Tightness in Contact Metric Manifolds

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I. Introduction

For quite some time now it has been clear that there are deep connections between the topology of 3-manifolds and Riemannian geometry.

More recently there have also been deep connections between the topology of 3-manifolds and contact geometry.

But there seems to be few results connecting important properties of contact structures (like tightness) and Riemannian geom.

In this talk we consider some steps toward such a connection.

Specifically I will discuss the ideas that go into the following theorems.

Th^{m} 1 (E-Komendarczyk-Massot):

> let (M,ξ,g) be a contact metric 3-manifold
> If g is complete and $\exists K>0$ such that the sectional curvature of g satisfies
> $$\frac{4}{9}K < sec(g) \leq K$$
> then the universal cover of (M,ξ) is (S^3,ξ_{std})

- Recall the classical sphere theorem has pinching constant $1/4$
- The classical sphere theorem said how "curvature can controle topology" here we see it can also "controle contact topology."

Thm 2 (EKM):

Let (M, ξ) be a contact 3-manifold weakly compatible with a complete Riemanian metric g.

If
$$\sec(g) \leq -d_g^2$$
then (M, ξ) is universally tight.

Here
$$d_g = \sup_M \| \nabla(\ln \theta')^{\perp} - \nabla \ln p \|$$
where

p is the length of a Reeb v.f.

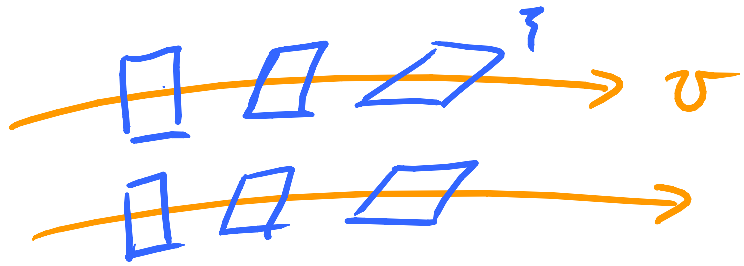
θ' is the instantanious rotation of ξ

- It is our hope this might be useful in finding tight contact structures on hyperbolic 3-mfds

II Metrics and Contact Structures

let ξ be a plane field on M

Recall if ξ is contact then
it will always twist as we flow
along a vector field $v \in \xi$



Given any metric g on M we can
measure this at a point $p \in M$
by taking a local frame v, u, n
where v, u an oriented orthonormal
from for ξ
n is a unit normal vector to ξ
let ϕ_t be the flow of v and

$$\theta(t) = \cos^{-1}\left(\frac{g\left((\phi_{-t})_* u, n\right)}{\|(\phi_{-t})_* u\|}\right)$$

the angle $\theta(t)$ must be increasing if ξ is contact

we call $\theta' = \theta'(0)$ the instantanious rotatuin of ξ

θ' is a function on M and ξ is contact $\Longleftrightarrow$ $\theta' > 0$

Remark: Setting $\alpha(\cdot) = g(n, \cdot)$ we have

$$\alpha \wedge d\alpha = \theta' \, vol_g \, .$$

so θ' depends only on ξ and g

definition: a metric g and a contact structure ξ are weakly compatible if there is a Reeb vector field R for ξ such that $\xi \perp_g R$

(recall R Reeb $\Longleftrightarrow$ R transverse to ξ and flow preserves ξ)

Prop:

let α be a contact form on M
g a Riemannian metric
R_α the Reeb field of α

Then the following are equivalent

1) $\boxed{R_\alpha \perp_g \xi}$

2) $\boxed{* d\alpha = \theta' \alpha}$

where $*$ is the Hodge star operator assoc. to g

3) $\boxed{g(u,v) = \frac{\rho}{\theta'} d\alpha(u, \phi(v)) + \rho^2 \alpha(u)\alpha(v)}$

where $\rho = \| R_\alpha \|$

J is complex str on ξ given by rot$^{\underline{n}}$ by $\pi/2$

$\phi : TM \to \xi$ is projection to ξ followed by J

Note: All terms in Th$^{\underline{m}}$ 2 are now defined

Observation:

If ξ and g are weakly compatible and n is the unit vector field normal to ξ then

$$\nabla_n n = -(\nabla \ln \rho)^{\xi}$$

where v^{ξ} is the component of v in ξ and $\rho = \|R\|$ where R is the Reeb field showing weak compat.

