Knots and Contact Geometry

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Basic Definitions

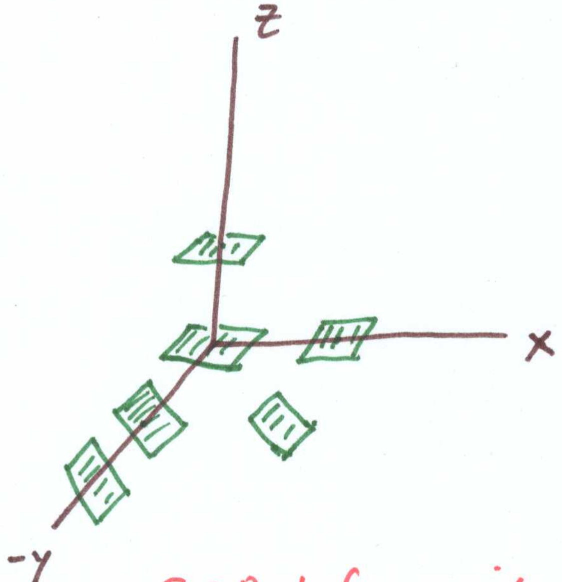
a hyperplane field ξ^{2n} on a manifold M^{2n+1} is a contact structure if there is (at least locally) a 1-form α such that

$$\boxed{\begin{array}{l} \xi = \ker\alpha \\ \alpha \wedge (d\alpha)^n \neq 0 \end{array}}$$

example:

on $\mathbb{R}^{2n+1}$ let $\alpha = dz - \sum_{i=1}^{n} y_i \, dx_i$.

and $\xi_{std} = \ker\alpha$



90° left twist

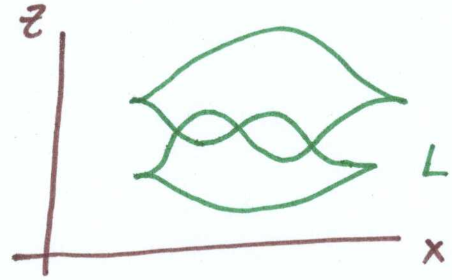
Th$^{\underline{m}}$ (Darboux): all contact structures are locally diffeomorphic to this one let (M, ξ) be a contact manifold

$L^n \subset M^{2n+1}$ is a <u>Legendrian</u> <u>submanifold</u>

if

$$\boxed{T_x L \subset \xi_x} \qquad \text{for all } x \in L$$

<u>example</u>: in $(\mathbb{R}^3, \xi_{std})$ project L to the xz-plane



(Front projection)

you recover the y-coordinate since along L you have

$$dz - y\,dx = 0$$

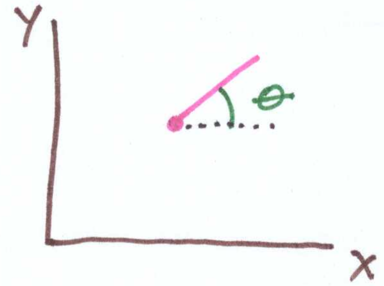
$$\Rightarrow \quad \boxed{y = \frac{dz}{dx}}$$

<u>Th$^{\underline{m}}$:</u>

any arc in a contact 3-manifold can be C^0-approximated by a Legendrian curve (rel end pts).

Natural Occurrences Of Contact Structures

consider the configuration space of a skate (or front wheel of a car)



(x,y) determine the position of the skate in the plane.

θ determines the angle it forms with the x-axis.

so the configuration space is

$$\boxed{W = \mathbb{R}^2 \times S^1}$$

note: 1) at a fixed point the skate can point in any direction.

2) skate can only move in the direction it is pointing (we assume it does not scrape)

so if $\gamma(t) = (x(t), y(t), \theta(t))$ is a motion of the skate then

$$\boxed{\frac{y'(t)}{x'(t)} = \tan \theta(t)}$$

if we set $\xi = \ker(\cos\theta \, dy + \sin\theta \, dx)$

then ξ is a contact structure on W.

and

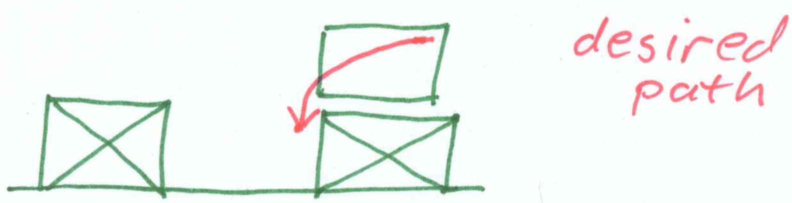
γ is a motion of the skate

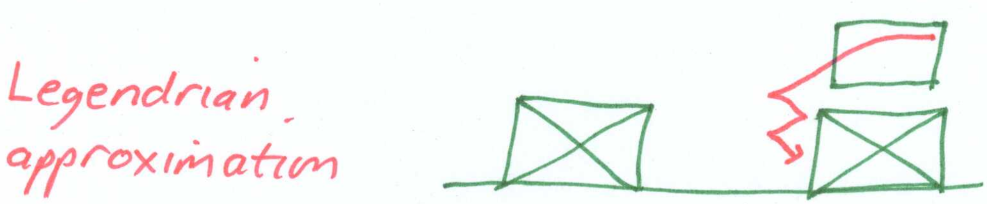
$\Longleftrightarrow$

γ parameterizes a Legendrian curve

<u>application:</u>

You can always (in theory) parallel park your car.





