CONTACT STRUCTURES ON OPEN 3-MANIFOLDS

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ABSTRACT

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In this thesis, we study contact structures on any open 3-manifold V which is the interior of a compact 3-manifold. To do this, we introduce proper contact isotopy invariants called the slope at infinity and the division number at infinity. We first prove several classification theorems for $T^2 \times [0, \infty)$, $T^2 \times \mathbb{R}$, and $S^1 \times \mathbb{R}^2$ using these concepts. This investigation yields infinitely many tight contact structures on $T^2 \times [0, \infty)$, $T^2 \times \mathbb{R}$, and $S^1 \times \mathbb{R}^2$ which admit no precompact embedding into another tight contact structure on the same space. Finally, we show that if every $S^2 \subset V$ bounds a ball or an S^2 end, then there are uncountably many tight contact structures on V that are not contactomorphic, yet are isotopic. Similarly, there are uncountably many overtwisted contact structures on V that are not contactomorphic, yet are isotopic.

Contents

List of Figures

Chapter 1

Introduction and Statements of the Main Results

Recently, there has been much work towards the classification of tight contact structures on compact 3-manifolds up to isotopy (relative to the boundary). In particular, Honda and Giroux provided several classification theorems for solid tori, toric annuli, torus bundles over the circle, and circle bundles over surfaces [Gi1, Gi2, Gi3, Ho2, Ho3]. In comparison, tight contact structures on open 3-manifolds have been virtually unstudied. Two main results dealing with open contact manifolds are due to Eliashberg. In [El1], Eliashberg shows that $\mathbb{R}^3$ has a unique tight contact structure. It is immediate from his proof that $S^2 \times [0, \infty)$ has a unique tight contact structure with a fixed characteristic foliation on $S^2 \times 0$. Therefore, the classification of tight contact structures on open manifolds with only S^2 ends can be reduced to the case of compact manifolds. In [El3], Eliashberg shows that, in contrast to the situation for S^2 ends, there are uncountably many tight contact structures on $S^1 \times \mathbb{R}^2$ that are not contactomorphic. The situation for closed 3-manifolds is different. Colin, Giroux, and Honda proved that an atoroidal 3-manifold supports finitely many tight contact structures up to isotopy [CGH]. Honda, Kazez, and Matić, and independently, Colin, show that an irreducible, toroidal 3-manifold supports countably infinitely many tight contact structures up to isotopy [HKM1, Co].

In this paper, we study tight contact structures on any open manifold V that is the interior of a compact manifold. Due to the failure of Gray's Theorem on open contact manifolds, we relegate ourselves to the study of tight contact structures up to *proper isotopy*, by which we mean isotopy of the underlying manifold rather than a one-parameter family of contact structures. When we say that two contact structures are isotopic, we will mean that they are connected by a one-parameter family of contact structures. We first introduce two new proper isotopy invariants which we call the *slope at infinity* and the *division number at infinity* of an end $\Sigma_g \times [0, \infty)$ of an open contact manifold. These invariants are most naturally defined for toric ends $T^2 \times [0, \infty)$, where we take our inspiration from the usual definition of the slope and division number of a convex torus. Using these invariants and Honda's work in [Ho2], we essentially classify tight contact structures on toric ends $T^2 \times [0, \infty)$. In particular, we show that there is a natural bijection between tight toric annuli and tight toric ends that *attain* the slope at infinity and have finite division number at infinity. However, we also show that for any slope at infinity there is an infinite family of tight toric ends which do not attain the slope at infinity and therefore do not come from closed toric annuli. Interestingly, these contact structures are strange enough that they cannot be properly embedded in another tight contact manifold. This yields the following

Theorem 1.0.1. *Let X be $T^2 \times [0, 1)$, $T^2 \times (0, 1)$ or $S^1 \times D^2$, where D^2 is the open unit disk. Let X' be another copy of X parametrized as $T^2 \times [0, \infty)$, $T^2 \times \mathbb{R}$ or $S^1 \times \mathbb{R}^2$. For each slope at infinity, there exist infinitely many tight contact structures on X with that slope, distinct up to proper isotopy, which do not extend to a tight contact structure on X'.*

This result stands in contrast to Eliashberg's original examples, all of which are neighborhoods of a transverse curve in S^3 and have a different slope at infinity. Using this embedding, it is easy to compactify his examples. Theorem 1.0.1 shows that, in general, finding such a nice compactification is not straightforward.

Finally, just as high torus division number is a problem in the classification of toric annuli, contact structures with infinite division number at infinity prove difficult to understand. However, we are able to use the notion of *stable disk equivalence* to partially understand this situation. Precise statements of all of these results are in Section 4. In Section 4.4, we use these results to reduce the classification of tight contact structures on $S^1 \times \mathbb{R}^2$ and $T^2 \times \mathbb{R}$ to the classification of the corresponding toric ends.

In the second half of the paper, we use the notion of the slope at infinity to prove a generalization of Eliashberg's result in [El3]:

Theorem 1.0.2. *Let V be any open 3-manifold which is the interior of a compact,* *connected 3-manifold M with nonempty boundary such that every embedded S^2 either bounds a ball or is isotopic to a component of ∂M. If ∂M contains at least one component of nonzero genus, then V supports uncountably many tight contact structures which are not contactomorphic, yet are isotopic.*

Eliashberg's proof involves computing the contact shape of the contact structures on $S^1 \times \mathbb{R}^2$, which in turn relies on a previous computation of the symplectic shape of certain subsets of $T^n \times \mathbb{R}^n$ done in [Si]. We bypass the technical difficulties of computing the symplectic shape by employing convex surface theory in the end of V. The first step in the proof is to put a tight contact structure on the manifold M with a certain dividing curve configuration on the boundary. To do this, we use the correspondence between taut sutured manifolds and tight contact structures covered in [HKM2]. We then find nested sequences of surfaces which allow us to construct a contact manifold (V, η_s) for every $s \in (-2, -1)$. We distinguish these contact structures up to proper isotopy by showing that they have different slopes at infinity. Since the mapping class group of an irreducible 3-manifold with boundary is countable (see [McC]), uncountably many of the η_s are not contactomorphic. To simplify the presentation of the proof, we first present the proof in the case when ∂M is connected in Section 5.2. We deal with the case of disconnected boundary in Section 5.3.

In [El1], Eliashberg declares a contact structures on an open 3-manifold V to be *overtwisted at infinity* if for every relatively compact $U \subset V$, each noncompact component of $V \setminus U$ is overtwisted. If the contact structure is tight outside of a compact set, then it is *tight at infinity*. He then uses his classification for overtwisted contact structures in [El2] to show that any two contact structures that are overtwisted at infinity and homotopic as plane fields are properly isotopic. In contrast to this result, we have the following:

Theorem 1.0.3. *Let V be any open 3-manifold which is the interior of a compact, connected 3-manifold M with nonempty boundary such that every embedded S^2 either bounds a ball or is isotopic to a component of ∂M. If ∂M contains at least one component of nonzero genus, then V supports uncountably many overtwisted contact structures which are tight at infinity and which are not contactomorphic, yet are isotopic.*

Chapter 2

Background

2.1 Basic Contact Geometry

In this section, we will summarize the most basic ideas and results in contact geometry that will be necessary for our work. Many of these ideas have been summarized in a more expanded form in survey articles and books. In particular, we refer the reader to [Ho2] and [Et]. We have followed several sections in [Ho2] and [Et] very closely in our exposition. We only consider contact geometry in dimension three, although many of the basic concepts have analogs in higher dimensions. Unless otherwise specified, M will be a 3-manifold. We refer the reader to [He] for facts about 3-manifolds.

Let ξ be a plane field on M, by which we mean a rank 2 distribution on M. Let $p \in M$ and let α be a 1-form defined in a neighborhood of p such that $\ker \alpha = \xi$. Then, ξ is a *contact structure at* p if $\alpha \wedge d\alpha \neq 0$ in a neighborhood of p. If ξ is a contact structure at p for every $p \in M$, then we call ξ a *contact structure* and (M, ξ) a *contact manifold*.

We now introduce three notions of equivalence of contact structures. Let (M_1, ξ_1) and (M_2, ξ_2) be two contact manifolds. We say that these contact manifolds are *contact diffeomorphic* or *contactomorphic* if there is a diffeomorphism $f\colon M_1 \to M_2$ such that $f_*(\xi_1) = \xi_2$. If the ξ_i are on the same manifold M, then we say that the ξ_i are *isotopic* if there is a family of contact structures η_t on M with $t \in [0, 1]$ such that $\eta_0 = \xi_1$ and $\eta_1 = \xi_2$. We say that the ξ_i are *properly isotopic* if there is a family of diffeomorphisms ϕ_t of M such that $\eta_t = \phi_{t*}(\xi_1)$. One can use *Moser's Method*, described in [Aeb], to show that on a closed 3-manifold, every isotopy is a proper isotopy. This is called Gray's Theorem. Gray's Theorem is not always true on open manifolds, since Moser's Method involves integrating a vectorfield.

One of the ubiquitous examples of a contact structure ξ_{std} on $\mathbb{R}^3$ is given as the kernel to the globally defined form $\alpha_{std} = dz + xdy$. This contact structure is shown in Figure 2.1. Note that this contact structure is vertically invariant. Another example on $\mathbb{R}^3$ is given by the kernel of the form $dz + r^2 d\theta$ given in cylindrical coordinates.

Given that contact structures are inherently geometric objects, it makes sense to try and understand them by trying to understand the curves and surfaces inside them. There are two natural classes of curves: Legendrian and transverse curves. A curve γ in M is *Legendrian* if γ is everywhere tangent to ξ and is *transverse* if γ is everywhere transverse to ξ. There is no natural dichotomy for surfaces in M. Algebraic manipulation of the condition that ξ must satisfy to be a contact structure shows, by Darboux's Theorem, that

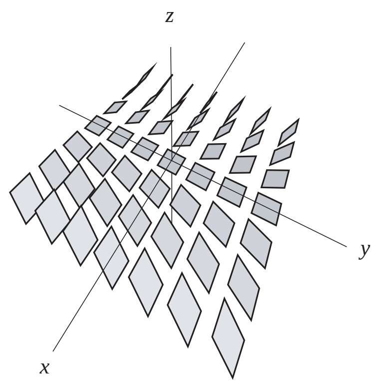


Figure 2.1: The contact structure $\ker(dz + xdy)$.

there is no open subset of a surface Σ which integrates ξ. This means that any surface Σ in a contact manifold has a singular, one-dimensional foliation defined at $p \in M$ by $T_p\Sigma \cap \xi_p$. This singular foliation, denoted Σ_ξ, is called the *characteristic foliation* of Σ.

We say that two Legendrian (transverse) curves γ_1 and γ_2 are *Legendrian (transversally) isotopic* if there is an isotopy through embedded Legendrian (transverse) curves that begins with γ_1 and ends with γ_2. One invariant of Legendrian curves up to Legendrian isotopy is the Thurston-Bennequin invariant. Let γ be a Legendrian curve in M, Σ be a surface with $\partial\Sigma = \gamma$, and v be a vectorfield defined locally along γ which is transverse to ξ. Let γ' be the curve obtained by moving γ slightly along v. Define the *Thurston-Bennequin invariant* of γ, written $tb(\gamma)$, to be the signed intersection number of γ' with Σ. We will give an easy way to compute this invariant after we introduce convex surfaces.

We now discuss the local stability of a contact structure. The first result in this direction is Darboux's Theorem:

Theorem 2.1.1. *Let (M, ξ) be any contact 3-manifold and $p \in M$. Then there exist neighborhoods N of p in M, and U of the origin in $\mathbb{R}^3$ and a contact diffeomorphism $f\colon (N, \xi|_N) \to (\mathbb{R}^3, \xi_{std}|_U)$.*

Darboux's Theorem is a simple manifestation of the idea in contact geometry (proved using Moser's Method) that if two contact structures agree on some compact subset, then they can be isotoped to agree on an open neighborhood of that subset. We now state the most commonly used results which rely on this fact.

Theorem 2.1.2. *Any two Legendrian (transverse) knots have contact diffeomorphic neighborhoods.*

Theorem 2.1.3. *Let (M_i, ξ_i) be a contact manifold and Σ_i and embedded surface for $i = 0, 1$. If there is a diffeomorphism $f\colon \Sigma_0 \to \Sigma_1$ that preserves the characteristic foliation, then f can be extended to a contact diffeomorphism in some neighborhood of Σ_0.*

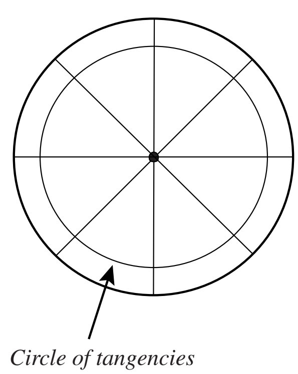


Figure 2.2: An overtwisted disk D with the characteristic foliation.