So if $(\nabla \ln \rho)^{\xi} = 0$ (in particular if ρ is constant) then the flow lines of R are <u>geodesics</u>.

Proof: for any u

$$0 = \nabla_u g(n,n) = 2g(\nabla_u n, n)$$

so $\nabla_n n$ is tangent to ξ

for any $v \in \xi$

$$\begin{aligned}
g(\nabla_u n, v) &= -g(n, \nabla_u v) \\
&= -g(n, \nabla_n v - \cancelto{0}{\nabla_v n}) \\
&= -g(n, [n,v]) \\
&= -\rho\,\alpha([n,v]) \\
&= \rho\left(\cancelto{0}{d\alpha(n,v)} - \cancelto{0}{n\cdot\alpha(v)} + v\cdot\alpha(n)\right) \\
&= \rho\left(v\cdot \tfrac{1}{\rho} g(n,n)\right) = \rho\, v\cdot \tfrac{1}{\rho} \\
&= -\rho\left(\tfrac{1}{\rho^2}\, d\rho(v)\right) \\
&= -d(\ln\rho)(v) \\
&= -g(\nabla \ln\rho, v)
\end{aligned}$$

note: $g(n,\cdot) = \rho\alpha(\cdot)$

so $\nabla_n n = -(\nabla \ln\rho)^{\xi}$

$\blacksquare$

definition: a contact structure ξ and a metric g are compatible if there is a contact 1-form α for ξ such that

$$\|\alpha\| = 1 \quad \text{and} \quad *d\alpha = \theta'\alpha$$

for some constant θ'

(that is g is compatible with ξ if the unit orthogonal to ξ is a Reeb field and the instantaneous rotation is constant)

note: the Reeb field for a contact str compatible with ξ is tangent to geodesics

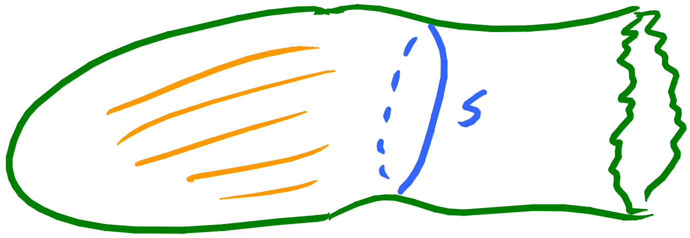
Remark 1) This is the same as Chern and Hamilton's definition from 1984 if $\theta' = 2$.

2) This form of compatibility has been extensively studied from a Riemannian geom. perspective.

III Convexity

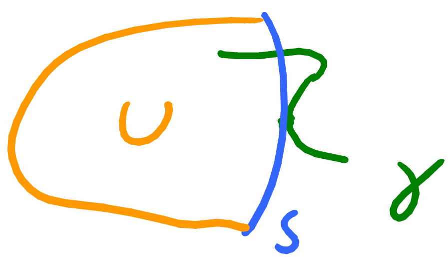
Suppose g is a metric on M^n

consider



U a domain bounded by S

We say S (or U) is **geodesically convex** if any geodesic γ tangent to S at p satisfies (locally) $\gamma \cap U = \{p\}$



lemma:

$$\boxed{\text{if } f : M \to \mathbb{R} \text{ s.t. } c \in \mathbb{R} \text{ a regular value, } f^{-1}(c) = S \text{ and } f^{-1}((-\infty, c]) = U \text{ then } S \text{ geod. convex} \iff \nabla^2 f(v,v) > 0 \quad v \in TS}$$

let $\mathrm{conv}(g) = \max_r \left\{ \begin{array}{l} B_r(p) \text{ geod convex} \\ \text{for all } p \in M \end{array} \right\}$

Thm:

$\boxed{\begin{array}{l} \text{If } K > 0 \text{ and } \sec(g) \le K \text{ then} \\ \qquad \mathrm{conv}(g) \ge \min\left\{ \mathrm{inj}(g), \frac{\pi}{2\sqrt{K}} \right\} \\ \text{where } \mathrm{inj}(g) \text{ is the injectivity radius.} \\ \text{If } \sec(g) \le 0 \text{ then } \mathrm{conv}(g) = \mathrm{inj}(g) \end{array}}$

