Figure 3.3: In this picture we are describing the Kirby moves of how connected summing with $S^2 \times S^2$ helps to cancel the 1-handle of the cork.

cobordism which is attaching a cork along the red knot on top Figure 3.2.

Let $D(W_1)$ be the double of W_1 along Y', that is glue an upside down copy of W_1 on top of W_1 along the boundary. If h is the 2-handle in W_1, then $D(W_1)$ is formed by attaching a 2- and 3-handle to W_1, with the 2-handle attached to a 0-framed meridian to h [34, Section 4.2]. Check Figure 3.4. Now by changing crossings on the attaching circle of h using the 0-framed meridian we can arrange that h is passing over the 1-handle geometrically once. Thus they cancel each other. And after the cancellation the 0-framed meridian h will cancel the 3-handle. And thus the resultant manifold $D(W_1) = Y \times I$. $\square$

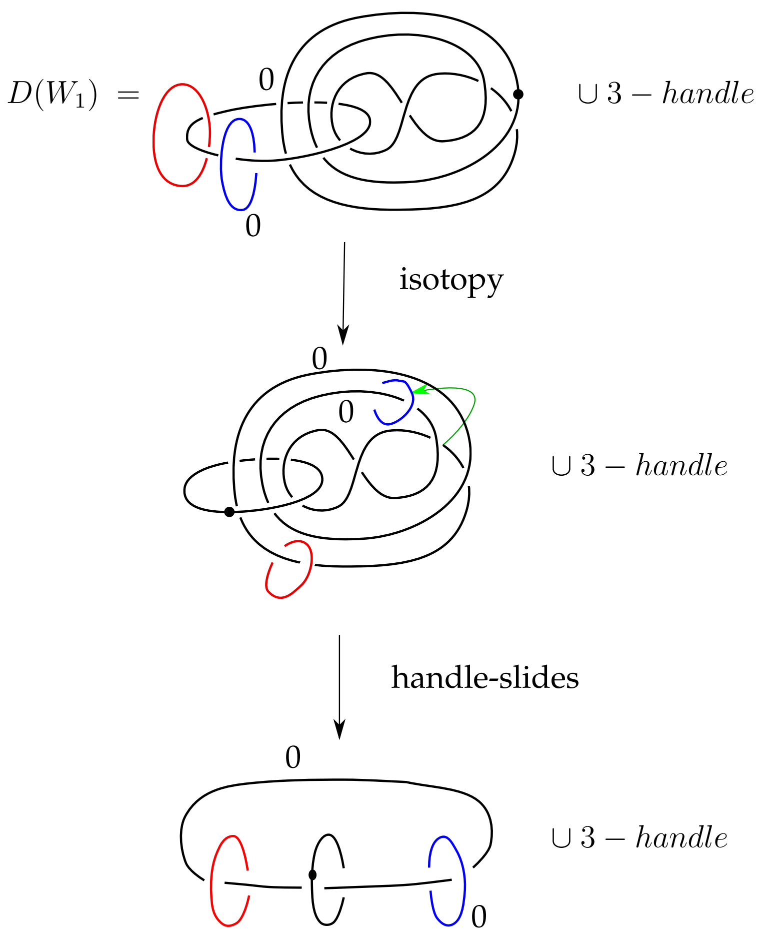

Figure 3.4: After isotopy, in the second picture the 0-framed blue 2-handle will help to resolve the crossings of the black 2-handle so that it can cancel the 1-handle. And after that the 3-handle will cancel the 0-framed unknotted blue 2-handle.

We now prove that 3-dimensional homology cobordism group is generated by Stein fillable manifolds

Proof of Corollary 9. Theorem 7 provides a homology cobordism from any manifold to a Stein manifold. Thus the homology cobordism groups are generated by Stein manifolds. $\square$

We now prove the existence of degree 1 map from a Stein fillable 3-manifold to a given 3-manifold.

Proof of Corollary 11. As we noticed previously that Y' smoothly embed in $Y \times I$ as a separating way. So, restriction to Y' of the projection map on $Y \times I$ onto Y will induces a degree 1 map from $Y' \to Y$. $\square$

CHAPTER 4

DISCUSSION

4.1 Embedding L–spaces in symplectic 4-manifolds

We now prove that smooth embeddings of L-space in symplectic 4-manifold is always separating.