Other Occurrences Of Contact Structures

- **PDE** (Sophus Lie 1872)

  given $F: \mathbb{R}^{2n+1} \to \mathbb{R}$

  finding $z: \mathbb{R}^n \to \mathbb{R}$ solving

  $$\boxed{F(x_1 \ldots x_n, \frac{\partial z}{\partial x_1} \ldots \frac{\partial z}{\partial x_n}, z) = 0}$$

  is equivalent to finding $u: \mathbb{R}^n \to \mathbb{R}^{2n+1}$

  s.t.

  $$\boxed{\begin{array}{l} F \circ u = 0 \\ \text{Image}(u) \text{ is Legendrian} \end{array}}$$ (in ξ_{std})

- **Riemannian Geometry**

  $$\begin{array}{ccc} TM & \cong_g & T^*M \\ \cup\!\vert & & \cup\!\vert \\ S(TM) & & S(T^*M) \end{array} \qquad \lambda = \Sigma p_i dq_i$$

  Geodesic Flow = Reeb Flow

- **Optics** via Huygen's Principle

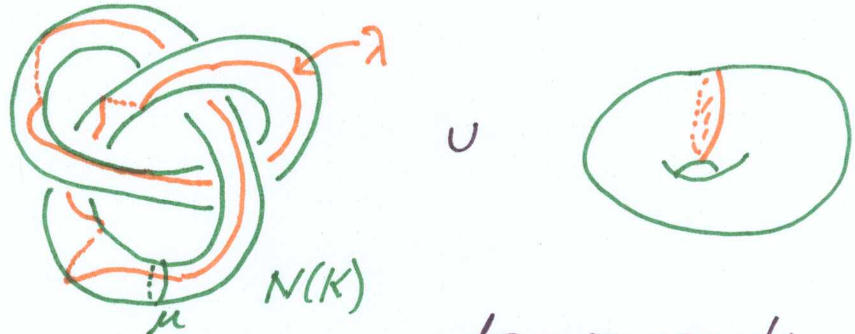
- **Thermodynamics** by work of Gibbs

Applications To Topology

- Kronheimer and Mrowka proved

$$\boxed{\text{Nontrivial Knots have Property P}}$$

recall: Dehn surgery on a knot $K \subset S^3$ is
- remove a nbhd $N(K)$ of K from S^3
- glue back $S^1 \times D^2$ so that $\{pt\} \times \partial D^2$ goes to a curve $\lambda \subset \partial(S^3 \setminus N(K))$



denote result $S^3_\lambda(K)$

K has Property P if non-trivial surgery yields a manifold with nontrivial fundamental group.

(i.e. $\lambda \neq \mu \Rightarrow \pi_1(S^3_\lambda(K)) \neq \{e\}$)

- Ozsváth and Szabó proved

If L is ,  or 

then $S^3_p(K) = S^3_p(L)$ for any p

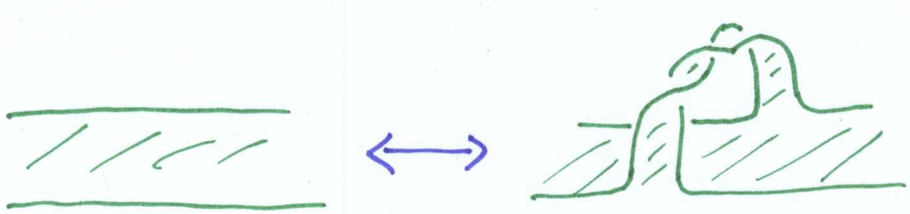
$\Rightarrow K = L$

(the $L =$  case was a conjecture of Gordon from 1970's and was first proven by Kronheimer-Mrowka-Ozsváth-Szabó about 2 month before above proof)

- Giroux and Goodman proved

Harer's Conjecture:

All fibered knots in S^3 are related by Hopf plumbing and deplumbing

Σ  Σ'

$K = \partial\Sigma$ $\qquad$ $K' = \partial\Sigma'$

- Invariants of Knots (and other things)
(Conormal Construction)

given $\qquad$ get

$$\mathbb{R}^3 \rightsquigarrow \text{unit } T^*\mathbb{R}^3 = \mathbb{R}^3 \times S^2 \quad \} \text{ contact str } \xi = \ker\lambda,\ \lambda = \sum p_i dq_i$$

$$\cup \qquad\qquad \cup$$

$$\underset{\text{knot}}{K} \rightsquigarrow L_K = \overset{\text{(unit)}}{\text{conormal}} \text{ to } K \quad \} \text{ Legendrian}$$

so invariants of L_K are invariants of K

Th$^{\underline{m}}$ (Ekholm-E-Ng-Sullivan):

$$\boxed{\text{The contact homology of } L_K \text{ is a well defined invariant of } K \text{ and can be computed from a braid representation of } K}$$