Now for symplectic convexity.

let (W, J) be an almost complex mfd

Ω a domain in W bounded by a hypersurface Σ

let $\mathcal{C} \subset T\Sigma$ be the complex tangencies to Σ i.e.

$$\boxed{\mathcal{C} = T\Sigma \cap J(T\Sigma)}$$

We say Σ (or Ω) is (strongly) pseudoconvex if ξ is a (positive) contact structure on Σ

If $f: W \to \mathbb{R}$ a function with $c \in \mathbb{R}$ a regular value s.t.

$$\Sigma = f^{-1}(c)$$
$$\Omega = f^{-1}(-\infty, c)$$

then

$$\boxed{\xi = \ker(-df \circ J)}$$

so

$$\xi \text{ a contact str.}$$
$$\Longleftrightarrow$$
$$L(v,v) > 0 \quad v \in \xi$$

where

$$L(v,w) = -d(df \circ J)(v,w)$$

is the Levi form

We are now ready for ...

IV Darboux Th$^{\underline{m}}$ with Estimates

let (M, ξ) be a contact manifold
g a metric on M

Set
$$\tau(g) = \sup_r \left\{ \begin{array}{l} \xi \text{ restricted to} \\ B_r(p) \text{ is tight } \forall p \end{array} \right\}$$

<u>tightness radius</u>
or <u>Darboux radius</u>

Th$^{\underline{m}}$ (E-Komendarczyk-Massot):

> If g is a metric compatible with (M, ξ), then
> $$\tau(g) \geq \operatorname{conv}(g)$$

<u>Remark:</u> If M compact it is easy to use Darboux + Lebesgue number to prove a lower bound on τ exists for <u>any</u> g, but <u>not true</u> if M <u>not compact</u> and <u>not computable</u> if g <u>not compatible</u>

For the proof we need to compare symplectic and geodesic convexity

Prop (EKM): let g be weakly compatible with (M, ξ)

S a surface in M cut out by f and U the sublevel set

$\Sigma = \mathbb{R}\times S \subseteq \mathbb{R}\times M$

$\Omega = \mathbb{R}\times U \subseteq \mathbb{R}\times M$

If R is the Reeb field showing weak compatibility and J is an almost complex str leaving ξ invariant and intertwining R and $\partial/\partial t$

then for any $v \in \mathcal{C}$ ← complex tangencies to Σ

we have

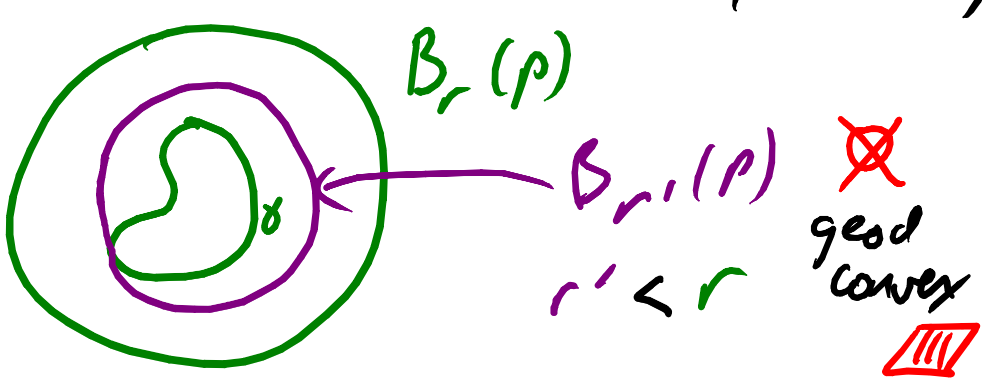
$$\boxed{\begin{aligned} L(v,v) = {} & \nabla^2 f(v,v) + \nabla^2 f(Jv, Jv) \\ & - \|v\|_g \left((\nabla \ln \theta')^\perp - \nabla \ln \rho, \nabla f \right) \end{aligned}}$$

The proof is a long computation

Proof of Th$^{\underline{m}}$ on tightness radius

- Fix a point $p \in M$ for all $r < \mathrm{conv}(g)$ we know $B_r(p)$ has geod convex boundary
- If v is a complex tangency $\mathcal{C}_r$ to $\partial(\mathbb{R} \times B_r(p))$ then $\nabla^2 f(v,v) \geq 0$ as is $\nabla^2 f(Jv, Jv)$ and one must be positive so $L(v,v) > 0$.