Proof of Theorem 3. Suppose Y is an L-space that smoothly embeds in a closed symplectic 4-manifold. We begin by showing it is separating. To this end we assume it is non-separating. Let X_1 be the compact manifold obtained from X by cutting along Y. Notice that $\partial X_1 = Y \sqcup -Y$, so we can glue two copies X_1^1, X_1^2 of X_1 along their boundaries to get a closed manifold X'. As constructed, X' is a double cover of X so, in particular, we can lift the symplectic form using the covering map and thus X' is symplectic. Let N be a neighbourhood of an arc in X_1^1 connecting its boundary components. Set $X_1' = \overline{X_1^1 - N}$ and $X_2' = X_1^2 \cup N$. Clearly $\partial X_i' = Y \# -Y$ which is an L-space. As X is symplectic and Y is a rational homology sphere, by using the Mayer-Vietoris sequence we can see that $b_2^+(X_i') = b_2^+(X) > 0$ (since X is symplectic, the cohomological element corresponds to the symplectic form produces an element of b_2^+). So $Y \# -Y$ is an admissible cut for a symplectic manifold X' which contradicts Remark 34.

Now when Y embeds in X in a separating manner then one of the component of $X - Y$ must have $b_2^+ = 0$ or we will get the same contradiction as before. $\square$

We now prove Theorem 13 that says an L-space that does not bound negative-definite 4-manifold cannot have an oriented cobordism embedding in a compact symplectic 4-manifold with convex boundary.

Proof of Theorem 13. Let Y be an L-space that does not bound negative definite 4-manifold. If Y embed in any symplectic 4-manifold with weakly convex boundary W then it has to be separating since otherwise we can cap off with a concave cap to get a closed symplectic manifold where Y is non-separating which contradicts Proposition 3. So Y has to be separating. When we cap off the upper boundary of W by a cap with $b_2^+ > 0$, since Y does not bound a negative definite 4-manifold, both the sides of Y have $b_2^+ > 0$. In particular Y is an admissible cut for a symplectic 4 manfiold with $b_2^+ > 1$ which is a contradiction by Remark 34. □

Proof of Corollary 15. Let Y' admit weakly fillable contact structure and Y be an L-space that does not bound a negative definite 4-manifold. If Y has an oriented cobordism embedding in $Y' \times [0, 1]$, then since Y' is weakly fillable Y has an oriented cobordism embedding in a symplectic 4 manfiold with weakly convex boundary, contradicting Theorem 13. □

We now show the existence of exotic manifolds with boundary and $b_2 = 1$ using the ideas above.

Proof of Corollary 16. Now start with B^4 and attach a 2-handle h along pretzel knot $K = P(-2, 3, 7)$ to get a 4-manifold W' with $S^3_9(K)$ (which is an L-space as mentioned in Remark 14) as its boundary. Attach a cork as Step 2 of the Theorem 2 and get W with $b_2^+(W) = 1$. After a cork-twist we can see that the 2-handle h now passing over the 1-handle of the cork and this will increase the contact framing of h by one as in Figure 4.2 and thus the resulting manifold W' will be Stein by Theorem 25. Before the cork-twist we had a smooth embedding of $S^3_9(K)$ in W. But by Theorem 13 $S^3_9(K)$cannot embed smoothly in W' so they are exotic pairs. □

4.2 Constructing rational ribbon cobordism

Now we begin the proof of Theorem 18 that says that given a compact 4-manifold with some specific conditions one can construct a Stein 4-manifold with same algebraic topology but different boundary.

Proof of the Theorem 18. Part i) Let X be a compact oriented 4-manifold with boundary Y and $b_1(X) = 0$. Turning a handle structure on X upside down, we can think of X as a cobordism from $-Y$ to $\emptyset$, this is indicated in Figure 4.1. Notice that in this upside-down cobordism all the 1-handles of X are converted into 3-handles and all the 3-handles become 1-handles. In the upside-down X, 1-handles are attached onto $-Y \times [0, 1]$ along $-Y \times \{1\}$, let us call this cobordism M_1. Notice that $b_1(X) = 0$ so the homology long exact sequence of the pair (X, Y) implies that there exists a minimal set of 2-handles such that if we attach those on top of M_1, and let us call it M_2, then $H_1(M_2, Y; \mathbb{Q})) = 0$. (Here by minimum we mean that if we take any 2-handle out from the set then $H_1(M_2, Y; \mathbb{Q}) \neq 0$.) Since we consider a minimal set of 2-handles for this construction, we have $H_2(M_2, Y; \mathbb{Q}) = 0$ as well because in this case the number of 1-handles of M_2 is the same as the number of 2-handles. Thus M_2 is a rational ribbon cobordism from Y to say Y_1 which is the top boundary of M_2, see the top right of Figure 4.1. Consider X_1 to be the handlebody obtained from X by taking out M_2 and turning what remains upside-down, this is indicated in the third picture Figure 4.1. Thus X_1 only has 1- and 2-handles with boundary Y_1. If this is Stein then we are done. If not then that implies it has some 2-handles whose smooth framing is bigger than that the contact framing minus 1 of the attaching circle in $\#S^1 \times S^2$ (in this case we can think of the top boundary Y_1 is obtained after attaching 2-handles on the boundary of 1-handlebody which is connected sum of $S^1 \times S^2$). To fix this framing issue, we repeatedly apply the Step 2 of the proof of Theorem 2. That is we attach a cork as in Figure 3.2 (where