Ng has shown:

- this detects the unknot
- contains the Alexander module/poly
- has relations to A-polynomial

⋮

3D Contact Geometry

a contact manifold (M^3, ξ) is called **overtwisted** if there is an embedded disk $D^2 \subset M$ such that

$$T_x D^2 = \xi_x \quad \text{for all } x \in \partial D^2$$

otherwise it is called **tight**.

examples:

- $\mathbb{R}^3$ $\quad \xi_{ot} = \ker(\cos r \, dz + r \sin r \, d\theta)$

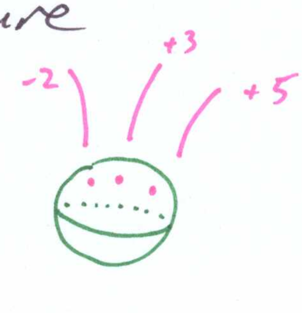
  $D = \{(r, \theta, z) \mid z = 0 \quad r \leq \pi\}$

  so $(\mathbb{R}^3, \xi_{ot})$ is overtwisted.

- $(\mathbb{R}^3, \xi_{std})$ is tight (see below)

facts:

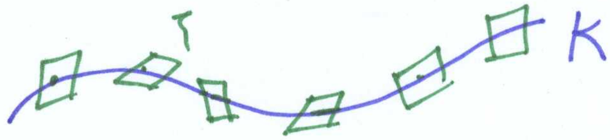
- Every 3-manifold has a contact structure — Lutz-Martinet '70
- Eliashberg '92: overtwisted contact structures on M $\overset{\text{1 to 1 corresp.}}{\longleftrightarrow}$ homotopy classes of plane fields on M
- E-Honda '01: not all 3-manifolds admit a tight contact structure



Legendrian Knots

recall a Legendrian knot in a contact 3-manifold (M, ξ) is a knot $K \subset M$ such that

$$\boxed{T_x K \subset \xi_x} \quad \text{for all } x \in K$$

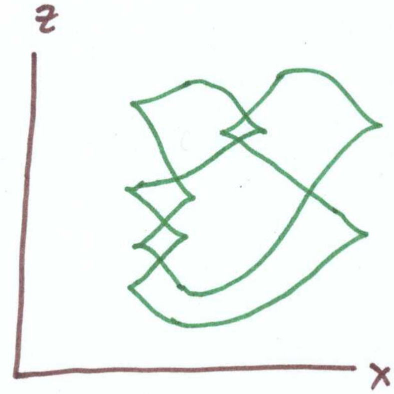


Main problems:

- Classify Legendrian knots upto isotopy through Legendrian knots.
- See what Legendrian knots tell you about contact geometry and topology.

Since Darboux's Theorem says all contact manifolds locally look like $(\mathbb{R}^3, \xi_{std})$ we will mainly study the first problem there. (though most results hold for more general contact manifolds)

If L is a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ then we can study its front projection that is its projection to the xz-plane.

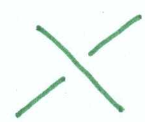


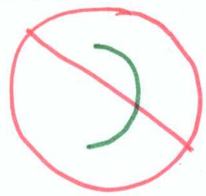
recall: along L

$$dz - y\,dx = 0$$

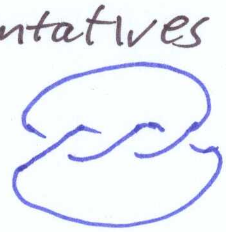
so $\boxed{y = \frac{dx}{dz}}$

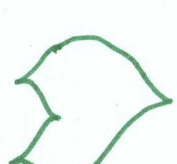
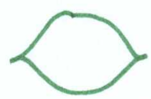
Thus

-  means  ( is never part of a front proj)
- no vertical tangencies

  cusps instead.

note: any knot type has Legendrian representatives

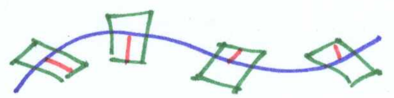
 $\Rightarrow$ 

question: are  and  isotopic?

There are two "classical" invariants of Legendrian knots (other than topological knot type!)

Thurston-Bennequin Invariant:

$tb(L)$ = framing induced on L by ξ



= writhe(L) − (# <)

in front proj.