  Thus $\mathcal{C}_r$ is pseudoconvex for $r < \mathrm{conv}(g)$
- If $\xi|_{B_r(p)}$ then Hofer's argument using holomorphic disks will give a closed Reeb orbit γ in $B_r(p)$ (since $\partial(\mathbb{R} \times B_r(p))$ is pseudoconvex holo disks inside here stay inside)
- But now



Recall

Th$^{\underline{m}}$ 2 (EKM):

let (M, ξ) be a contact 3-manifold weakly compatible with a complete Riemanian metric g.

If
$$\sec(g) \leq -d_g^2$$
then (M,ξ) is universally tight.

Where $d_g = \sup_M \| \underbrace{\nabla(\ln \theta')^{\perp} - \nabla \ln p}_{D_g} \|$.

Proof:

- pull everything back to the universal cover $\tilde{M} = \mathbb{R}^3$.
- As above we just need to see for what r is $\partial B_r(p)$ geod convex
- Fix p let $r_p : M \to \mathbb{R} : x \mapsto d(p,x)$

  if $K>0$ and $\sec(g) \leq -K$ then it is known that $\nabla^2 r_p \geq \underbrace{\sqrt{K} \coth(\sqrt{K} r)}_{ct_K} g$

- Note for $v \in C_r$ we can write it as $v = v^{\xi} + an + b\,\partial/\partial t$ so

$$L(v,v) = \nabla^2 r_p(v,v) + \nabla^2 r_p(Jv,Jv) - g(D_g, \nabla r_p)\|v\|^2$$
$$\geq (\|v\|^2 + a^2 + b^2)\, ct_K(r) - \|D_g\|\,\|v\|^2$$
$$\geq (ct_K(r) - d_g)\|v\|^2$$

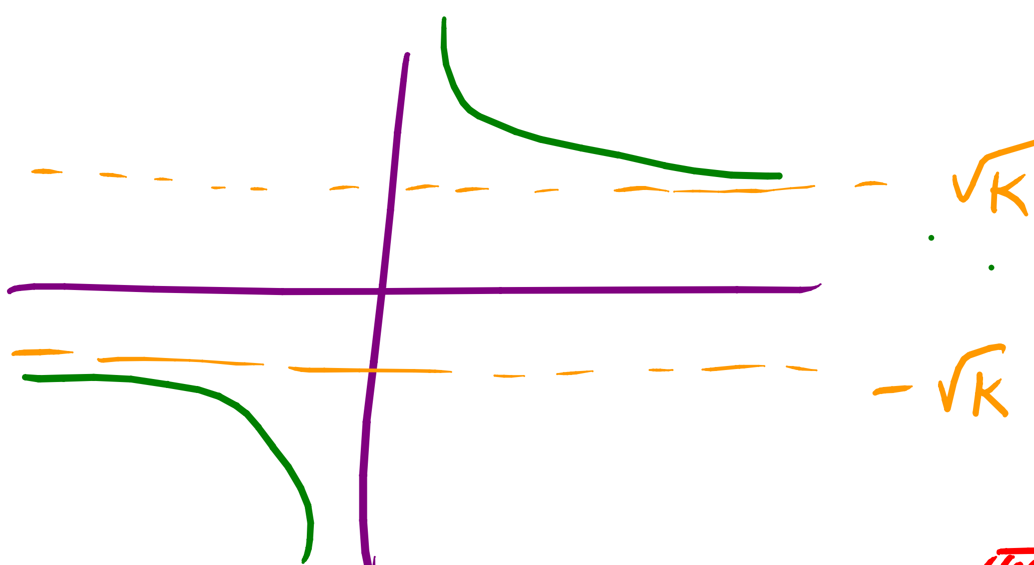
- So $\partial(\mathbb{R} \times B_r(p))$ pseudo-convex if