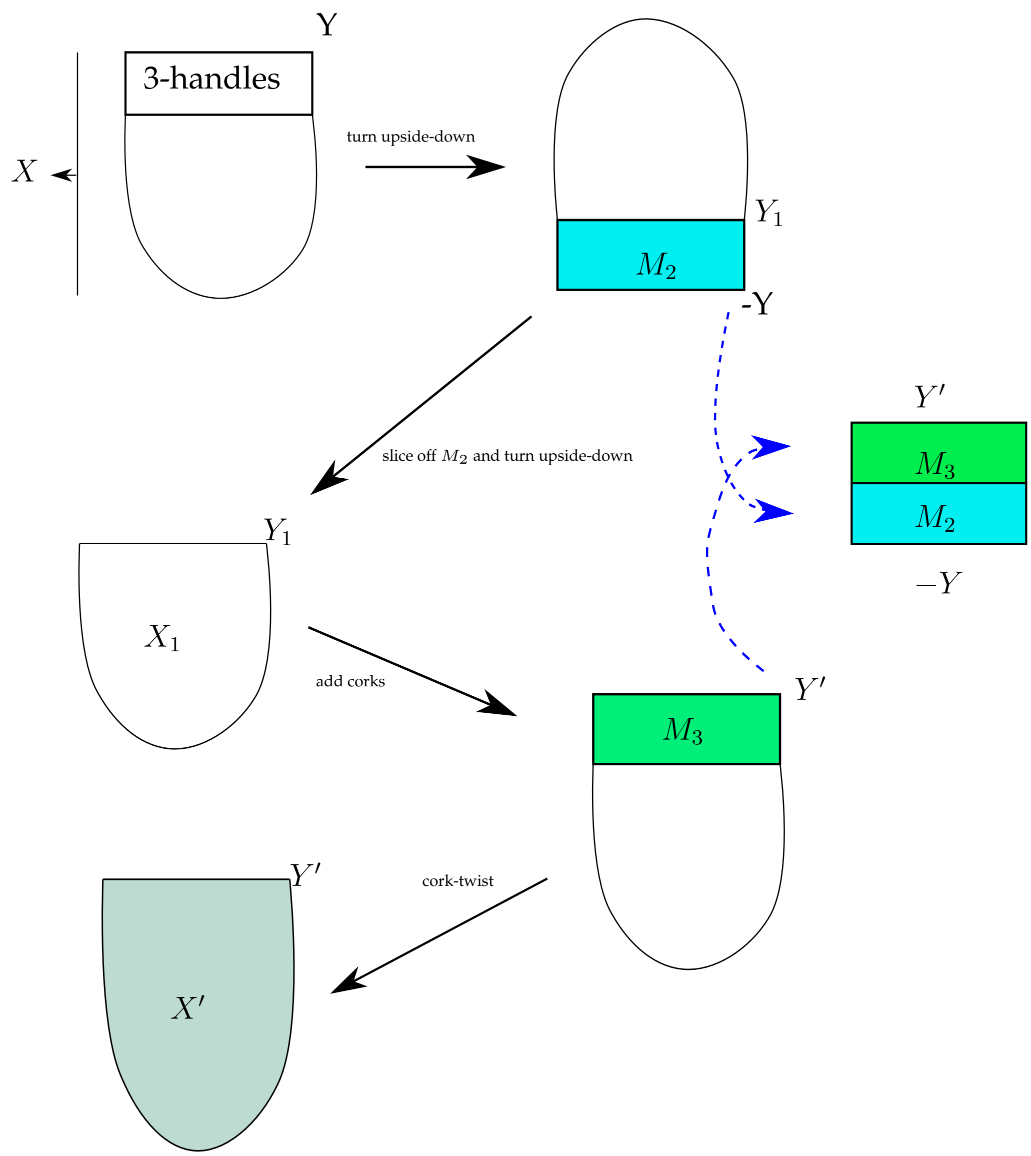

Figure 4.1: A schematic of the construction of a ribbon cobordism from Y to a Stein fillable Y'.

the red curve there is the handle that needs its Thurston–Bennequin invariant increased). We then do a cork twist that exchange the 1-and 2-handles. We claim this has the effect of increasing the contact framing of the original attaching sphere of the 2-handle by 1. To see this, notice that if a knot passing over 1-handle then in the front projection diagram of a knot we are actually deleting two consecutive right and left cusps by connecting them through a 1-handle and thus we are increasing the contact framing. See Figure 4.2. But this process does not change