Rotation number:

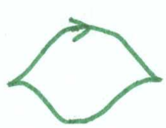
$r(L)$ = Euler number of $\xi|_\Sigma$ relative to v

(where Σ^2 a sfc with $\partial\Sigma = L$ and $v \in TL$ orients L)

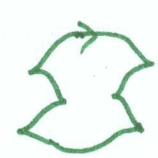
$= \frac{1}{2}$(# < and >) $- \frac{1}{2}$(# < and >)

in front proj

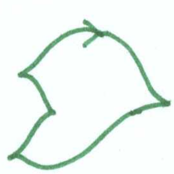
Examples:



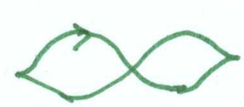
$tb = -1$
$r = 0$



$tb = -3$
$r = 0$

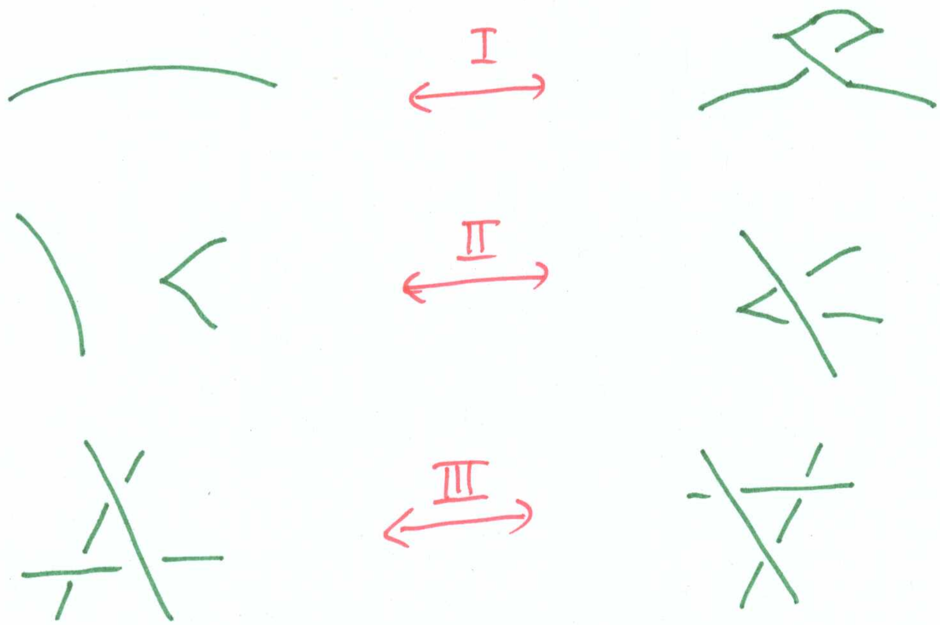


$tb = -2$
$r = -1$

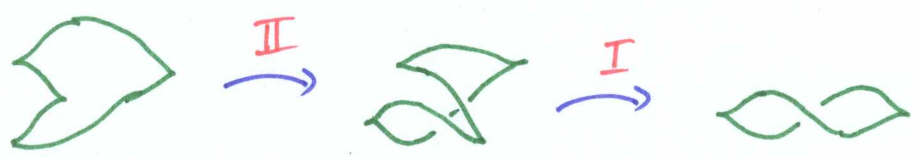


$tb = -2$
$r = -1$

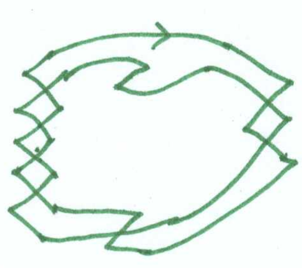
"Reidemeister Moves"



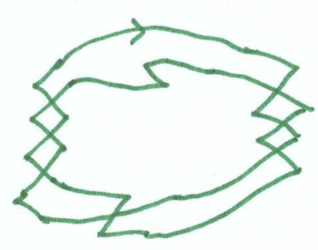
example:



exercise: Show



and



are isotopic.

note: $tb = -15$
$r = -2$

Geography and Botany

given a topological knot type K in $\mathbb{R}^3$

let $\mathcal{L}(K)$ be the set of Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ topologically isotopic to K

define $\boxed{\Phi : \mathcal{L}(K) \to \mathbb{Z} \times \mathbb{Z} : L \mapsto (r(L), tb(L))}$

classifying Legendrian knots is equivalent to

Geography: determine image Φ

+

Botany: determine $\Phi^{-1}(r,t)$ for all $(r,t) \in \text{im}\,\Phi$

If Φ is injective then we say K is

Legendrian simple (of course this means Legendrian knots in the knot type K are determined by tb and r)

Fact: if $(r,t) \in$ image Φ for any K then

$\boxed{r + t \text{ is odd}}$ (congruent mod 2 to $\chi(\Sigma)$)