We now introduce a fundamental dichotomy in contact geometry. We say that a contact manifold (M, ξ) is *overtwisted* if there exists a disk D in M, called an *overtwisted disk*, with characteristic foliation as shown in Figure 2.2. We say that (M, ξ) is tight if it has no overtwisted disk.

2.2 Convex Surfaces

2.2.1 Definitions and Basic Results

As one might suspect, trying to prove anything using characteristic foliations can get quite tricky. Therefore, we often work with surfaces which are embedded in a particularly nice way with respect to the contact structure. A vectorfield v in a contact manifold (M, ξ) is called a *contact vectorfield* if the flow of the vectorfield is a one-parameter family of contact diffeomorphisms. A surface Σ in M is *convex* if there is a contact vectorfield v transverse to Σ. It turns out that v need only be defined on a neighborhood of Σ, since one can always write down an extension to a contact vectorfield defined on the whole of M. Note that by definition a convex surface Σ has a natural product neighborhood structure $\Sigma \times (0, 1)$ in which the contact structure is vertically invariant.

It turns out that convex surfaces are very easy to find. In fact, every surface is C^∞-close to a convex surface [Gi]. This is also true for a surface with Legendrian boundary, as long as the twisting of the contact planes relative in the framing given by the surface is nonpositive [Ka].

The important point about a convex surface is that we can distill all of the data about the contact structure ξ in a neighborhood of the surface into a collection of curves on the surface. We do this as follows. Let Σ be a convex surface and let v be the contact vectorfield transverse to Σ. Let Γ be the collection of points $p \in \Sigma$ where $v(p) \subset \xi_p$. In [Gi], Giroux shows that this is generically a multi-curve, by which we mean a one-dimensional submanifold of Σ. Γ also satisfies the following conditions:

1. $\Sigma \setminus \Gamma = \Sigma_+ \cup \Sigma_-$, where Σ_+ and Σ_- are disjoint subsurfaces.

2. Σ_ξ is transverse to Γ

3. There is a vectorfield w and a volume form ω on Σ such that

   (a) w directs Σ_ξ in the sense that it is contained in Σ_ξ and zero only where Σ_ξ is singular.

   (b) the flow of w expands ω on Σ_+ and contracts ω on Σ_-.

   (c) w points transversally out of Σ_+

We momentarily forget about the contact structure and just think about singular one-dimensional foliations on Σ. We say that a multi-curve Γ *divides* a singular one-dimensional foliation $\mathcal{F}$ on Σ if $\mathcal{F}$ and Γ satisfy the above conditions, where Σ_ξ is replaced by $\mathcal{F}$. If Γ divides $\mathcal{F}$, then we call Γ a collection of *dividing curves*. If Σ is a convex surface, then we denote the dividing curves corresponding to Σ_ξ by Γ_Σ. The power of convex surfaces to distill information about contact structures into the collection of dividing curves is based on the following theorem.

Theorem 2.2.1 (Giroux Flexibility [Gi]). *Suppose* $\mathcal{F}$*and* Σ_ξ *are both divided by the same multi-curve* Γ*. Then inside any neighborhood* N *of* Σ *there is an isotopy* $\Phi_t\colon \Sigma \to N, t \in [0,1]$ *of* Σ *such that*

1. $\Phi_0 =$ *inclusion of* Σ *into* N,
2. $\Phi_t(\Sigma)$ *is a convex surface for all* t,
3. Φ_t *does not move* Γ,
4. $(\Phi_1(\Sigma))_\xi = \Phi_1(\mathcal{F})$.

This result is often referred to as *Giroux Flexibility*. A frequently used application of Giroux Flexibility is the *Legendrian Realization Principle*.

Corollary 2.2.2 (Legendrian Realization Principle [Ka]). *Let* Σ *be a convex surface and* C *be a multicurve on* Σ*. Assume* $C \pitchfork \Gamma_\Sigma$ *and* C *is* nonisolating*, i.e., each connected component of* $\Sigma \setminus C$ *nontrivially intersects* Γ_Σ*. Then there is an isotopy (as in the Giroux Flexibility Theorem) such that* $\varphi_1(C)$ *is Legendrian.*

When we say “LeRP”, we will mean “apply the Legendrian Realization Principle” to a collection of curves. We will use this as a verb and call this process “LeRPing” a collection of curves. It can be shown that for a Legendrian curve γ on Σ convex, $tb(\gamma) = -\frac{1}{2}\sharp\Gamma \cap \gamma$.

We now state Giroux’s Criterion for determining whether or not a convex surface has a neighborhood that is tight.

Theorem 2.2.3 (Giroux's Criterion). *Let Σ be a convex surface in (M, ξ). A vertically invariant neighborhood of Σ is tight if and only if $\Sigma \neq S^2$ and Γ_Σ contains no curves that are contractible on Σ or $\Sigma = S^2$ and Γ_Σ is connected.*

2.2.2 Bypasses

Given a convex surface Σ, it is natural to ask what happens to the dividing set as we isotop the surface through the manifold. It turns out that the dividing set changes in a "discrete" fashion, and the simplest change happens by attaching a *bypass*.

Let Σ be a convex surface and α be a Legendrian arc in Σ which intersects Γ_Σ in three points p_1, p_2, p_3, where p_1 and p_3 are endpoints of α. A *bypass* is a convex half-disk D with Legendrian boundary, where $D \cap \Sigma = \alpha$ and $tb(\partial D) = -1$. α is called the *arc of attachment* of the bypass, and D is said to be a bypass *along* α or Σ (Figure 2.3).

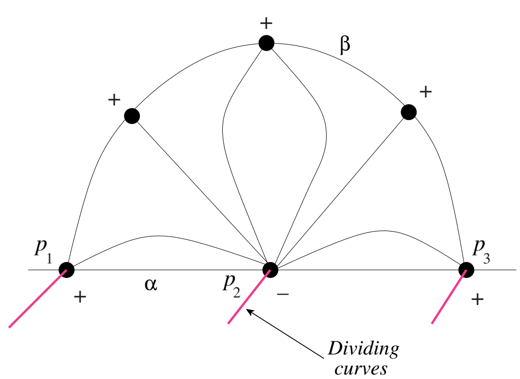
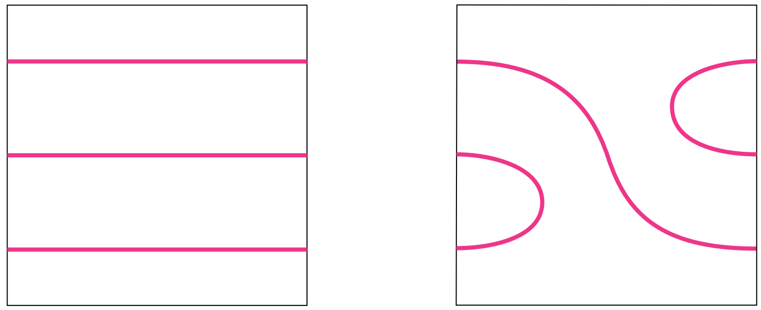


Figure 2.3: A bypass.

Each bypass carries a natural sign, so we say there are *positive* and *negative bypasses*. We show in Figure 2.4 how the dividing set of a convex surface changes after isotoping the surface across the bypass.



(a) (b)

Figure 2.4: The effect of attaching a bypass to a surface Σ from the front to form a surface Σ'. The dividing set Γ_Σ is (a) and $\Gamma_{\Sigma'}$ is (b).

Finally, we describe a way of finding bypasses. This technique is known as the *Imbalance Principle* and can be proved by using LeRP and Giroux Flexibility.

Theorem 2.2.4 (Imbalance Principle). *Let S be a convex surface with Legendrian boundary.*

1. *If $S = D^2$ so that $tb(\partial S) < -1$, then there exists a bypass along ∂S. Similarly, if $S \neq D^2$, $tb(\partial S) \leq -1$, and Γ_S is boundary parallel, then there exists a bypass along ∂S.*

2. *Let $S = S^1 \times [0,1]$. If $tb(S^1 \times \{1\}) < tb(S^1 \times \{0\})$, then there is a ∂-parallel arc and hence a bypass along $S^1 \times \{1\}$.*

2.2.3 Convex Tori and the Farey Graph

In this section, we discuss convex tori, our first example of a convex surface, and explicitly describe how bypass attachment changes the dividing set of the convex torus. The dividing set of a convex torus must, first of all, be nonempty. This follows from some facts about characteristic foliations which we will not go into here. By Giroux's Criterion, if the torus has any contractible dividing curves, then the ambient contact manifold is overtwisted. If the torus is in a tight contact manifold, it cannot have any contractible dividing curves. Therefore, the dividing set of a convex torus with no contractible dividing curves consists of an even number, $2n$, of parallel curves, all having some slope s. We say that such a torus has *slope* s and *division number* n. Figure 2.5 shows two convex tori. The right-hand side torus is said to be in *standard form*. We will often attach bypasses along the curves labeled as *Legendrian ruling cuves*. Giroux Flexibility tells us that we can always put a convex torus into standard form. Moreover, we can find ruling curves of any rational slope that is different from the slope of the convex torus.

Suppose that the division number of a convex torus is greater than 1. If we attach a bypass along a ruling curve, then one can check that the division number goes down by one. If the division number is 1, then we can keep track of the change in slope by using the Farey graph (Figure 2.6). The Farey graph can be described as follows: The set of vertices of the Farey graph is $\mathbb{Q} \cup \{\infty\}$ on $\partial\mathbb{H}$. (More precisely, fix a fractional linear transformation f from the upper half-plane model of hyperbolic space to the unit disk model $\mathbb{H}$. Then the set of vertices is the image of $\mathbb{Q} \cup \{\infty\}$ under f.) There is a unique

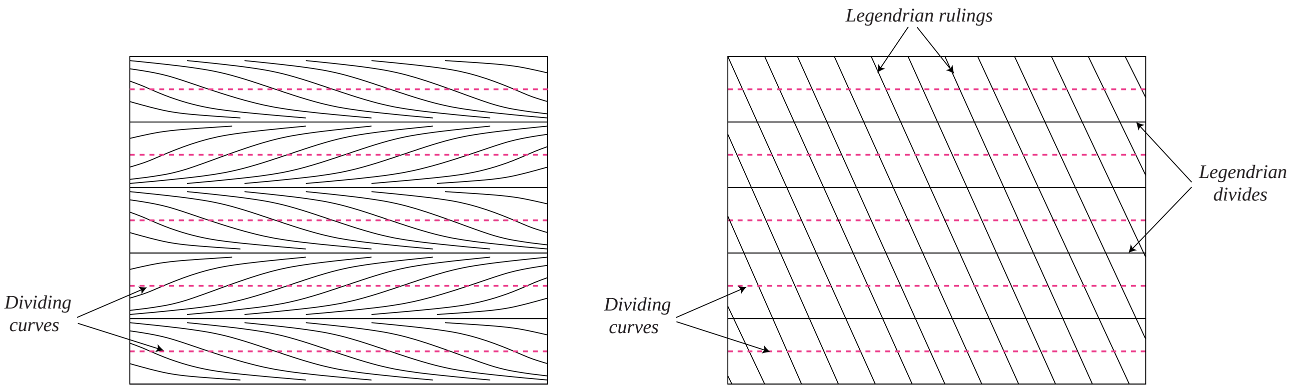


Figure 2.5: The left-hand side is a torus with nonsingular, Morse-Smale characteristic foliation. The right-hand side is a torus in standard form. Here the sides are identified and the top and bottom are identified.

edge between $\frac{p}{q}$ and $\frac{p'}{q'}$ if and only if the corresponding shortest integer vectors form an integral basis for $\mathbb{Z}^2$. (The edge is usually taken to be a geodesic in $\mathbb{H}$.) The usefulness of the Farey graph is contained in the following result which is due to Honda.

Theorem 2.2.5 (Honda [Ho2]). *Let $s = slope(\Gamma_{T^2})$. If a bypass is attached along a closed Legendrian ruling curve of slope s', then the slope s'' on the resulting convex surface is obtained as follows: Let $[s', s) \subset \partial\mathbb{H}$ be the counterclockwise interval from s' to s. Then s'' is the point on $[s', s)$ which is closest to s' and has an edge to s.*

See Figure 2.7 for an illustration of how to use this result.

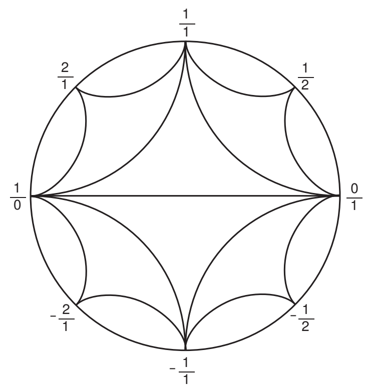


Figure 2.6: The Farey graph.