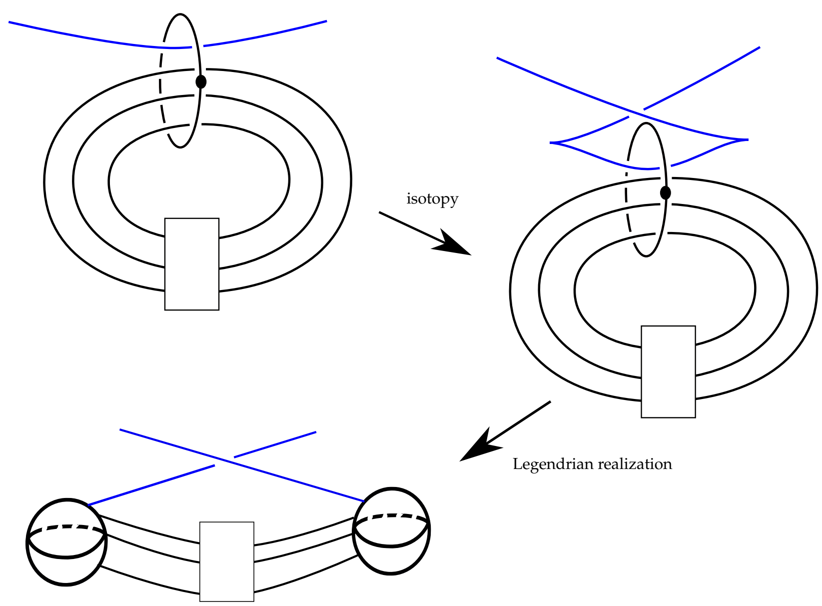

Figure 4.2: The contact framing of blue knot increased by +1 after a cork-twist.

the smooth surgery coefficient. Let us consider the cobordism X_2 obtained by attaching a suitable number of corks to X_1 so that the manifold X_2' obtained by applying the cork twists is Stein. The manifold X_2 and X_2' are homeomorphic as the cork-twist homeomorphism can always be extend as homeomorphism on the 4-manifold by the result of Freedman [5]. Observe $b_2(X_2') = b_2(X_1) = b_2(X)$. Let Y' be the top boundary of X_2'. Then there is a homology ribbon cobordism M_3 from Y_1 to Y' which is given by attaching the above corks to the top of X_1, see the fourth picture in Figure 4.1. Glue this cobordism on top of M_2 to get our desired ribbon rational homology cobordism $M = M_2 \cup M_3$ from Y to Y' with Y' Stein fillable.

For part ii), let X is a compact manifold with connected boundary Y which is a $\mathbb{Q}HS^3$, then we consider a handle decomposition of $X = X_0 \cup X_1 \cup X_2 \cup X_3$ where X_i contains handles of index i. Consider the minimum set of 1-handles which generate the free part of $(H_1(X;\mathbb{Q}))$. Let $\overline{X}$ be the manifold obtained from X by doing surgery on those 1-handles. (In Kirby calculus this is equivalent of replacing those dotted 1-handles with 0-framed unknotted 2-handles.) We will now show that this surgery operation does not change the b_2 (or more precisely the intersection form). As Y is a $\mathbb{Q}HS^3$, $H_1(\overline{X};\mathbb{Q}) = 0 = H_3(\overline{X};\mathbb{Q})$. But we are not doing anything with the 3-handles of X, so the only way the third homology of $\overline{X}$ vanishes with $\mathbb{Q}$ co-efficients is if the 3-handles cancel the 2-handles in homology. And thus from cellular homology, we can see that $b_2(X) = b_2(\overline{X})$. Also notice that the above surgery does not change the non-torsion elements of $H^2(X;\mathbb{Z})$ so they have the same intersection form. Now apply the proof of Theorem 18 on $\bar{X}$ and we get the desired Stein manifold X' with boundary Y'. $\square$

CHAPTER 5

FUTURE DIRECTIONS AND QUESTIONS:

Here I will summarize all the important questions that arrived in the above discussions.

Question 5. *Does every L-space bound a definite 4-manifolds?*

Question 6. *Does rational homology cobordism group $\Theta_{\mathbb{Q}}$ genrated by L-spaces?*

Question 7. *Is $\Sigma(2,3,11)$ rationally homology cobordant to some L-space?*

Question 8. *Does every 3-manifolds admit a Floer cap?*

Question 9. *Does every 3-manifold embed smoothly in a separating way to some closed 4-manifold with non-trivial mix map or Seiberg–Witten solution?*

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