Th$^{\underline{m}}$ (Bennequin '82):

for a Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ we have

$$\boxed{tb(L) + |r(L)| \leq -\chi(\Sigma)}$$

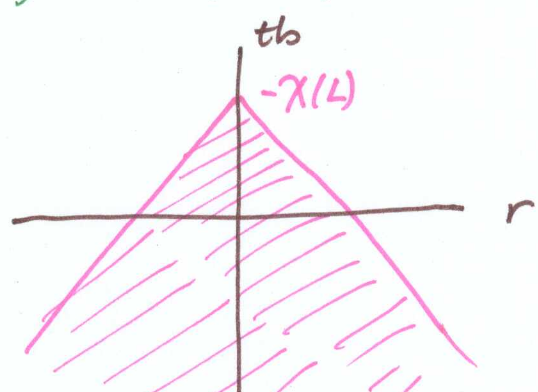
for any Σ with $\partial\Sigma = L$

Eliashberg extended this to any tight (M, ξ)

remark: methods of proof

- braid/knot theory or
- surfaces (convex) or
- holomorphic curves or
- Seiberg-Witten theory or
- Heegaard Floer theory ....

note: this says image$(\Phi : \mathcal{L}(K) \to \mathbb{Z}\times\mathbb{Z})$ contained in

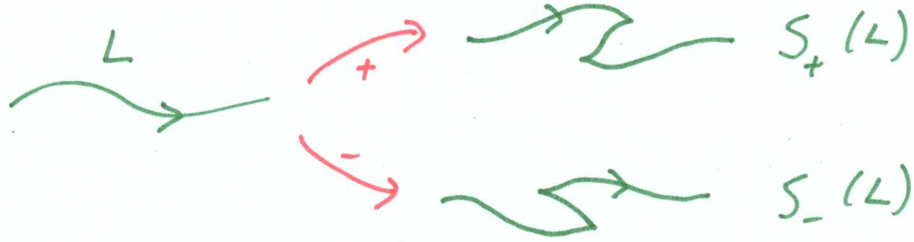


observation: a contact manifold (M, ξ) is tight

$$\Longleftrightarrow$$

image$(\Phi : \mathcal{L}(K) \to \mathbb{Z}\times\mathbb{Z})$ is bounded above for any K

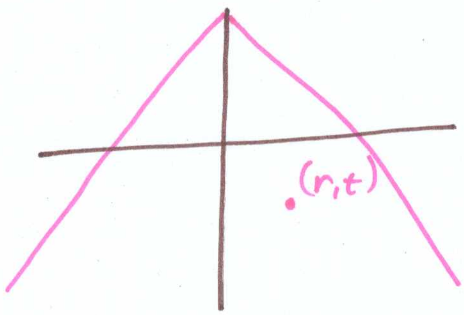
- given a Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ we can stabilize L



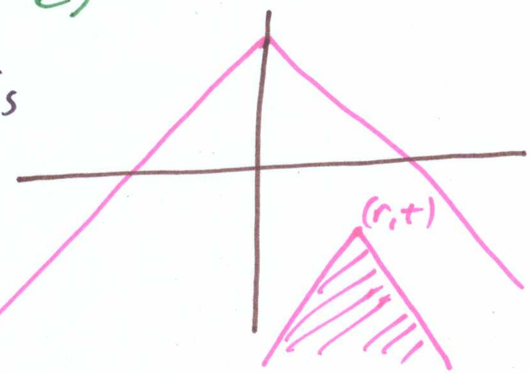
note: $tb(S_\pm(L)) = tb(L) - 1$

$r(S_\pm(L)) = r(L) \pm 1$

so if $(r,t) \in \text{image}(\Phi : \mathcal{L}(K) \to \mathbb{Z} \times \mathbb{Z})$



then so is



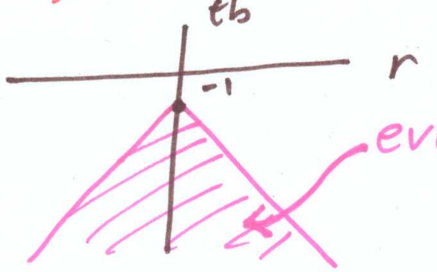
- note: $f : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \longmapsto (-x, y, -z)$ preserves ξ_{std}, is isotopic to the identity and $f(L)$ and L have front proj. related by reflection

so $\text{image}(\Phi : \mathcal{L}(K) \to \mathbb{Z} \times \mathbb{Z})$ is symmetric about r-axis

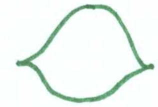
(this is not necessarily true in general manifold)

Legendrian Simple Knots

- (Eliashberg-Fraser '95) the unknot is Legendrian simple



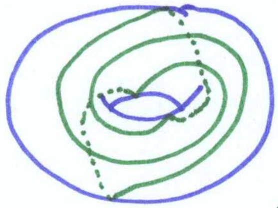
every (r,t) with $r+t$ odd realized



all stabilizations of this one

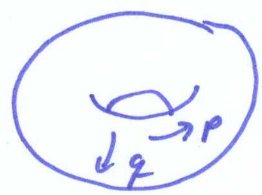
- (E-Honda '01) all torus knots are Legendrian simple