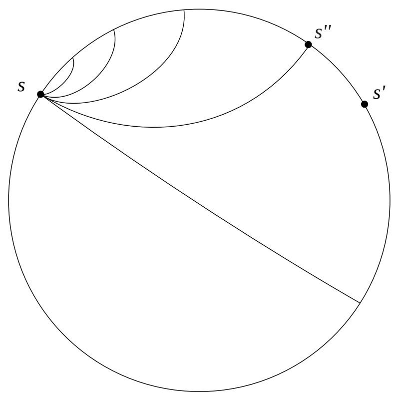


Figure 2.7: Bypass attachment using the Farey graph.

2.3 Known Classification Results

In this section, we will outline the basic classification results in contact geometry for closed manifolds and highlight the ones which we will build upon later. We group these results into theorems concerning classification of tight contact structures and theorems concerning the classification of overtwisted contact structures.

2.3.1 Tight, Compact Contact Manifolds

The result which gets most any classification theorem off the ground is Eliashberg's Theorem [El4].

Theorem 2.3.1 (Eliashberg [El4]). *Fix a characteristic foliation $\mathcal{F}$ adapted to $\Gamma_{\partial B^3} = S^1$. Then there is a unique tight contact structure on B^3 up to isotopy relative to ∂B^3.*

Much of the progress towards classifying tight contact structures on compact manifolds is due to the development of convex surface theory, the usefulness of which is due to Eliashberg's result. In general, the scheme for determining an upper bound on the number of tight contact structures on a Haken 3-manifold (a Haken 3-manifold can be cut up along successive surfaces until all one is left with are balls) up to isotopy is as follows: Make each successive cutting surface convex and cut along it, keeping track of the dividing set. Then, once you are left with balls, you invoke the result of Eliashberg, and then examine all possible dividing sets on the cutting surfaces as you glue them back together.

We will now proceed to classification results for $T^2 \times I$. We first identify the most basic building block for a contact structure on this space: the *basic slice*. Consider $T^2 \times [0,1]$ with convex boundary conditions $\#\Gamma_0 = \#\Gamma_1 = 2$, $s_0 = \infty$, and $s_1 = 0$. Here we write $\Gamma_i = \Gamma_{T^2 \times \{i\}}$ and $s_i = \text{slope}(\Gamma_i)$. (Using a diffeomorphism of $T^2 \times I$, there is an analogous result when the shortest integer vectors corresponding to s_0 and s_1 form an integral basis for $\mathbb{Z}$.) Then there are exactly two tight contact structures (up to isotopy relative to the boundary) which are *minimally twisting* (i.e., every convex torus T' isotopic to $T^2 \times \{i\}$ has slope($\Gamma_{T'}$) in the interval $(0, +\infty)$). These $T^2 \times [0,1]$ layers are *basic slices*. These two tight contact structures are formed by attaching a bypass along a ruling curve of slope 0, so each has a sign which depends on the sign of the bypass that is attached. We call a basic slice *positive* (*negative*) *basic slice* if it is formed by attaching a positive (negative) bypass.

Given any tight, minimally twisting contact structure on $T^2 \times I$ with convex boundary, each component of which has division number 1, one can act by an element of $SL_2(\mathbb{Z})$ so that the slope of $T^2 \times \{0\}$ is -1 and the slope of $T^2 \times \{1\}$ is $-\frac{p}{q}$. We now write $-\frac{p}{q}$ using the continued fraction expansion

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 \cdots - \frac{1}{r_k}}},$$

where $r_i \leq -2$. We will abbreviate this continue fraction expansion as $(r_0, r_1, \ldots, r_{k-1}, r_k)$. Let A be a convex annulus with boundary on ruling curves of slope 0 on opposite boundary components. By the Imbalance Principle, there is a bypass along $T^2 \times \{1\}$. We attach this bypass. After bypass attachment, we obtain a convex torus T' isotopic to T, such that T and T' cobound a $T^2 \times I$. Denote $slope(\Gamma_{T'}) = -\frac{p'}{q'}$. One can prove that, in fact, $-\frac{p'}{q'}$ has continued fraction expansion $(r_0, r_1, \ldots, r_{k-1}, r_k + 1)$. In terms of the Farey graph, we have moved counterclockwise. We successively peel off $T^2 \times I$ layers according to the Farey tessellation. The sequence of slopes is given by the continued fraction expansion, or, equivalently, by the shortest sequence of counterclockwise arcs in the Farey tessellation from $-\frac{p}{q}$ to -1. Once slope -1 is reached, we can peel off no more layers, and are done with our factorization.

We group the basic slices into *continued fraction blocks*. Each block consists of all the slopes whose continued fraction representations are of the same length. It turns out that we may *shuffle* basic slices which are in the same continued fraction block. This, along with some work from [Ho2] that we suppress for brevity, means that the contact structure in a continued fraction block depends only on the number of positive (or negative) basic slices in a continued fraction block. Keeping careful track of the continued fraction expansion together with what we have said, implies

Theorem 2.3.2 (Honda [Ho2]). *Let Γ_{T_i}, $i = 0, 1$, satisfy $\#\Gamma_{T_i} = 2$ and $s_0 = -1$, $s_1 = -\frac{p}{q}$, where $p > q > 0$. Then the number of tight contact structures with these boundary conditions, up to isotopy relative to the boundary, is equal to* $|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)(r_k)|$.

The main point to take away from this Theorem is that the isotopy classification of tight contact structures on $T^2 \times I$ boils down to counting the number of positive basic slices in each continued fraction block. This, together with the boundary conditions, determines the contact structure up to isotopy relative to the boundary. For the case of nonminimally twisting, tight contact structures, we refer the reader to [Ho2].

2.3.2 Tight, Open Contact Manifolds

Before beginning to discuss tight, open contact manifolds, we should say something about the equivalence relation up to which we are classifying the contact structures. We now illustrate that classifying contact structures on open manifolds up to isotopy is in many cases uninteresting. For example, consider $S^1 \times \mathbb{R}^2$. Given any contact structure on this space, one can always find a transverse knot in the same isotopy class as $S^1 \times \{0\}$. Since all transverse knots are locally the same, all contact structures on $S^1 \times \mathbb{R}^2$ are isotopic. It is therefore more reasonable to classify contact structures up to proper isotopy, which, as we shall see, is a more interesting equivalence relation.

In [El1], Eliashberg proves that the proper isotopy classification of contact structures on $\mathbb{R}^3$ coincides with the isotopy classification of contact structures on S^3. In particular, this means that there is a unique tight contact structure on $\mathbb{R}^3$ and the overtwisted contact structures are countable. Eliashberg's proof essentially shows the following:

Theorem 2.3.3 (Eliashberg). *Let $\mathcal{F}$ be a foliation on $S^2 \times \{0\}$ with a tight neighborhood. Then $S^2 \times [0, \infty)$ has a unique tight contact structure ξ with $S^2 \times \{0\}_\xi = \mathcal{F}$.*

This means that when looking at contact structures on open manifolds with S^2 ends, one can essentially ignore the contact structure in the S^2 end.

The first result for a manifold other that $S^2 \times [0, \infty)$ is Eliahsberg's existence result for $S^1 \times \mathbb{R}^2$ [El3]. These contact manifolds are all found as open subsets of S^3 with the unique, tight contact structure given by the 1-form $\alpha = \rho_1^2 d\phi_1 + \rho_2^2 d\phi_2$, where $(\rho_1 e^{i\phi_1}, \rho_2 e^{i\phi_2})$ are the standard complex coordinates from $\mathbb{C}^2$ restricted to S^3. For a positive $\delta < 1$, let $U_\delta = \{(z_1, z_2) | \rho_1 < \delta\}$, an open torus in S^3.

Theorem 2.3.4 (Eliashberg [El3]). *U_δ is contact diffeomorphic to $U_{\delta'}$ if and only if the difference $\frac{1}{\delta^2} - \frac{1}{\delta'} \in \mathbb{Z}$.*

One direction is fairly straightforward. If $\frac{1}{\delta^2} - \frac{1}{\delta'} = -k$ with k a positive integer, then the map $(\rho_1, \phi_1, \phi_2) \to (\frac{\rho_1}{\sqrt{1+k\rho_1^2}}, \phi_1 - k\phi_2, \phi_2)$ is contact and sends U_δ onto U'_δ.

The reverse direction is not nearly as straightforward. Eliashberg's proof relies on an invariant of a contact manifold V^{2n-1} called the *contact shape*, which is a subset $H^1(V;\mathbb{R})$ and is computed with respect to a manifold M^n. One computes the contact shape by looking at the symplectization of the contact manifold and computing the *symplectic shape* with respect to M. The *symplectic shape* of an exact, symplectic manifold $(X, \omega = d\lambda)$ with respect to M and a map $\alpha\colon\ H^1(X;\mathbb{R}) \to H^1(M;\mathbb{R})$ is the collection of $f^*(\lambda) \in H^1(M;\mathbb{R})$ where $f\colon\ M \hookrightarrow X$ is an embedding such that $f^* = \alpha$.

Eliashberg derives the formulaic relationship between δ and δ' by computing the contact shape of the open region in $\Gamma_{\delta,\delta'} = U_\delta \setminus \overline{U}_{\delta'}$ (assuming $\delta > \delta'$) with respect to T^2. To do this, he embeds $\Gamma_{\delta,\delta'}$ inside $T^2 \times S^1$, the unit cotangent bundle of T^2. He then relies on a computation of Sikorav [Si] about the symplectic shape of open subsets of the cotangent bundle of T^n which have the form $T^n \times A$, where $A \subset R^n$ is open (the symplectization of the embedding of $\Gamma_{\delta,\delta'}$ has this form). The k in the statement of the theorem is the number of Dehn twists the contact diffeomorphism performs about the meridinal disks in U_δ and shows up when inducing maps on the level of cohomology.

We mention these techniques to contrast the methods in this thesis. While the contact shape is in the same circle of ideas as Gromov's result about intersections of exact Lagrangian manifolds [Gr], our main tool will be convex surface theory, which is of a very different nature.

2.3.3 Overtwisted Contact Manifolds

We now turn our attention to overtwisted contact structures. The most fundamental result in this area is Eliashberg's classification of overtwisted contact structures:

Theorem 2.3.5 (Eliashberg [El2]). *Classification of overtwisted contact structures on a closed manifold M up to isotopy coincides with the homotopical classification of plane fields. If $\partial M \neq \emptyset$, then isotopy and homotopy are fixed in a neighborhood of ∂M.*

On an open manifold V, an overtwisted contact structure can be *tight at infinity* or *overtwisted at infinity*. An overtwisted contact structure is *tight at infinity* if there exists a compact set $K \subset V$ such that $V \setminus K$ is tight. Analogously, an overtwisted contact structure is *overtwisted at infinity* if for every compact set $K \subset V$ the manifold $V \setminus K$ is overtwisted. These notions were introduced by Eliashberg [El1] so that he could prove a similar theorem for open manifolds.

Theorem 2.3.6 (Eliashberg [El1]). *Let ξ_1 and ξ_2 be two contact structures overtwisted at infinity on an open 3-manifold M. If ξ_1 and ξ_2 are homotopic as plane fields, then they are properly isotopic.*

The proof of this result is essentially a repeated application of Eliahsberg's homotopy classification of overtwisted contact structures on compact manifolds. We now sketch the proof. Begin with and exhaustion $U_1 \subset U_2 \subset \cdots \subset M$ of M by open domains with smooth boundaries such that U_i is relatively compact inside U_{i+1} for each i and both contact structures are overtwisted on $U_i \setminus \overline{U}_{i-1}$. Set $U_0 = \emptyset$. We inductively construct a contact structure τ on V as follows: set $\tau = \xi_1$ on U_1; for each i, τ coincides with ξ_2 near ∂U_{2i} and with ξ_1 near ∂U_{2i+1}; τ is homotopic relative to the boundary to ξ_1 (ξ_2) on $U_{2i+1} \setminus U_{2i-1}$ ($U_{2i} \setminus U_{2i-2}$); τ is overtwisted on $U_{i+1} \setminus \overline{U}_i$. According to Eliashberg's homotopy classification of overtwisted contact structures, there is an isotopy on $U_{2i} \setminus U_{2i-2}$ fixed near the boundary taking τ to ξ_2. Similarly, there is an isotopy on $U_{2i+1} \setminus U_{2i-1}$ fixed near the boundary from τ to ξ_1. Hence, we have inductively constructed a proper isotopy on V from τ to either ξ_i. Clearly, such a proof will not work for manifolds that are tight at infinity, since we cannot construct an exhaustion that satisfies the desired properties.

2.4 Taut Sutured Manifolds and Tight Contact Structures

For the reader's convenience, we list some of the definitions and results in [HKM2] which we will need later. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. $R(\gamma)$ denotes $\partial M \setminus int(\gamma)$. Each component of $R(\gamma)$ is oriented. $R_+(\gamma)$ is defined to be those components of $R(\gamma)$ whose normal vectors point out of M and $R_-(\gamma)$ is defined to be $R(\gamma) \setminus R_+(\gamma)$. Each component of $A(\gamma)$ contains a *suture* which is a homologically nontrivial, oriented simple closed curve. The set of sutures is denoted $s(\gamma)$. The orientation on $R_+(\gamma)$, $R_-(\gamma)$, and $s(\gamma)$ are related as follows. If $\alpha \subset \partial M$ is an oriented arc with $\partial\alpha \subset R(\gamma)$ that intersects $s(\gamma)$ transversely in a single point and if $s(\gamma) \cdot \alpha = 1$, then α

must start in $R_+(\gamma)$ and end in $R_-(\gamma)$.