$$2\, ct_K(r) - d_g > 0$$

We are assuming $d_g \leq \sqrt{K}$ so

$$2\, ct_K(r) - d_g \geq ct_K(r) - \sqrt{K}$$

and



V Finding Overtwisted disks

Th^m (E-Komendarczyk-Massot):

let (M, ξ) be a contact manifold compatible with g

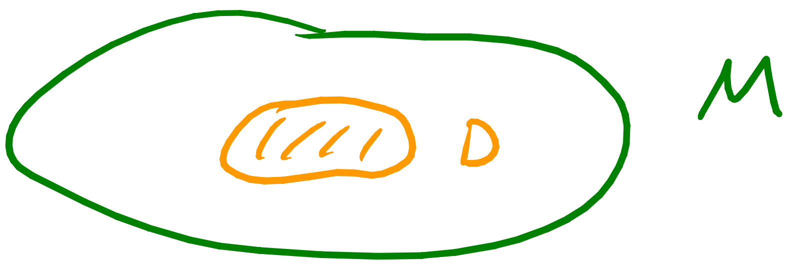
If $r < \text{inj}_p(g)$ and ξ is overtwisted on $B_r(p)$ then $\partial B_r(p) = S_r(p)$ contains an overtwisted disk

So while we cannot guarantee $\xi|_{B_r(p)}$ is tight, we can clearly see when it is not.

We prove this later, but for now we use it to prove the contact sphere theorem.

Proof of Contact Sphere Theorem:

- Recall (M, ξ) is compatible with g and $\exists K > 0$ s.t. $\frac{4}{9} K < \sec(g) \leq K$
- we can pull back to the universal cover so $\pi_1 = 1$ $\therefore$ ordinary sphere th$^{\underline{m}}$ says $M = S^3$
- Now assume ξ overtwisted so let D be an overtwisted disk



- we can rescale so that $K = 1$
- Bonnet-Meyer's Th$^{\underline{m}}$ says

$$\operatorname{diam}(M) < \pi / \sqrt{4/9} = \frac{3\pi}{2}$$

- We also know by Klingenberg that

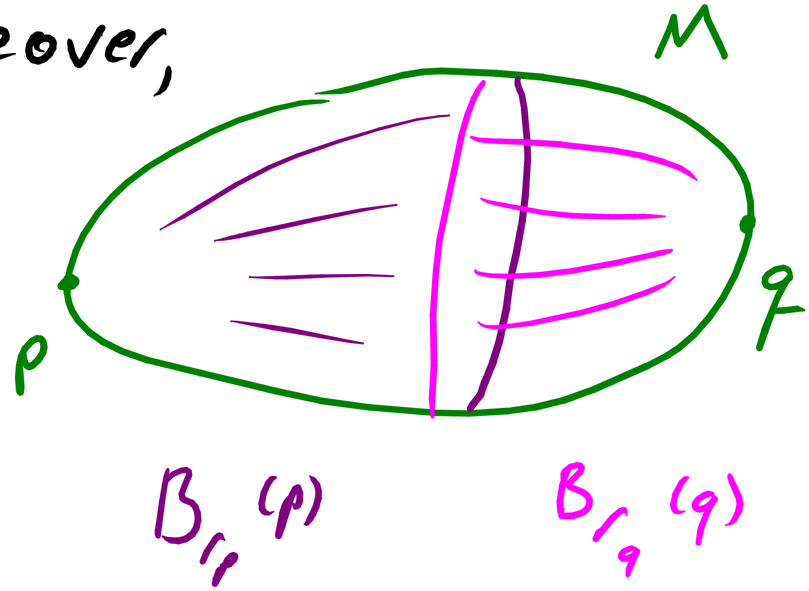
$$\operatorname{inj}(g) \geq \pi / \sqrt{1} = \pi$$

and then from above

$$\operatorname{conv}(g) \geq \frac{\pi}{2\sqrt{1}} = \pi / 2$$

- Using a standard Toponogov comparison argument we see ~~that~~ if $p, q \in M$ such that $d(p,q) = \mathrm{diam}(M)$ then there is an $r_p < \pi$, $r_q < \frac{\pi}{2}$ s.t. $M = B_{r_p}(p) \cup B_{r_q}(q)$