a torus knot sits on standard torus

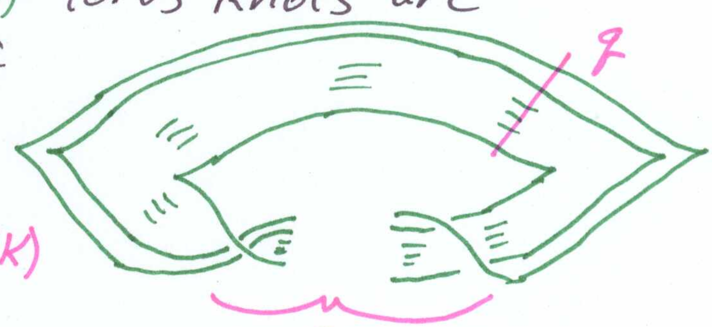


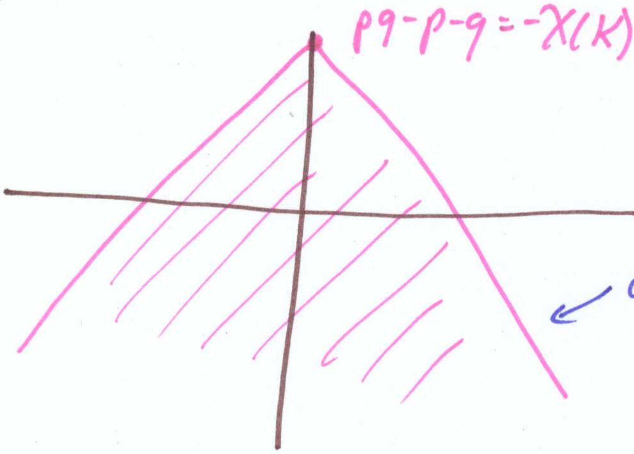
(2,3) torus knot

in general determined by (p,q)



if $p, q > 0$ then all (p,q) torus knots are stabilizations of



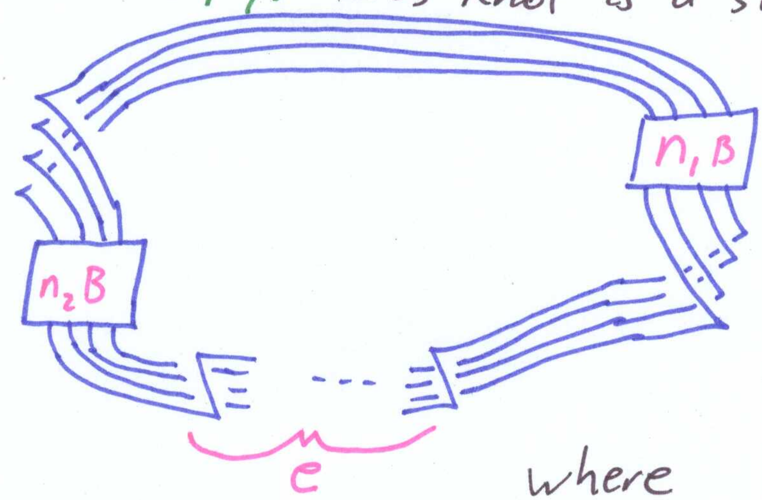


all allowable (r,t) realized.

if $p>q>0$ write $p=(n_1+n_2+1)q+e$

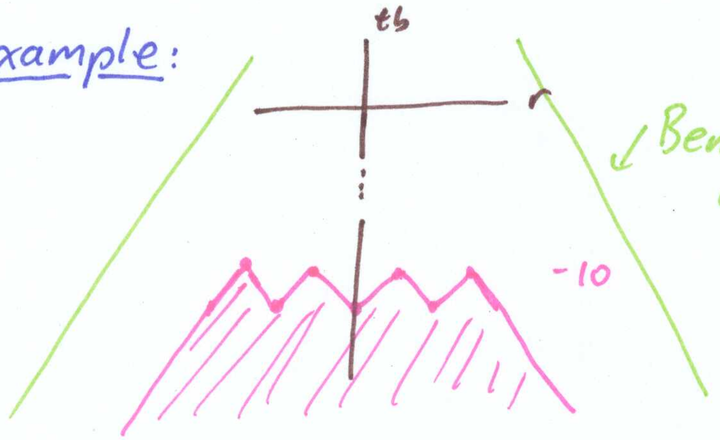
$0<e<q$

any Legendrian $(-p,q)$ torus knot is a stabilization of



where B =

example:

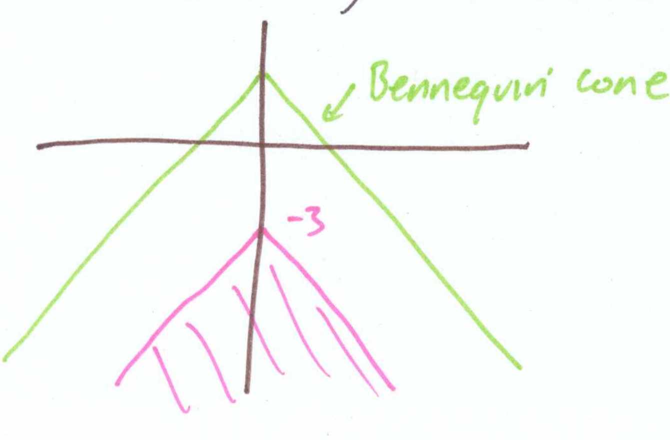


note: · Bennequen bound and cone way off

· you can make arbitrarily many peaks and deap valleys

• (E-Honda '01)