A *sutured manifold with annular sutures* is a sutured manifold (M, γ) such that ∂M is nonempty, every component of γ is an annulus, and each component of ∂M contains a suture. A sutured manifold (M, γ) with annular sutures determines an *associated convex structure* (M, Γ), where $\Gamma = s(\gamma)$. For more on this correspondence, see [HKM2].

A transversely oriented codimension-1 foliation $\mathcal{F}$ is *carried by* (M, γ) if $\mathcal{F}$ is transverse to γ and tangent to $R(\gamma)$ with the normal direction pointing outward along $R_+(\gamma)$ and inward along $R_-(\gamma)$, and $\mathcal{F}|_\gamma$ has no Reeb components. $\mathcal{F}$ is *taut* if each leaf intersects some closed curve or properly embedded arc connecting $R_-(\gamma)$ to $R_+(\gamma)$ transversely.

Let S be a compact oriented surface with components $S_1, \ldots, S_n$. Let $\chi(S_i)$ be the Euler characteristic of S_i. The *Thurston norm of* S is defined to be

$$x(S) = \sum_{\chi(S_i)<0} |\chi(S_i)|.$$

A sutured manifold (M, γ) is *taut* if

1. M is irreducible.

2. $R(\gamma)$ is Thurston norm minimizing in $H_2(M, \gamma)$; that is, if S is any other properly embedded surface with $[S] = [R(\gamma)]$, then $x(R(\gamma)) \leq x(S)$.

3. $R(\gamma)$ is incompressible in M.

The following is due to Gabai [Ga] and Thurston [Th].

Theorem 2.4.1. *A sutured manifold* (M, γ) *is taut if and only if it carries a transversely oriented, taut, codimension-*1 *foliation* $\mathcal{F}$.

We require the following result due to Honda, Kazez, and Matić [HKM2].

Theorem 2.4.2. *Let* (M, γ) *be an irreducible sutured manifold with annular sutures, and let* (M, Γ) *be the associated convex structure. The following are equivalent.*

1. (M, γ) *is taut.*

2. (M, γ) *carries a taut foliation.*

3. (M, Γ) *carries a universally tight contact strucuture.*

4. (M, Γ) *carries a tight contact structure.*

Chapter 3

The End of an Open Contact Manifold and Some Invariants

3.1 Definitions of the Invariants

Let (V, ξ) be any open contact 3-manifold that is the interior of a compact 3-manifold M such that ∂M is nonempty and contains at least one component of nonzero genus. Fix an embedding of $V \hookrightarrow int(M)$ so that we can think of V as $M \setminus \partial M$. Choose a boundary component $S \subset \partial M$ and let $\Sigma \subset M \setminus \partial M$ be an embedded surface isotopic to S in M. Note that S and Σ bound a contact manifold $(\Sigma \times (0,1), \xi)$. We call such a manifold, along with the embedding into V, a *contact end* corresponding to S and ξ. Let $Ends(V, \xi; S)$ be the collection of contact ends corresponding to S and ξ.

Let $S \subset \partial M$ be a component of nonzero genus and let $\lambda \subset S$ be a separating, simple closed curve which bounds a punctured torus T in S. Fix a basis B of the first homology of T. Let $\Sigma \subset V$ be a convex surface which is isotopic to S in M and contains a simple closed curve γ with the following properties:

1. γ is isotopic to λ on Σ, where we have identified Σ and S by an isotopy in M.

2. γ intersects Γ_Σ transversely in exactly two points.

3. γ has minimal geometric intersection number with Γ_Σ.

Call any such surface *well-behaved* with respect to S and λ. Note that there exists a simple closed curve $\mu \subset \Gamma_\Sigma$ which is contained entirely in T. Let the *slope* of Σ, written $slope(\Sigma)$, be the slope of μ measured with respect to the basis B of the first homology of T. When S is a torus, we omit all reference to the curve λ as it is unnecessary for our definition.

Let $E \in Ends(V, \xi; S)$. Let $\mathcal{C}(E)$ be the set of all well-behaved convex surfaces in the contact end E. If $\mathcal{C}(E) \neq \emptyset$, then define the *slope* of E, to be

$$slope(E) \;=\; \sup_{\Sigma \in \mathcal{C}(E)} (slope(\Sigma)).$$

Here we allow sup to take values in $\mathbb{R} \cup \{\infty\}$. Note that $Ends(V, \xi; S)$ is a directed set, directed by reverse inclusion and that the function $slope\colon\; Ends(V, \xi; S) \to \mathbb{R} \cup \infty$ is a net. If $\mathcal{C}(E)$ is nonempty for a cofinal sequence of contact ends and this net is convergent, then we call the limit the *slope at infinity* of $(V, \xi; S, \lambda, B)$ or the *slope at infinity* of (V, ξ) if S, λ, and B are understood from the context. If the slope at infinity exists, then we say that this slope is *attained* if for each $E \in Ends(V, \xi; S)$ there exists a $\Sigma \in \mathcal{C}(E)$ with that slope. Note that any slope that is attained must necessarily be rational.

Let $\Sigma \in \mathcal{C}(E)$. Define the *division number* of Σ, written $div(\Sigma)$ to be half the number of dividing curves and arcs on T. When Σ is a torus, this is the usual torus division number. If $\mathcal{C}(E) \neq \emptyset$, then let

$$div(E) = \min_{\Sigma \in \mathcal{C}(E)} (div(\Sigma)).$$

Note that $div\colon\ Ends(V, \xi, S) \to \mathbb{N} \cup \{\infty\}$ is a net, where we endow $\mathbb{N} \cup \{\infty\}$ with the discrete topology. If $\mathcal{C}(E)$ is nonempty for a cofinal sequence of contact ends, then we call the limit the *division number at infinity* of $(V, \xi; S, \lambda, B)$ or the *division number at infinity* of (V, ξ) if S, λ, and B are understood from the context. Note that the slope at infinity and the division number at infinity are proper isotopy invariants.

3.2 An Example

In this section, we compute the slope and infinity and the division number at infinity for Eliashberg's examples of uncountably many open, solid tori that are not contact diffeomorphic. We will show that the U_δ are, in fact, distinct up to proper isotopy. First, we examine $\partial\overline{U}_\delta$. Note that there are natural coordinates on T^2 given by ϕ_1 and ϕ_2. These coordinates induce coordinates on the homology of $\partial\overline{U}_\delta$. In these coordinates, the characteristic foliation of $\partial\overline{U}_\delta$ is linear and has slope $\delta^2/(\delta^2-1)$. We call a torus with linear characteristic foliation *pre-Lagrangian*. We will show that, in fact, the slope at infinity of

U_δ is $\delta^2/(\delta^2-1)$ and the division number at infinity is 1.

To compute the slope at infinity, we must have several facts at our disposal. First, every $T^2 \times I$ in S^3 with the standard tight contact structure is minimally twisting. This fact follows from the fact that $T^2 \times \{0\}$ must bound a solid torus S which does not contain the $T^2 \times \{1\}$. If $T^2 \times I$ we not minimally twisting, then we could find an overtwisted meridinal disk inside $S \cup T^2 \times I$.

The second fact is the following: Given a pre-Lagrangian torus $T \subset S^3$ with rational slope s and an open neighborhood N of T, one can perturb T to be a convex torus $T' \subset N$ such that $slope(T') = s$ and $div(T') = 1$. To prove this, first recall that the characteristic foliation on T determines the contact structure in a neighborhood of T. We construct the standard model for this neighborhood as follows: Consider $\mathbb{R}^3$ with the contact structure $\xi = \ker(dz + xdy)$. If we quotient $\mathbb{R}^3$ by $z \mapsto z+1$ and $y \mapsto y+1$, we will get $M = \mathbb{R} \times T^2$. Since the contact structure is preserved by this action, ξ will induce a contact structure on M. Clearly, there exists a diffeomorphism taking T to $\{0\} \times T^2$ that preserves the characteristic foliations. Therefore, they have contact diffeomorphic neighborhoods. Now, perturb $\{0\} \times T^2$ into $\Sigma = \{(f(z), y, z)\}$ via f, the graph of which is shown in Figure 3.1. The resulting torus Σ will be convex and have characteristic foliation identical to the torus on the left-hand side of Figure 2.5.

Given these facts, the computation of the slope at infinity is fairly straightforward. Note that, given any neighborhood N of $\partial\overline{U}_\delta$,we can find pre-Lagrangian tori inside $N \cap U_\delta$ with rational slopes that are arbitrarily close to $\delta^2/(\delta^2-1)$. We then perturb each

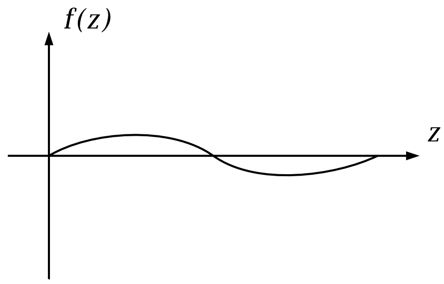


Figure 3.1: The graph of f.

of these tori to be convex. Choose a sequence T_i of such convex tori with division number 1 and slopes converging to $\delta^2/(\delta^2-1)$. To show that the slope at infinity is well-defined in this case (and therefore that our sequence actually computes the slope at infinity), assume that there is another sequence of convex tori S_i that leave every compact set of U_δ, have division number 1, and have slopes converging to some number r other than $\delta^2/(\delta^2-1)$. By the minimal twisting condition, $r < \delta^2/(\delta^2-1)$. Let $\epsilon = \delta^2/(\delta^2-1)-r$. There exists an m such $|slope(T_m) - \delta^2/(\delta^2-1)| \leq \epsilon/4$ and an n such that $|slope(S_n) - r| \leq \epsilon/4$ and S_n lies outside of the compact set bounded by T_n inside U_δ. But, the existence of T_m and S_n violates minimal twisting, since S_n lies outside of the compact set bounded by T_n inside U_δ. Hence, the S_i do not exist, so the slope at infinity is well-defined and is equal to $\delta^2/(\delta^2-1)$. Given the existence of the family of tori T_i, we see that the division number at infinity is 1.

Chapter 4

Classification Theorems for Tight Toric Ends

In this section, we study tight contact structures on toric ends. We say that a toric end is *minimally twisting* if it contains only minimally twisting toric annuli. We first show that it is possible to refer to the slope at infinity and the division number at infinity for toric ends.

Proposition 4.0.1. *Let* $T^2 \times [0, \infty)$ *be a tight toric end. Then the division number at infinity and the slope at infinity are defined.*

Proof. First note that $\mathcal{C}(E)$ is nonempty for any end E since the condition for being well-behaved is vacuously true for tori. Also, note that the division number at infinity exists by definition.

If there exists a nested sequence of ends E_i such that $slope(E_i) = \infty$, then the slope at infinity is ∞. Otherwise, there exists an end $E = T^2 \times [0, \infty)$ such that for no end $F \subset E$ is $slope(F) = \infty$. This means that E is minimally twisting. Without loss of generality, assume $T_i = T^2 \times i$ is convex with slope s_i. Note that the s_i form a clockwise sequence on the Farey graph and are contained in a half-open arc which does not contain ∞. Since $slope(F) \leq s_i$ for any end $F \subset T^2 \times [i, \infty)$, our net is convergent, so the slope at infinity is defined. □

4.1 Tight, minimally twisting toric ends with irrational slope at infinity

In this section, we study tight, minimally twisting toric ends $(T^2 \times [0, \infty), \xi)$ with *irrational* slope r at infinity and with convex boundary satisfying $div(T^2 \times 0) = 1$ and $slope(T^2 \times 0) = -1$. Unless otherwise specified, all toric ends will be of this type.

We first show how to associate to any such toric end a function $f_\xi \colon \mathbb{N} \to \mathbb{N} \cup \{0\}$. There exists a sequence of rational numbers q_i on the Farey graph which satisfies the following:

1. $q_1 = -1$ and the q_i proceed in a clockwise fashion on the Farey graph.
2. q_i is connected to q_{i+1} by an arc of the graph.
3. The q_i converge to r.
4. The sequence is minimal in the sense that q_i and q_j are not joined by an arc of the

graph unless j is adjacent to i.

We can form this sequence inductively by taking q_2 to be the rational number which is closest to r on the clockwise arc of the Farey graph $[-1, r]$ between -1 and r and has an edge of the graph from -1 to q_2. Similarly, construct the remaining q_i. Any such sequence can be grouped into *continued fraction blocks*. We say that $q_i, \ldots, q_j$ form a *continued fraction block* if there is an element of $SL_2(\mathbb{Z})$ taking the sequence to $-1, \ldots, -m$. We call m the *length* of the continued fraction block. We say that this block is *maximal* if it cannot be extended to a longer continued fraction block in the sequence q_i. Since r is irrational, maximal continued fraction blocks exist. Denote these blocks by B_i. To apply this to our situation, we need the following.