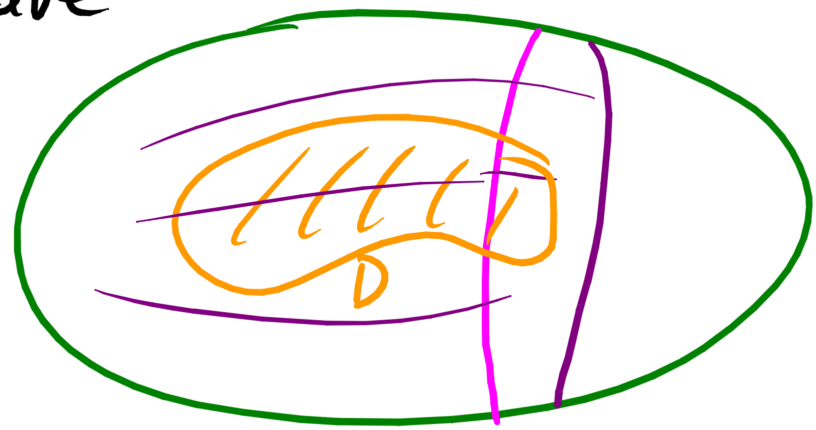
moreover,



- We can assume the ot disk $D \not\ni q$

- Th$^{\underline{m}}$ above says $B_{r_q}(q)$ is a standard contact ball so there is a contact vector field v whose flow pushes any point $\neq q$ into small nbhd of $\partial B_{r_q}(q) \subset \mathrm{int}\, B_{r_p}(p)$

so we have



$D \subset B_{r_p}(p)$

- Thus our last th$^{\underline{m}}$ says $\partial B_{r_p}(p)$ contains an overtwisted disk D'

but $D' \subset \partial B_{r_p}(q) \subset B_{r_q}(q)$

$\otimes$ tightness of $\xi|_{B_{r_q}(q)}$

$\therefore$ ξ tight ct str on S^3

so Eliashberg $\Rightarrow$ ξ is standard

$\blacksquare$

now for the proof of

Thm (E-Komendarczyk-Massot):

let (M,ξ) be a contact manifold compatible with g

If $r < \mathrm{inj}_p(g)$ and ξ is overtwisted on $B_r(p)$ then $\partial B_r(p) = S_r(p)$ contains an overtwisted disk

for the proof we need

lemma:

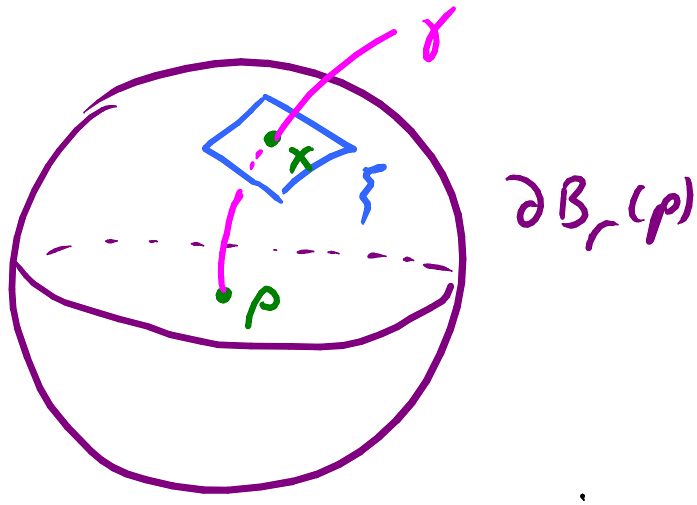
With notation as in the theorem

If $r < \mathrm{inj}_p(g)$ then the characteristic foliation $\xi(\partial B_r(p))$ has only 2 singular points

(and they are $\gamma \cap \partial B_r(p)$ where γ is the Reeb flow line through p)

Proof:

- suppose $x \in \partial B_r(p)$ is a singular pt
- So we have



- let γ be the geodesic starting at p such that $\gamma(r) = x$
- by the Gauss lemma we know

$$T_x(\partial B_r(p)) = \xi_x$$

is orthogonal to $\gamma'(r)$

- thus $\gamma'(r) = R$ the Reeb field and since the Reeb flow is tangent to geodesics we see γ is a Reeb flow line through p