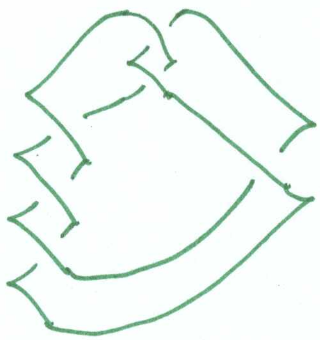
the figure 8 knot is Legendrian simple



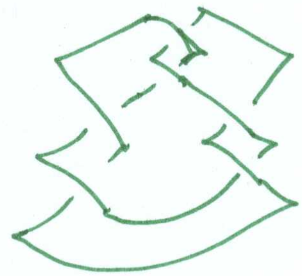
all are stabilizations of

not all knots are Legendrian simple

the first examples were



and



they both have $tb = 1$ and $r = 0$ but are not Legendrian isotopic

to distinguish them one needs contact homology

(developed by Chekanov, Eliashberg-Hofer)

(extensions by E-Ng-Sabloff)

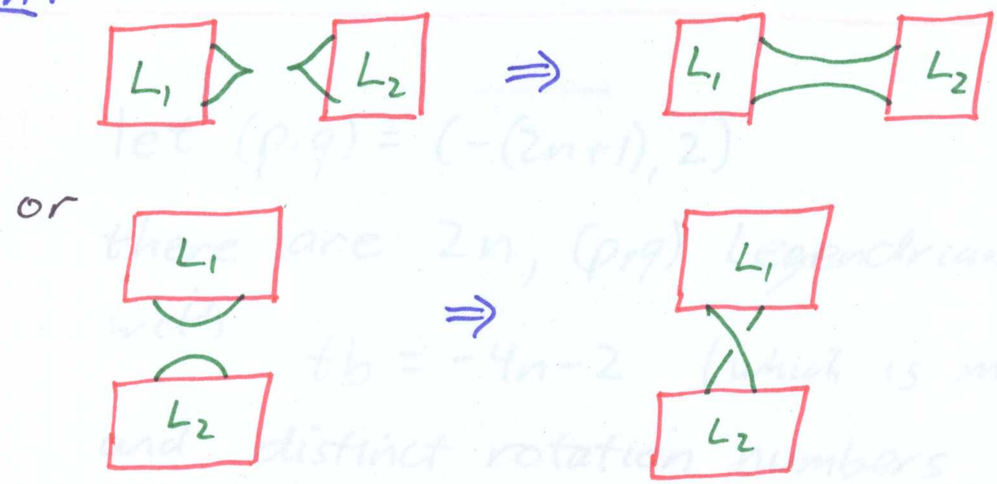
more recently you can use Heegaard-Floer homology.

Connect Sums

topologically:

$K_1 \# K_2$

Legendrian:



$L_1 \# L_2$

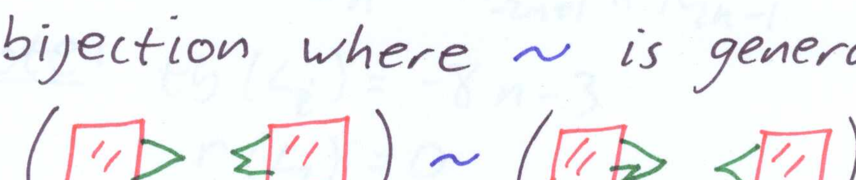
it is not obvious but this is well-defined

Th^m (E–Honda '03):

let $K = K_1 \# K_2$ then

$$\frac{\mathcal{L}(K_1) \times \mathcal{L}(K_2)}{\sim} \longrightarrow \mathcal{L}(K)$$

is a bijection where $\sim$ is generated by

1. 

2. any obvious topological symmetry

   (e.g. $K_1 = K_2$ then $(L_1, L_2) \sim (L_2, L_1)$

Corollary:

given any positive integer n there is a topological knot type K and distinct Legendrian knots $L_1 \ldots L_n$ in $\mathcal{L}(K)$ with the same tb and r.