Proposition 4.1.1. *There exists a nested sequence of convex tori T_i with $div(T_i) = 1$ such that $slope(T_i) = q_i$. Moreover, any such sequence must leave every compact set.*

Proof. By the definition of slope at infinity, for any ϵ, there is an end E such that $slope(E)$ is within ϵ of r. This means that there is a convex torus T in E with slope lying within 2ϵ of r. Note that since our toric end is minimally twisting and has slope r at infinity, $slope(T) \in [-1, r)$. We attach bypasses to T so that $div(T) = 1$. The toric annulus bounded by $T^2 \times 0$ and T contains the tori T_i with q_i lying couterclockwise to $slope(T)$. Fix these first T_i. Choose another torus T' outside of the toric annulus with slope even closer to r. Again, adjust the division number of T' so that it is 1 and factor the toric annulus bounded by T and T' to find another finite number of our T_i. Proceeding in this fashion, we see we have the desired sequence of T_i. Any such sequence must leave every compact set by the definition of the slope at infinity. For, if not, then we could find a torus T in any end with $slope(T) > r$, which would show that the slope at infinity is not r. $\square$

This factors the toric end according to our sequence of rationals. We say that a consecutive sequence of T_i form a continued fraction block if the corresponding sequence of rationals do. Each maximal continued fraction block B_i determines a maximal continued fraction block of tori which we also call B_i. We think of B_i as a toric annulus.

To each continued fraction block, we let n_j be the number of positive basic slices in the factorization of B_i by T_j. Define $f_\xi\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ by $f_\xi(j) = n_j$. To show that the function f_ξ is independent of the factorization by T_i, suppose T_i' is another factorization with the same properties as T_i. Let B_j' denote the corresponding continued fraction blocks. Fix j. There exists n large such that the toric annulus A bounded by T_n and T_1 contains the continued fraction blocks B_j and B_j'. Extend the partial factorization of A by B_j'. Recall that one can compute the relative Euler class via such a factorization and that it depends on the number of positive basic slices in each continued fraction block [Ho2]. Therefore, B_j and B_j' must have the same number of positive basic slices.

Given an irrational number r, let $\mathcal{F}(r)$ denote the collection of functions $f\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ such that $f(i)$ does not exceed one less than the length of B_i. We can now state a complete classification of the toric ends under consideration.

Theorem 4.1.2. *Let $(T^2 \times [0, \infty), \xi)$ be a tight, minimally twisting toric end with convex boundary satisfying $div(T^2 \times 0) = 1$ and $slope(T^2 \times 0) = -1$. Suppose that the slope at infinity is irrational. To each such tight contact structure, we can assign a function*

$f_\xi\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ *which is a complete proper isotopy (relative to the boundary) invariant. Moreover, given any* $f \in \mathcal{F}(r)$*, there exists a toric end* $(T^2 \times [0,\infty), \xi)$ *such that* $f_\xi = f$*.*

Proof. If $f_\xi = f_{\xi'}$, then we can shuffle bypasses within any given continued fraction block so that all positive basic slices occur at the beginning of the block. Since the number of positive basic slices in any continued fraction block is the same, it is clear that they are properly isotopic.

It is a straightforward application of the gluing theorem for basic slices in [Ho2] to show that we can construct a toric annulus corresponding to the desired continued fraction blocks. The fact that they stay tight under gluing follows from the fact that overtwisted disks are compact. ☐

Corollary 4.1.3. *Let* $(T^2 \times [0,1), \xi)$ *be a tight, minimally twisting toric end with irrational slope* r *at infinity. Suppose* $f_\xi(i)$ *is not maximal or minimal for an infinite number of numbers* i*. Then there does not exist any tight, toric end* $(T^2 \times [0,\infty), \eta)$ *such that* $\xi|_{T^2\times[0,1)} = \eta|_{T^2\times[0,1)}$*.*

Proof. Assume that there were an inclusion $\phi\colon (T^2 \times [0,1), \xi) \to (T^2 \times [0,\infty), \eta)$. Perturb $T^2 \times \{2\}$ to be convex of slope b. Choose a convex torus $\phi(T')$ of slope a. As before, we have a minimal, clockwise sequence of rationals q_j for $1 \le j \le n$ on the Farey graph such that $q_1 = a$, $q_n = b$, and q_i is joined to q_{i+1} by an arc of the graph. Let q_m be the rational closest to q_1 such that r lies clockwise to q_1 and counterclockwise to q_m. By our assumption on f_ξ, there exists a continued fraction block of tori $T_{j_1}, \dots, T_{j_k} \subset (T^2 \times [0,1), \xi)$ which contains both positive and negative basic slices.

Moreover, we can assume that the corresponding sequence of rationals lies clockwise to q_{m-1} and counterclockwise to q_m. Perturb tori T_{in} and T_{out} in $(T^2 \times [0, \infty), \eta)$ to be convex of slopes q_{m-1} and q_m, respectively, such that the basic slice bounded by T_{in} and T_{out} contains $\phi(T_{j_1}), \ldots, \phi(T_{j_k})$. This is a contradiction, since a basic slice cannot be formed by gluing basic slices of opposite signs unless the contact structure η is overtwisted [Ho2]. □

4.2 Tight, minimally twisting toric ends with rational slope at infinity

We now consider tight, minimally twisting toric ends $(T^2 \times [0, \infty), \xi)$ with *rational* slope r at infinity and with convex boundary satisfying $div(T^2 \times 0) = 1$ and $slope(T^2 \times 0) = -1$. Unless otherwise specified, all toric ends will be of this type. We first deal with the situation when the slope at infinity is *not* attained.

We show how to every toric end under consideration we can assign a function

$$f_\xi \colon \{1, \ldots, n(r)\} \times \{1, -1\} \to \mathbb{N} \cup \{0, \infty\}.$$

We proceed in a fashion similar to the irrational case. Given r rational, there exists a sequence of rationals q_i satisfying the following:

1. $q_1 = -1$ and the q_i proceed in a clockwise fashion on the Farey graph.

2. q_i is connected to q_{i+1} by an arc of the tesselation.

3. The q_i converge to r, but $q_i \neq r$ for any i.

4. The sequence is minimal in the sense that q_i and q_j are not joined by an arc of the tesselation unless j is adjacent to i.

We construct such a sequence inductively just as in the irrational case, except we never allow the rationals q_i to reach r. Note that such a sequence breaks up naturally into $n-1$ finite continued fraction blocks B_i and one infinite continued fraction block B_n (*i.e.*, B_n can be taken to the negative integers after action by $SL_2(\mathbb{Z})$). Note that n is completely determined by r. Just as in the irrational case, there exist nested covex tori T_i with $div(T_i) = 1$ and $slope(T_i) = q_i$. We can argue as in the irrational case to show that these tori must leave every compact set of the toric end. We will also refer to the collection of tori T_i corresponding to B_i by the same name.

We will now construct f_ξ. Let $f_\xi(i, \pm 1)$ be the number of positive (negative) basic slices in the continued fraction block B_i. Of course, for a finite continued fraction block, $f_\xi(i, 1)$ determines $f_\xi(i, -1)$. However, this is clearly not the case for B_n.

As in the irrational case, let $\mathcal{F}(r)$ be the collection of functions $f\colon\ \{1, \ldots, n(r)\} \times \{1, -1\} \to \mathbb{N} \cup \{0, \infty\}$ such that $f_\xi(i, 1) + f_\xi(i, -1) = |B_i| - 1$ for $i \leq n-1$, where $|B_i|$ is the length of B_i, and at least one of $f_\xi(n(r), \pm 1)$ is infinite.

Theorem 4.2.1. *Let $(T^2 \times [0, \infty), \xi)$ be a tight, minimally twisting toric end with convex boundary satisfying $div(T^2 \times 0) = 1$ and $slope(T^2 \times 0) = -1$. Suppose that the slope at inifinity is rational and is not attained. To each such tight contact structure, we can assign a function $f_\xi : \{1, \ldots, n(r)\} \times \{1, -1\} \to \mathbb{N} \cup \{0, \infty\}$ which is a complete proper* *isotopy (relative to the boundary) invariant. Moreover, for any $f \in \mathcal{F}(r)$, there exists a tight, minimally twisting toric end $(T^2 \times [0, \infty), \xi)$ with slope r at infinity which is not realized such that $f = f_\xi$.*

Proof. Suppose $f_\xi = f'_\xi$. As in the irrational case, we can adjust our factorization of the finite continued fraction blocks so that all of the positive basic slices occur first in each continued fraction block. Therefore, we can isotope the two contact structures so that they agree on the first $n - 1$ continued fraction blocks.

We now consider the infinite basic slice. Without loss of generality, we may assume that the infinite basic slices for ξ and ξ' are toric ends $(T^2 \times [0, \infty), \xi)$ and $(T^2 \times [0, \infty), \xi')$ with $slope(T^2 \times \{0\})$, $div(T^2 \times \{0\})$, and infinite slope at infinity that is not realized. The corresponding factorization is then given by nested tori T_i and T'_i such that $slope(T_i) = slope(T'_i) = -i$ and $div(T_i) = 1$. We now construct model toric ends $\xi_n^\pm$ and ξ_{alt} and show that any infinite basic slice is properly isotopic to one of the models. Let $B_i^\pm$ be the positive (negative) basic slice with $slope(T^2 \times 0) = -i$ and $slope(T^2 \times 1) = -i - 1$. Let $\xi_n^\pm$ be the toric end constructed as $B_1^\pm \cup \cdots \cup B_n^\pm \cup B_{n+1}^\mp \cup \cdots$. Let ξ_{alt} be $B_1^+ \cup B_2^- \cup B_3^+ \cup \cdots$. First consider the case when $f_\xi(n, 1) = m$. There exists N large so that the toric annulus bounded by T_1 and T_N contains *at least* m positive basic slices and m negative basic slices. By shuffling bypasses in this toric annulus, we can rechoose our factorization so that all positive bypass layers occur first in our factorization. This toric end is clearly properly isotopic to ξ_m^+. We handle the case when $f_\xi(n, -1) = m$ similarly. Now, suppose that $f_\xi(n, \pm 1) = \infty$. Fix some number k. Choose N_1 large enough that the toric annulus bounded by T_1 and T_{N_1} contains at least k positive and k negative basic slices. By shuffling bypasses in this toric annulus, we can arrange for the first $2k$ basic slices in the factorization to be alternating. There exists an isotopy ϕ_t^1 such that ϕ_0^1 is the identity and $\phi_{1*}^1(\xi)$ agrees with ξ_{alt} in the first $2k$ basic slices. Call the pushed forward contact structure by the same name. There exists N_2 large such that T_{2k} and T_{N_2} bound a toric annulus with k positive and k negative basic slices. Leaving the first $2k$ tori in our factorization fixed, we can shuffle bypases in the toric annulus bounded by T_{2k} and T_{N_2} so that signs are alternating. Choose an isotopy ϕ_t^2 as before such that ϕ_t^2 is the identity on the toric annulus bounded by T_1 and T_{2k} and takes the second $2k$ basic slices of ξ onto those of ξ_{alt}. Continuing in this fashion, we can construct ϕ_t^n which is supported on K_n compact such that $K_i \subset K_{i+1}$ and $T^2 \times [0,\infty) = \cup K_i$. Hence we have an isotopy taking ξ to ξ_{alt}. The existence result follows immediately from Honda's gluing results for toric annuli [Ho2]. □

Corollary 4.2.2. *Let $(T^2 \times [0,1), \xi)$ be a tight, minimally twisting toric end that does not attain a rational slope r at infinity. Suppose $f_\xi(n(r) \times \{1\})$ and $f_\xi(n(r) \times \{-1\})$ are nonzero. Then there does not exist any tight, toric end $(T^2 \times [0,\infty), \eta)$ such that $\xi|_{T^2\times[0,1)} = \eta|_{T^2\times[0,1)}$.*

Proof. Assume that there were such an inclusion $\phi\colon (T^2 \times [0,1), \xi) \to (T^2 \times [0,\infty), \eta)$. Let T_i be the first torus in the factorization of the infinite continued fraction block of $(T^2 \times [0,1), \xi)$. By definition, there exists another torus T_j with $j > i$ such that T_i and T_j bound basic slices of both signs. By the definition of the slope at infinity and the precompactness condition, there exists a convex torus T outside of the toric annulus bounded by $\phi(T_i)$ and $\phi(T_j)$ which has slope r. Note that $\phi(T_i)$ and T bound a continued fraction block which is formed by gluing basic slices of opposite signs. This implies that $(T^2 \times [0, \infty), \eta)$ is overtwisted [Ho2]. □

Corollary 4.1.3 and Corollary 4.2.2 will be essential to proving Theorem 1.0.1. We now consider tight, minimally twisting toric ends that realize the slope at infinity and have finite division number at infinity.