$\blacksquare$

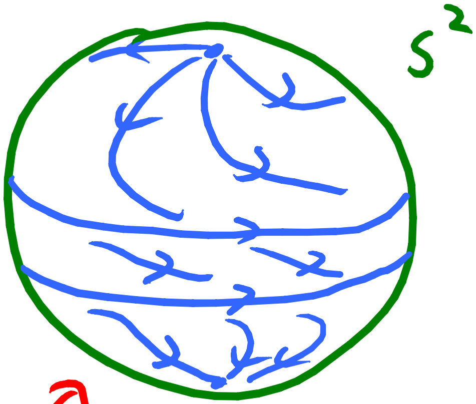
We call a surface Σ in (M, ξ)

ξ-convex if there is a contact vectorfield transverse to Σ

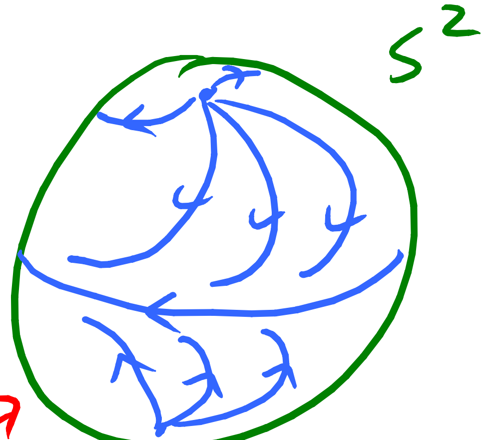
We call a sphere S simple if

ξS contains only two singular points (we call the positive one the north pole and the other the south pole)

we call ξS almost horizontal if, in addition, all closed leaves in ξS are oriented as the boundary of the disk containing the north pole



almost horiz.



not almost horiz.

Lemma (Giroux 1991):

If ξS is simple then

$$\begin{pmatrix} \xi S \text{ is} \\ \xi\text{-convex} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \xi S \text{ has no} \\ \text{degenerate} \\ \text{closed leaves} \end{pmatrix}$$

we are now ready for

Proposition (EKM):

B a ball in (M, ξ)

B a union of a point p and spheres S_t $t \in (0,1]$

1) $\left.\begin{array}{l} \xi S_t \text{ all simple} \\ \xi|_B \text{ tight} \end{array}\right\} \Longrightarrow \xi S_t$ almost horiz. $\forall t$

2) all ξS_t almost horiz $\Longrightarrow$ $\xi|_B$ tight.

3) ξS_t all simple and $\xi|_B$ overtwisted then $\exists$ t_0 such that

ξS_t has a closed leaf for $t \geq t_0$

$\xi|_{B_t}$ tight for $t < t_0$

Remark: The proof of the theorem clearly follows from the lemmas and proposition item 3)

Proof of Proposition

1) is obvious

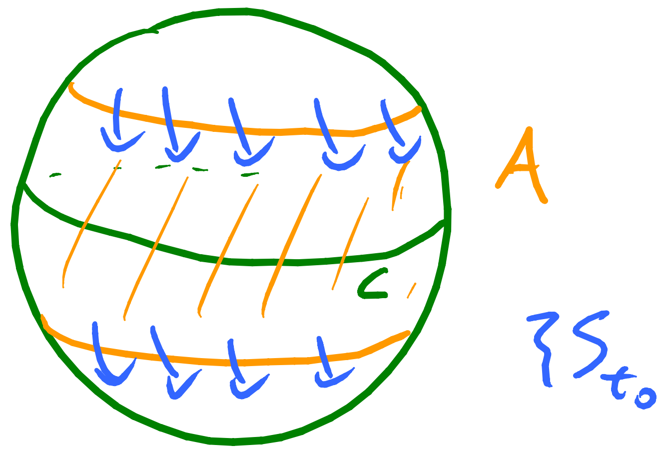
2) • recall that for small t, B_t will be tight by Darboux's Theorem.

• So ξS_t has no closed orbits
∴ trivially almost horiz.