Theorem 4.2.3. *Tight, minimally twisting toric ends with finite division number d at infinity that realize the slope r at infinity are in one-to-one correspondence with tight, minimally twisting contact structures on $T^2 \times [0, 1]$ with $T^2 \times i$ convex, $slope(T^2 \times 0) = -1$, $slope(T^2 \times 1) = r$, $div(T^2 \times 0) = 1$, and $div(T^2 \times 1) = d$ up to isotopy relative to $T^2 \times 0$.*

Proof. Let $(T^2 \times [0, \infty), \xi)$ be such a toric end. By the definition of division number at infinity and slope at infinity, there exists a convex torus T with the following properties:

1. $div(T) = d$

2. $slope(T) = r$

3. Any other convex torus T' lying in the noncompact component of $T^2 \times [0, \infty) \setminus T$ satisfies $div(T') \geq d$.

Any such torus will necessarily have slope r. Let A be the toric annulus bounded by $T^2 \times 0$ and T. We know that any other torus T' with the same properties as T bounds a toric annulus A' that is topologically isotopic to A. By the definition of T and T' there exists a torus T'' outside of A and A' that has the same properties as T. Since ξ is minimally twisting, T' and T'' bound a vertically invariant toric annulus. Similarly, T and T'' bound a vertically invariant toric annulus. We can use these toric annuli to isotope A and A' to the same toric annulus in our toric end. This yields the desired correspondence. Given a tight, minimally twisting contact structures on $T^2 \times [0,1]$ with $T^2 \times i$ convex, $slope(T^2 \times 0) = -1$, $slope(T^2 \times 1) = r$, $div(T^2 \times 0) = 1$, and $div(T^2 \times 1) = d$, we obtain a toric end by removing $T^2 \times 1$. ☐

We say that two convex annuli $A_i = S^1 \times [0,1]$ with Legendrian boundary, $tb(S^1 \times 0) = -1$ and $tb(S^1 \times 1) = -m$ are *stabily disk equivalent* if there exist disk equivalent convex annuli $A_i' = S^1 \times [0,2]$ such that $tb(S^1 \times 1) = -1$, $tb(S^1 \times 2) = -n < -m$, and $A_i = S^1 \times [0,1] \subset A_i'$.

Theorem 4.2.4. *Let $(T^2 \times [0,\infty), \xi)$ be a tight, minimally twisting toric end with $slope(T^2 \times 0) = \infty$, slope ∞ at infinity, and division number ∞ at infinity. Then we can associate to ξ a collection of nested families of convex annuli $A_i = S^1 \times [0,i]$ with Legendrian boundary such that $tb(S^1 \times 0) = -1$, $tb(S^1 \times i+1) = tb(S^1 \times i) + 1$ such that any two annuli A_i and A_i' in different families are stabily disk equivalent.*

Proof. To construct such annuli, simply choose a factorization of the toric end by tori T_i such that $T_1 = T^2 \times 0$, $slope(T_i) = \infty$, $div(T_{i+1}) = div(T_i) + 1$ and the T_i leave every compact set. Let A_1 be the convex annulus with boundary on T_1 and T_2. Choose A_1' a horizontal convex annulus between T_2 and T_3 which shares a boundary component with

A_1. Let $A_2 = A_1 \cup A_1'$. Continuing in this fashion, we construct a sequence of nested annuli A_i. Now, choose any other factorization by tori T_i' satisfying the same properties as the T_i and let A_i' be the corresponding sequence of convex annuli. We will show that A_i is stabily disk equivalent to A_i'. Choose N large so that the toric annulus bounded by T_1 and T_N contains A_i and A_i'. Let A be a convex annulus between the $S^1 \times i \subset A_i'$ and a horizontal Legendrian curve on T_N. Let $A' = A_i' \cup A$. Honda's result in [Ho2] implies that A and A' are disk equivalent. $\square$

Corollary 4.2.5. *Any tight, minimally twisting toric end* $(T^2 \times [0,\infty), \xi)$ *with* $slope(T^2 \times 0) = \infty$*, slope* ∞ *at infinity, and division number* ∞ *at infinity embeds in a vertically invariant neighborhood of* $T^2 \times 0$*.*

Proof. Honda's model [Ho2] for increasing the torus division number can be applied inductively on a vertically invariant neighborhood of $T^2 \times 0$ to create the desired sequence of nested tori T_i and corresponding annuli A_i. The contact structure on the toric annulus bounded by T_1 and T_i is uniquely determined by A_i [Ho2]. $\square$

We are lead to the following question:

Question 4.2.6. *What are necessary and sufficient conditions for two toric ends with infinite division number at infinity to be properly isotopic?*

4.3 Nonminimally twisting, tight toric ends

In this section, we deal with tight toric ends $(T^2 \times [0,\infty), \xi)$ with $slope(T^2 \times 0) = 0$, $div(T^2 \times 0) = 1$, and are not minimally twisting. We first recall Honda's classification for nonminimally twisting tight contact structures on $T^2 \times [0,1]$ in [Ho2]. He constructs a family $\xi_n^{\pm}$ of tight, rotative contact structures on $T^2 \times [0,1]$ with $slope(T^2 \times i) = 0$ and $div(T^2 \times i) = 1$ and shows that this is a complete and nonoverlapping list of contact structures satisfying these conditions. We define the *rotativity* of a tight toric end ξ with $slope(T^2 \times 0) = 0$ and $div(T^2 \times 0) = 1$ to be the maximum n such that there is an embedding $e\colon (T^2 \times [0,1], \xi_n^{\pm}) \hookrightarrow (T^2 \times [0,\infty), \xi)$ with $e(T^2 \times 0) = T^2 \times 0$. If no maximum exists, then we say that ξ has *infinite rotativity*. If n is the rotativity of ξ, then ξ_n^+ and ξ_n^- cannot both be embedded in ξ. Assume for contradiction that there are two such embeddings e_+ and e_-, respectively. Then the images of these two embeddings are contained in a common toric annulus $T^2 \times [0,1]$, where $div(T^2 \times \{1\}) \geq 1$. If $div(T^2 \times \{0\}) = 1$, then it follows from the factorization theorems concerning such toric annuli in [Ho2] that these embeddings cannot coexist. If $div(T^2 \times \{0\}) > 1$, then the closure of the exterior of the image of e_+ inside $T^2 \times [0,1]$ is a nonrotative outer layer (similarly for e_-). By Honda's work in [Ho2], the images of e_+ and e_- must therefore have the same sign. But, this contradicts the very existence of the two different embeddings e_+ and e_-. Hence, we can refer to the *sign of rotativity* as well. We construct two more nonminimally twisting toric ends $\xi_\infty^{\pm}$. Set $(T^2 \times [0,\infty), \xi_\infty^{\pm}) = \cup_{i=1}^{\infty}(T^2 \times [0,1], \xi_2^{\pm})$.

Theorem 4.3.1. *Let $(T^2 \times [0,\infty),\xi)$ be a tight toric end that is not minimally twisting and such that $slope(T^2 \times \{0\}) = 0$ and $div(T^2 \times \{0\}) = 1$.*

1. *Assume that ξ has finite rotativity and that the slope at infinity is s and is not attained. Then ξ is uniquely determined by n and the sign of rotativity. Moreover, ξ is universally tight.*

2. *Assume that ξ has finite rotativity, the slope at infinity is s and is attained, and the division number at infinity is $k < \infty$. Such ξ are in one-to-one correspondence with tight, toric annuli $T^2 \times [0,1]$ with $slope(T^2 \times \{0\}) = 0$, $slope(T^2 \times \{1\}) = s$, $div(T^2 \times \{0\}) = 1$, and $div(T^2 \times \{1\}) = k$, up to isotopy relative to $T^2 \times \{0\}$. Moreover, all such ξ are universally tight.*

3. *Assume that ξ has slope s at infinity and infinite division number at infinity (the rotativity must necessarily be finite). We can factor ξ into a toric annulus $T^2 \times [0,1]$ with $slope(T^2 \times \{1\}) = s$ and $div(T^2 \times \{1\}) = 1$ and a minimally twisting, toric end $T^2 \times [1,\infty)$. Moreover, the contact structure on the toric annulus is uniquely determined by ξ, and ξ is universally tight. To the toric end $T^2 \times [1,\infty)$, we can assign a family of annuli A_i as in Theorem 4.2.4 that is unique up to stable disk equivalence.*

4. *Assume that ξ has infinite rotativity. Then ξ is properly isotopic relative to the boundary to either ξ^+_∞ or ξ^-_∞, so the sign of rotativity is defined in the infinite case as well. Moreover, the $\xi^\pm_\infty$ are universally tight.*

Proof. First, consider the case of finite rotativity when the slope at infinity is not attained. Assume that the sign of rotativity is $+$. Factor off a toric annulus ξ_n^+. What remains is a minimally twisting, toric end. Based on previous classification results for these ends, it suffices to determine the number of positive basic slices in each continued fraction block. Note that the sign of the basic slices in the continued fraction blocks is determined by the sign of rotativity, just as in [Ho2]. Since all basic slices have the same sign, ξ is universally tight just as in [Ho2]. The proof of the case when the slope at infinity is attained and the division number at infinity is finite is essentially identical to the analogous case when the toric end is minimally twisting.

The case of infinite division number at infinity is similar to previous cases. We first show that the toric annulus in the factorization is unique. Choose two such factorizations by tori T and T'. These tori are contained in a larger toric annulus $T^2 \times [0,2]$. T and $T^2 \times \{2\}$ and T' and $T^2 \times \{2\}$ bound nonrotative outer layers. By [Ho2], we know that the toric annuli bounded by T and $T^2 \times \{0\}$ and T' and $T^2 \times \{0\}$ must therefore be the same. The fact that ξ is universally tight is virtually identical to the previous cases. The statement concerning the minimally twisting toric annulus follows from the proof of Theorem 4.2.4.

Now, assume ξ has infinite rotativity. First, note that we cannot have two embeddings $e_n^{\pm}\colon (T^2 \times [0,1], \xi_n^{\pm}) \hookrightarrow (T^2 \times [0,\infty), \xi)$ with $e_n^{\pm}(T^2 \times 0) = T^2 \times 0$ as previously discussed. Since ξ has infinite rotativity, there exists a sequence of, say, positive embeddings $e_n\colon (T^2 \times [0,n], \xi_n^+) \hookrightarrow (T^2 \times [0,\infty), \xi)$ with $e_n(T^2 \times 0) = T^2 \times 0$. Moreover, we can take this sequence of embeddings to be nested in the sense that $e_n = e_{n+1}$ on $[0, n]$. This follows immediately by factoring a toric annulus containing the images of e_n and e_{n+1}. Note that any sequence of such embeddings must necessarily leave any compact set. We can use this sequence of embeddings to construct a proper isotopy of ξ with ξ_∞^+ as in the proof of Theorem 4.2.1. Again, the fact that $\xi_\infty^\pm$ are universally tight follows from the fact that nonminimally twisting toric annuli are universally tight. ☐

Corollary 4.3.2. *Let $(T^2 \times [0, \infty), \xi)$ be a tight toric end that is not minimally twisting and has finite rotativity. Then $(T^2 \times [0, \infty), \xi)$ embeds into a toric annulus $(T^2 \times [0, 1], \eta)$ with convex boundary.*

Proof. In the case when the slope at infinity is not attained, the obstruction to finding an embedding, the mixing of signs of basic slices in continued fraction blocks, is not present. Therefore, such embeddings exist and are straightforward to construct using the techniques in [Ho2]. In the case when the slope is attained and the division number is finite, the embedding comes for free. When the division number at infinity is infinite, one must use the folding trick in a vertically invariant neighborhood of a convex torus described in [Ho2] and already used in our discussion of minimally twisting toric ends with infinite division number at infinity. ☐

4.4 Classifying Tight Contact Structures on $S^1 \times \mathbb{R}^2$ and $T^2 \times \mathbb{R}$

We now show that in many cases the classification of tight contact structures on $S^1 \times \mathbb{R}^2$ and $T^2 \times \mathbb{R}$ reduces to the classification of toric ends.

4.4.1 Factoring tight contact structures on $S^1 \times \mathbb{R}^2$

Let $(S^1 \times \mathbb{R}^2, \xi)$ be a tight contact structure and let r be the slope at infinity. Consider the collection of points on the Farey graph of the form $1/n$ where $n \in \mathbb{Z}$. Let $s(r) = 1/n$ be the point closest to r (when traversing the Farey graph counterclockwise from r) that is realized as the slope of a convex torus T topologically isotopic to $S^1 \times S^1$. We can then factor $(S^1 \times \mathbb{R}^2, \xi)$ into $(S^1 \times D^2, \xi)$ and $(T^2 \times [0, \infty), \xi)$. To see that this factorization is unique, consider any other torus T' satisfying the same conditions as T. Both T and T' lie in a common solid torus S with convex boundary. Note that the toric annuli bounded by ∂S and T and by ∂S and T' are identical by the uniqueness of such factorizations on solid tori. This proves the following:

Theorem 4.4.1. *Tight contact structures on* $(S^1 \times \mathbb{R}^2, \xi)$ *with nonzero slope at infinity are in one-to-one correspondence with isotopy classes relative to the boundary of tight, minimally twisting toric ends* $(T^2 \times [0, \infty), \eta)$ *with* $div(T^2 \times \{0\}) = 1$*. Tight contact structures on* $(S^1 \times \mathbb{R}^2, \xi)$ *with slope zero at infinity are in one-to-one correspondence with isotopy classes relative to the boundary of tight, minimally twisting toric ends* $(T^2 \times$

$[0, \infty), \eta)$ *which do not attain the slope at infinity.*

4.4.2 Factoring tight contact structures on $T^2 \times \mathbb{R}$

In this section, we deal with tight contact structures on $T^2 \times \mathbb{R}$. Any convex, incompressible torus $T \subset T^2 \times \mathbb{R}$ produces a factorization of $T^2 \times \mathbb{R}$ into $T^2 \times (-\infty, 0]$ and $T^2 \times [0, \infty)$. We identify $T^2 \times (-\infty, 0]$ with $T^2 \times [0, \infty)$ via reflection about the origin in $\mathbb{R}$ to obtain a negative contact structure on $T^2 \times [0, \infty)$. We change this to a positive contact structure by reflecting across the $(1, 0)$ curve in T^2. Let $(T^2 \times [0, \infty), \xi_\pm)$ and $(T^2 \times [0, \infty), \xi'_\pm)$ be two factorizations corresponding to two different convex tori T and T' with division number 1 and slope s. We see that by keeping track of the I-twisting of a toric annulus in $T^2 \times \mathbb{R}$ containing T and T', we can obtain $(T^2 \times [0, \infty), \xi_\pm)$ from $(T^2 \times [0, \infty), \xi'_\pm)$ as follows: Remove a (possibly) rotative $T^2 \times [0, 1]$ with $div(T^2 \times i) = 1$ and $slope(T^2 \times i) = s$ from the boundary of $(T^2 \times [0, \infty), \xi_+)$ (or $(T^2 \times [0, \infty), \xi_-)$). Apply a suitable diffeomorphism to $T^2 \times [0, 1]$. Then, glue $T^2 \times [0, 1]$ to the boundary of $(T^2 \times [0, \infty), \xi_-)$ (or $(T^2 \times [0, \infty), \xi_+)$). We call this procedure *shifting the rotativity* between $(T^2 \times [0, \infty), \xi_+)$ and $(T^2 \times [0, \infty), \xi_-)$.

Theorem 4.4.2. *Let $(T^2 \times \mathbb{R}, \xi)$ be a tight contact manifold which contains a convex, incompressible torus T with $div(T) = 1$ and $slope(T) = s$. Then the factorization of $(T^2 \times \mathbb{R}, \xi)$ into toric ends $(T^2 \times [0, \infty), \xi_\pm)$ is unique up to shifting the rotativity between the two toric ends.*

Theorem 4.4.2 shows that the classification of contact structures on $T^2 \times \mathbb{R}$ reduces to the study of toric ends if there is a convex, incompressible torus T with $div(T) = 1$. If $(T^2 \times \mathbb{R}, \xi)$ contains no such torus, then the situation is much more subtle.

Question 4.4.3. *If $(T^2 \times \mathbb{R}, \xi)$ contains no convex, incompressible torus with division number 1, then what is the relationship between two factorizations by convex, incompressible tori of minimal torus division number?*

Our previous discussion of $T^2 \times [0, \infty)$, $T^2 \times \mathbb{R}$, and $S^1 \times \mathbb{R}^2$ proves Theorem 1.0.1.

Chapter 5

Proof of Theorem 1.0.2 and Theorem 1.0.3

Before beginning the proof of Theorem 1.0.2, we prove a result which allows us to choose the dividing set on ∂M nicely. Let Σ be a genus n surface. In Figure 5.1, we specify α_i, β_i, and λ_j for a genus 3 surface. For a higher genus Σ, make the analogous specification.

Lemma 5.0.4. *Let M be any 3-manifold with connected boundary of genus n. Let K be the kernel of the map $H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ induced from inclusion. There exists an identification of ∂M with Σ such that the α_i form a basis for $K \subset H_1(\partial M;\mathbb{Q})$ as vector space over $\mathbb{Q}$. Moreover, there exist integers n_i and embedded, orientable surfaces Σ_i such that $\partial\Sigma_i$ consists of n_i parallel copies of α_i.*

Proof. Let S_1 be the first cutting surface in a Haken decomposition for M. We may assume that no collection of components of ∂S_1 is separating in ∂M and that S_1 is ori-

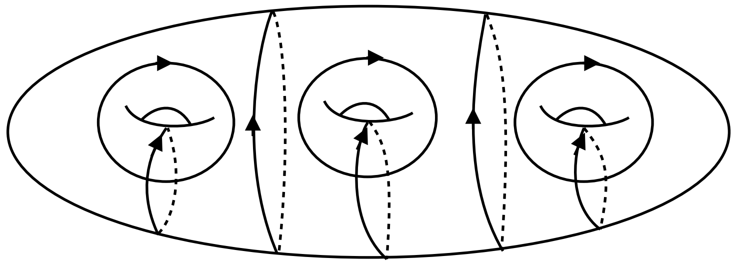


Figure 5.1: For $1 \leq i \leq 3$, let the α_i be the half-hidden, nonseparating, simple, closed curves and let β_i be the nonseparating curves such that $\alpha_i \cdot \beta_i = 1$ (with subscript increasing from left to right). Let the λ_j be the two separating curves again labeled left to right.

entable [He]. We may also assume that ∂S_1 consists of parallel copies of a nonseparating, simple closed curve that we identify with α_1. If ∂S_1 is not all parallel, then two boundary components b_1 and b_2 can be chosen so that there exists an arc μ joining the b_i that does not intersect any other components of ∂S_1. Let A be a small annular neighborhood of μ. Since ∂S_1 is nonseparating, we can choose μ so that $S_1 \cup A$ is an oriented surface with the b_i replaced by a new boundary component homologous to $b_1 + b_2$. We can continue this process until the boundary components of S_1 consist of n_1 copies of simple closed curve which we identify with α_1. Form a new 3-manifold M_1 by attaching a 2-handle H_1 to ∂M along α_1. Let S_2 be the first surface in a Haken decomposition for M_1. We may assume that ∂S_2 consists of m_2 copies of a nonseparating, simple closed curve $\gamma \subset \partial M_1$ which do not intersect the two disks $\partial H_1 \cap \partial M_1$. Since $\partial S_2 \subset M$, we can identify γ with α_2. Note that S_2 may intersect H_1. If we cannot isotop the interior of S_2 to be disjoint from H_1, then we may assume that the intersection consists of k disjoint disks

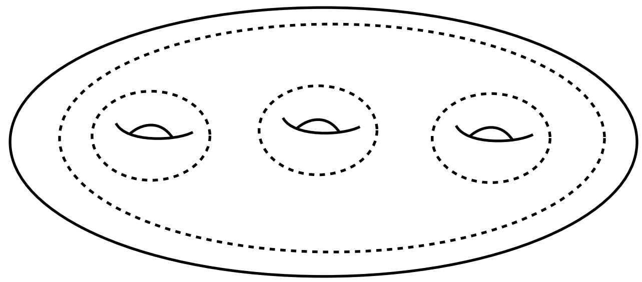


Figure 5.2: The collection of curves Γ is diffeomorphic to the collection of curves shown above.

D_i on S_2. Moreover, we can assume that the disks all have the same sign of intersection with the cocore of H_1. For, if two disks had different signs of intersection, then we could find two adjacent such disks, remove the disks, and identify the boundaries to reduce the intersection of S_2 with H_1. Note that ∂S_1 consists of n_1 copies of the attaching curve for H_1. Therefore, we can take k copies of S_1 and n_1 copies of S_2, remove the kn_1 disks $kS_2 \cap H_1$ from $n_1 S_2$, and use the kn_1 boundary components of kS_1 to cap off these boundary components, possibly reversing the orientation of S_1 if necessary. This operation shows that the class $n_2\alpha_2 \in K$, where $n_2 = n_1 m_2$. Attach another handle H_2 to M_1 along α_2 to form a new manifold M_2. Continuing in this fashion, we find n integers n_i and an identification of ∂M with Σ such that $n_i\alpha_i \in K$. The α_i are clearly linearly independent and thus generate K since $dim_{\mathbb{Q}}(K) = n$ [He]. ☐

Given any 3-manifold with connected boundary, we identify ∂M with the genus n surface Σ as specified in Lemma 5.0.5. We now describe the collection of curves $\Gamma \subset \partial M$ which will be the dividing set of a universally tight contact structure on M. Let γ_1 be a simple closed curve homologous to $\alpha_1 - 2\beta_1$ and let γ_i be a simple closed curve homologous to $\alpha_i - \beta_i$ for $2 \leq i \leq n$. Finally, let γ_{n+1} be a simple closed curve homologous to $-(\gamma_1 + \cdots + \gamma_n)$. Note that this collection of curves is diffeomorphic to the collection of curves shown in Figure 5.2.

Lemma 5.0.5. *Let M be any irreducible 3-manifold with connected boundary of nonzero genus. Then there exists a universally tight contact structure on M such that ∂M is convex and Γ divides ∂M.*

Proof. Let (M, γ) be the sutured 3-manifold with annular sutures $s(\gamma) = \Gamma$. We will show that (M, γ) is a taut sutured 3-manifold. We then invoke the result in [HKM2] which says that M also supports a universally tight contact structure with ∂M convex and $\Gamma_{\partial M} = \Gamma$.

To prove that (M, γ) is taut, it suffices to show that M is irreducible, $R(\gamma)$ is Thurston norm-minimizing in $H_2(M, \gamma)$ among all other orientable surfaces in the same relative homology class, and $R(\gamma)$ is incompressible in M. By assumption, M is irreducible. We now show $R(\gamma)$ is incompressible. Suppose not. Then the Loop Theorem [He] says that there exists an embedded disk $(D, \partial D) \subset (M, \partial M)$ such that ∂D is homotopically nontrivial in $R(\gamma)$. Since $R(\gamma)$ consists of two planar surfaces and ∂D is embedded, ∂D must also be homologous to $\pm(\gamma_{i_1} + \cdots + \gamma_{i_j})$ where $1 \leq i_1, i_j \leq n$ are distinct. There exist $q_i \in \mathbb{Q}$ such that $\pm(\gamma_{i_1} + \cdots + \gamma_{i_j}) = q_1\alpha_1 + \cdots + q_n\alpha_n$ since ∂D is nulhomologous in M. Take the intersection pairing of each side with α_i to arrive at a contradiction.

We show that $R(\gamma)$ is Thurston norm-minimizing in $H_2(M, \gamma)$. Let $S = \cup S_i$ be any orientable surface homologous to $R(\gamma)$ in $H_2(M, \gamma)$. Without loss of generality, we assume that $\partial S \subset int(A(\gamma))$. Fix an annulus $A(s) \subset A(\gamma)$ about the suture s (s is a homologically nontrivial simple closed curve in $A(s)$). Note that $\partial R(\gamma)$ intersects $A(s)$ in two oriented circles isotopic to s, where one comes from $R_+(\gamma)$ and the other comes from $R_-(\gamma)$. These circles must have the same orientation since the orientation of $R_+(\gamma)$ agrees with the orientation on ∂M and the orientation on $R_-(\gamma)$ does not. Consider the intersection of S with $A(s)$. If any two curves of $\partial S \cap A(s)$ have opposite orientation induced from S, then we can find two such curves which are adjacent. We then identify these curves and isotop them off of ∂M to reduce the number of boundary components of S. We continue this procedure until $\partial S \cap A(s)$ consists of two curves with the same orientation, which agrees with the orientation of $\partial A(s)$ induced from $R(\gamma)$. Note that the orientation on and number of these remaining curves in $\partial S \cap A(s)$ is completely determined by the assumption that $[S] = [R(\gamma)]$ in $H_2(M, \gamma)$. To summarize, we may assume that ∂S intersects each annulus of $A(\gamma)$ in exactly two essential curves with the same orientation induced from S, which agrees with the orientation of the boundary of the annulus induced $R(\gamma)$ (see Figure 5.3).