• If there are no closed orbits in ξS_t for all t then by Giroux's lemma above all the S_t are ξ-convex
From this it is easy to argue that $\xi|_B$ is tight

• Suppose t_0 is the smallest t so that ξS_t has a closed orbit

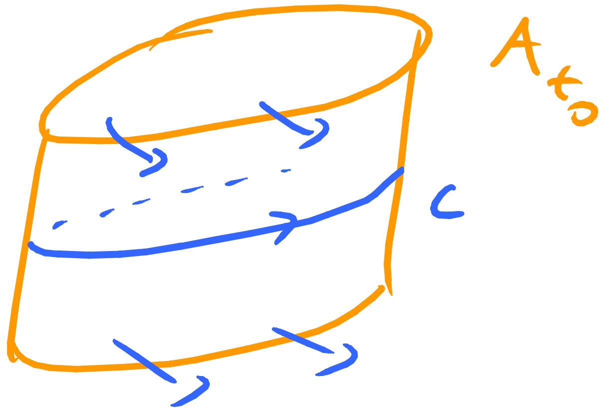
- The closed orbit C of $\{S_{t_0}$ must be degenerate
  (and we can assume there is just one)
- we can find a nbhd A of C on S_{t_0} so that



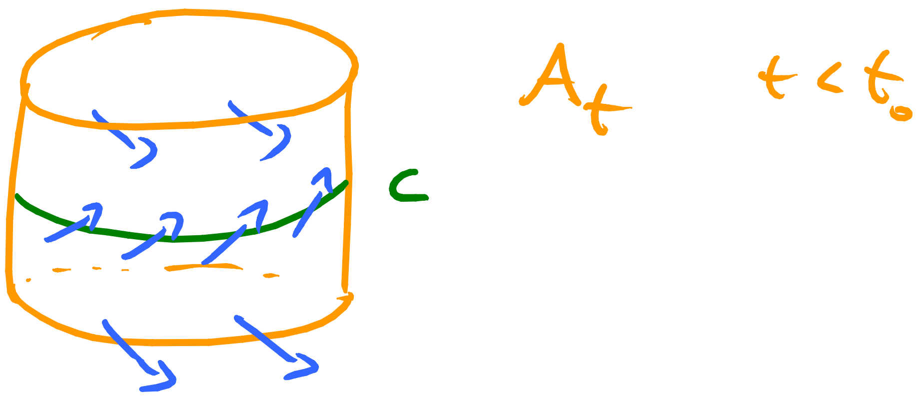
- We can map $A \times [t_0-\varepsilon, t_0+\varepsilon]$ into B so that $A_t = A \times \{t\}$ maps to S_t as above and $\{p\} \times [t_0-\varepsilon, t_0+\varepsilon]$ is Legendrian

  note A_t has no closed leaves for $t < t_0$
- recall the contact planes along $\{p\} \times [t_0-\varepsilon, t_0+\varepsilon]$ rotate in a left handed manner

• Since ξS_{t_0} is almost horizontal



just before t_0 we see



so by Poincaré-Bendixson there must be a closed leaf on A_t

$\otimes$ the fact that t_0 smallest such t

3) is similar, just uses the idea that closed leaves born near north pole must go "east to west" and die "west to east" (similarly for south pole)

for simple foliations

VI Furthur Directions

Can you use $Th^m Z$ to construct tight contact structures on hyperbolic 3-manifolds?

- given g a hyperbolic metric on M you can look at all metrics near g they will all have negative curvature
- for each metric, there will be lots of curle eigenforms

$$*d\alpha = c\alpha$$

some constant c

if $\alpha \neq 0$, $c \neq 0$, then α defines a contact structure and its tight if

$$\sec(g) \leq -\sup \| \nabla \ln \| R_\alpha \| \|^2$$

questions:

1) Can you find a non sing eigen form?

2) If you can does it satisfy ≤?

3) If not can you use, say, the Altschuler flow to improve the constant on rt hand side?

by taking a generic metric you can assume the zeros of α are isolated

a local computation shows any nbhd of a singular point is overtwisted so you can't do a "local surgery" to get a tight contact structure.

[What about higher dimensions?

The notion of compatible is more complicated (need integrable CR-structure)

But we do have similar estimates on plastikstufe free geodesic balls.

We have ideas for a bound on the size of Darboux balls but...

Thanks

for

Your

attention