We assume that our curves are exactly as in Figure 5.2. Recall that $\partial R(\gamma) = \cup_{j=1}^{n+1} \gamma_j \cup \cup_{j=1}^{n+1} \gamma_j$. Let S_i be a component of S. We now show that $\partial S_i = \cup_{j=1}^{n+1} \gamma_j$ or $\partial S_i = \cup_{j=1}^{n+1} \gamma_j \cup \cup_{j=1}^{n+1} \gamma_j$ as oriented manifolds. Note that ∂S_i is the union of some subset of the oriented curves $\{\gamma_1, \gamma_1, \ldots, \gamma_{n+1}, \gamma_{n+1}\}$. Since $\partial S_i \subset K$, $\partial S_i = q_1\alpha_1 + \cdots + q_n\alpha_n$. For $1 \leq j \leq n$, take the intersection pairing of both sides of this expression with α_j to

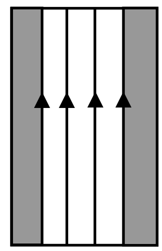


Figure 5.3: The white region is an annular suture. The grey region is $R(\gamma)$. The two vertical lines in the annulus are boundary curves of S with orientation induced from S. The arrows on $\partial R(\gamma)$ denote the orientation induced from $R(\gamma)$.

see that γ_j and γ_{n+1} must occur together (if they occur at all) in ∂S_i. This shows that $\partial S_i = \cup_{j=1}^{n+1}\gamma_j$ or $\partial S_i = \cup_{j=1}^{n+1}\gamma_j \cup \cup_{j=1}^{n+1}\gamma_j$. We say such surfaces are of type I or II, respectively. If S_i is of type I, then S consists of two such surfaces, and if S_i is of type II, then $S = S_i$. In either case, $x(S) \geq x(R(\gamma))$, with equality when S is planar. □

5.1 Construction of the contact structures when ∂M is connected

Let (M, η) be the universally tight contact manifold given by Lemma 5.0.5. When we refer to well-behaved surfaces, we will mean well-behaved with respect to $(M, \eta; \partial M, \lambda_1, \{\alpha_1, \beta_1\})$. Let S_1 be the first cutting surface in a hierarchy for M with boundary α_1. Recall that in the proof of Lemma 5.0.4 we chose S_1 so that ∂S_1 consists of n_1 copies of α_1, so S_1 is well-groomed, as defined in [Ga]. Via the correspondence between sutured manifold decompositions and convex decompositions, we may assume that S_1 is the first cutting surface in a convex decomposition for M and has ∂-parallel dividing curves (see [HKM2]). Since $tb(S_1) \leq -2$, there is a bypass abutting ∂M along α_1. After attaching this bypass to ∂M, we have a $\Sigma \times [0,1]$ slice with convex boundary, where Σ is a genus n surface, n is the genus of the boundary of M, and $\Sigma \times \{1\} = \partial M$. Let (Y, η) denote this contact manifold. Note that after attaching this bypass, the dividing curves consist of n $(-1,1)$ curves on each of the tori summands and another simple closed curve which is homologous to the sum of the other n.

We now construct an embedding of Y into S^3 with the standard tight contact structure. Fix g disjoint Darboux balls in S^3 labeled B_i, where g is the genus of the slice Y. In B_1, we have a convex torus T_1 with slope -2. One can find such a torus in S^3 and then remove a point from S^3 to get such a torus in $\mathbb{R}^3$. In each of the remaining B_i, we have a convex torus with slope -1. On T_1, LeRP a curve m_1 which bounds a disk in T_1 containing a single arc of the dividing set. On each of the other T_i, LeRP a curve l_i containing a disk in T_i with a single arc of Γ_{T_i} and LeRP a curve m_i which is disjoint from l_i and bounds a disk with a single arc of the same dividing curve that l_i intersects. Now, remove the disks bounded by the l_i and m_i on T_i and join l_i to m_{i+1} by a convex annulus A_i. This yields a convex genus n surface. Inside B_1, we have a compressing disk for T_1. By the Imbalance principle, there is a bypass along this compressing disk. Attaching this bypass yields the desired embedding of Y. Note that we can arrange for the sign of this bypass to agree with the sign of the bypass we attached to ∂M.

Fix a real number $r \in (-2,-1)$. Let q_i be an infinite sequence of rationals constructed in Section 4 such that $q_1 = -1$ and $q_i \neq r$.

Proposition 5.1.1. *There exists a sequence $\Sigma_i \subset Y = \Sigma \times [0,1]$ of well-behaved surfaces such that $slope(\Sigma_i) = q_i$ and $\Sigma_1 = \Sigma \times \{0\}$.*

Proof. We will prove our results for the embedding of $Y \subset S^3$. LeRP copies l_i of λ_1 on $\Sigma \times \{i\}$ such that $tb(l_i) = -1$. Let $A \subset Y$ be a convex annulus between l_0 and l_1. l_i separates $\Sigma \times \{i\}$ into a punctured torus P_i and a punctured genus $n-1$ surface. Cap off the P_i in S^3 with convex disks D_i to obtain tori T_i such that $slope(T_1) = -2$ and $slope(T_0) = -1$. There exists an incompressible torus T in the toric annulus bounded by the T_i such that $div(T) = 1$ and $slope(T) = q_2$ [Ho2]. Let d_2 be a Legendrian divide on T. d_2 can be Legendrian isotoped within the toric annulus bounded by the T_i so that is does not intersect the $D^2 \times [0,1]$ we used to cap off the thickened punctured torus bounded by $P_0 \cup P_1 \cup A$. This can be seen be working in a model for $D^2 \times [0,1]$, a standard neighborhood of a Legendrian arc. Hence, there exists a Legendrian isotopy taking d_2 to a curve in Y that is homologous to $a_2\alpha_1 + b_2\beta_1$, where $q_2 = b_2/a_2$. LeRP a curve d_2' in the same homology class on Σ_1 such that $d_2' \cap \Gamma_{\Sigma_1}$ is minimal. Let $A_2 \subset Y$ be a convex annulus between d_2 and d_2'. By our choice of d_2, $\Gamma_{A_2} \cap d_2 = \emptyset$ and $\Gamma_{A_2} \cap d_2' \neq \emptyset$, so there exists a bypass along d_2'. Attaching this bypass to Σ_1 yields Σ_2. Now, repeat the previous argument for q_3 and the slice bounded by Σ_2 and $\Sigma \times \{1\}$ to obtain Σ_3. These surfaces are well-behaved by construction. ☐

Let Σ_i be as in Proposition 5.1.1. Let (Y_i, η) be the genus n slice bounded by Σ_i and Σ_{i+1} in Y. Construct a contact structure η on $\Sigma \times [0, \infty)$ by taking $\Sigma \times [i, i+1]$ to be

Y_i. Let (V, η_r) be obtained from (M, η) by peeling off $Y \setminus \Sigma_1$ from (M, η) and attaching $(\Sigma \times [0, \infty), \eta)$ in the obvious way. Note that (V, η_r) is tight by construction since it embeds into (M, η).

Lemma 5.1.2. *Let $s, t \in (-2, -1)$. Then (V, η_s) and (V, η_t) are in the same isotopy class of contact structures.*

Proof. There exists a convex surface $S \subset V$ such that $V \setminus S = V' \cup S \times (0, \infty)$, where V' is diffeomorphic to V and $\eta_s|_{V' \cup S} = \eta_t|_{V' \cup S}$. This follows from the construction of (V, η_s) and (V, η_t). We claim that $\eta_s|_{S \times [0,\infty)}$ and $\eta_t|_{S \times [0,\infty)}$ are isotopic rel $S \times 0$. We can assume that $S \times [0, 1)$ is a one-sided vertically invariant neighborhood of our convex surface $S \times 0$. Hence, in particular, η_t and η_s agree on $S \times [0, 1)$. Form a new contact structure η_t^λ as follows: Extend the vertically invariant neighborhood $S \times [0, 1)$ of η_t to $S \times [0, \lambda)$, and on $S \times [\lambda, \infty)$ take η_t^λ to be $\eta_t|_{S \times [1,\infty)}$. Define η_s^λ similarly. By construction, $\eta_t^\infty = \eta_s^\infty$. Hence, (V, η_s) and (V, η_t) are in the same isotopy class of contact structures. ☐

5.2 Proof of Theorem 1.0.2 and Theorem 1.0.3 when ∂M is connected

In order to show that V supports uncountably many tight contact structures that are not contactomorphic, we will first show that the (V, η_s) are distinct up to proper isotopy. Theorem 1.0.2 then follows immediately since the mapping class group of any 3-manifold with boundary is countable ([McC]). To achieve this, we use the idea of the slope at infinity introduced in Section 3.

Proposition 5.2.1. *The net* $slope\colon\ \mathcal{C}(Ends(V, \eta_s; \partial M)) \to \mathbb{R} \cup \{\infty\}$ *is convergent, so the slope at infinity is defined. Moreover, the slope at infinity of* η_s *is* s *for all* $s \in (-2, -1)$.

Proof. We first show that there is an $E \in Ends(V, \eta_s)$ such that for all $F \subset E$, $slope(F) \leq s$. Choose $E \subset int(Y)$. We will be now working in S^3. Let $F \subset E$ and suppose for contradiction that $slope(F) > s$. Then, there exists $\Sigma \in \mathcal{C}(E)$ such that $slope(\Sigma) > s$. Let Σ_i be the family of surfaces given by Proposition 5.1.1. There exists an i such that Σ is contained in the genus n slice bounded by Σ_1 and Σ_i. LeRP a copy of λ_1 on Σ, Σ_1 and Σ_i and cap off the punctured tori bounded by these curves with convex disks. This yields a toric annulus $T^2 \times [0, 1] \subset S^3$ which contains a convex, incompressible torus T such that $slope(T) > slope(T^2 \times \{1\})$. No such $T^2 \times I$ can exist in S^3 (see [Ho2]). Therefore, such a Σ could not exist. Similarly, one can show that $slope(F) < s$ leads to a contradiction as in the example from Section 3.2. The existence of the family Σ_i now implies that the slope at infinity is s. ☐

By the proper isotopy invariance of the slope at infinity, there are uncountably many tight contact structures that are not properly isotopic, or even contact diffeomorphic, on V. This concludes the proof of Theorem 1.0.2 in the case of connected boundary. The proof of Theorem 1.0.3 is now immediate. For each η_s, simply choose a transverse curve in V and introduce a Lutz twist. Since the contact structures is identical outside of a compact set, the slope at infinity is unchanged.

5.3 Proof of Theorem 1.0.2 and Theorem 1.0.3 when ∂M is disconnected

Before proceeding with the proof, we will need the following technical result.

Lemma 5.3.1. *For every nonzero, positive integer n, there exists an irreducible 3-manifold M_n with connected, incompressible boundary of genus n.*

Proof. Let Σ_g be an orientable surface of genus g. If $n = 2m$, let $F \subset \Sigma_n$ be a once punctured genus m surface. Form a manifold M_n by identifying $F \times \{0\}$ with $F \times \{1\}$ on $\Sigma_n \times [0, 1]$. It is straightforward to show that $\Sigma_n \times \{0\}$ and $\Sigma_n \times \{1\}$ are incompressible in M_n. Using the incompressibility of these surfaces and the irreducibility of $\Sigma_n \times [0, 1]$, it is routine to show that M_n has incompressible boundary and is irreducible. If $n = 2m - 1$, let $F \subset \Sigma_m$ be an annular neighborhood of a nonseparating simple closed curve. Form a manifold M_n by identifying $F \times \{0\}$ with $F \times \{1\}$ on $\Sigma_m \times [0, 1]$. It is again straightforward to show that $\Sigma_m \times \{0\}$ and $\Sigma_n \times \{1\}$ are incompressible in M_n. Irreducibility and incompressibility of the boundary follow as before. □

Let $\partial M = \cup_{i=1}^{n} S_i$ where the S_i are the connected components of ∂M and S_1 is of nonzero genus. Let S_j be any component different from S_1. If S_j is compressible, compress it, and continue doing so until we have a collection of spheres and incompressible surfaces. We are now in the situation where every boundary component, besides possibly

S_1, is incompressible or a sphere. Fill in each sphere with a ball and onto each incompressible component of genus n, excluding S_1 if it happens to be incompressible, glue in an irreducible manifold with connected, incompressible boundary of genus n (such manifolds exist by Lemma 5.3.1). It is straightforward to show that the resulting manifold is irreducible since we are gluing irreducible manifolds (after filling any spheres in the boundary) along incompressible surfaces. Call the resulting manifold M'. We are now in the case of connected boundary. Put a tight contact structure on M' as before and attach a bypass along α_1 so that we have factored off a $\partial M' \times [0, 1]$ slice Y. Topologically, the closure of $M' \setminus Y$ is again M'. Without intersecting Y, remove each of the manifolds we glued in after perturbing the gluing surfaces to be convex. Reconstruct M by gluing the boundary components back together along the compressing disks. To ensure that the resulting manifold is tight, choose the compressing disks to be convex with Legendrian boundary and with a single arc in the dividing set [Ho1]. We now have a tight contact structure on M and a bypass layer Y along S_1 which is identical to the case of connected boundary. To form the (V, η_s) remove all the boundary components except for S_1 and construct the ends in Y as before. The calculation of the slope at infinity is identical to the case of connected boundary. As in the case of connected boundary, the proof of Theorem 1.0.3 is immediate after introducing a Lutz twist along a transverse curve in V.

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