Proof: let $(p,q) = (-(2n+1), 2)$

there are $2n$, (p,q) Legendrian torus knots with

$$tb = -4n-2 \quad \text{(which is max possible)}$$

and distinct rotation numbers

(odd numbers between $-2n+1$ and $2n-1$)

let K_r be the knot with $r(K_r) = r$

set

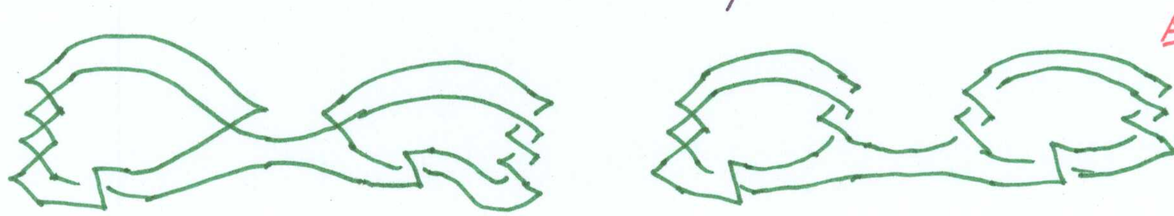
$$L_1 = K_{-1} \# K_1$$
$$L_3 = K_{-3} \# K_3$$
$$\vdots$$
$$L_n = K_{-2n+1} \# K_{2n-1}$$

note: $tb(L_i) = -8n-3$

$r(L_i) = 0$

but L_i distinct by the theorem

example:



Remark:

1. All the "computable contact homology" invariants of examples same.
   (and I bet all Heegaard-Floer invts same.)
2. All above examples become the same after one stabilization.

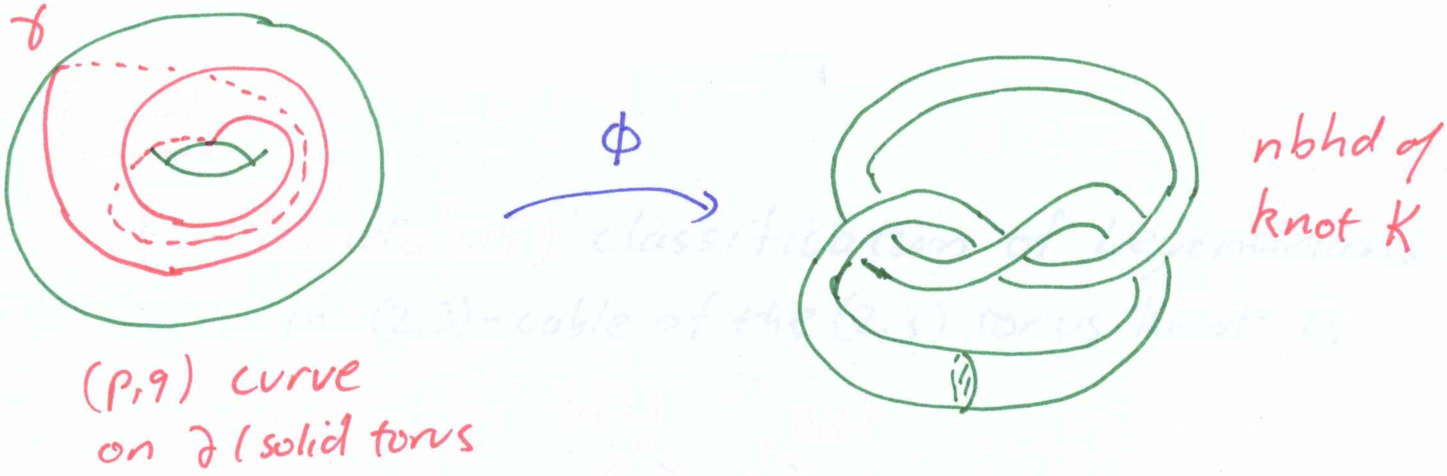
Corollary:

$\boxed{\text{given any positive integer } n \text{ there is a topological knot type } K \text{ containing 2 Legendrian knots } L_1, L_2 \text{ with same } tb \text{ and } r \text{ that remain distinct when stabilized } m \text{ or fewer times}}$

Idea of Proof

find negative torus knots whose geography has deap valleys

Cables



the (p,q)-cable of K is $\phi(\gamma)$

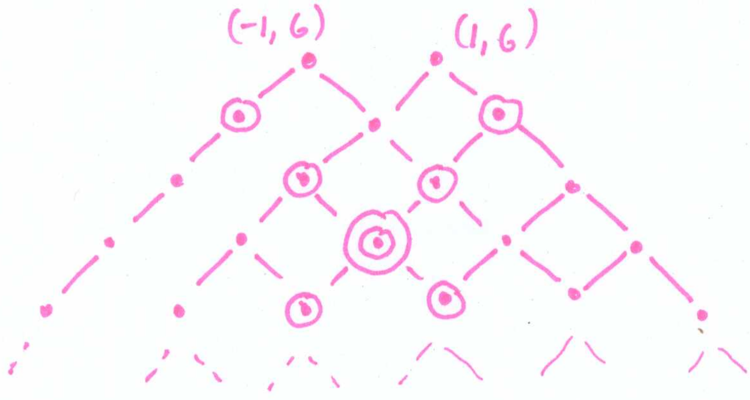
By work of E-Honda and Tosun we know a lot about when cables of Legendrian simple knots are Legendrian simple

example
- all cables of negative torus knots are Legendrian simple.
- all negative cables (and many positive cables) of positive torus knots are Legendrian simple

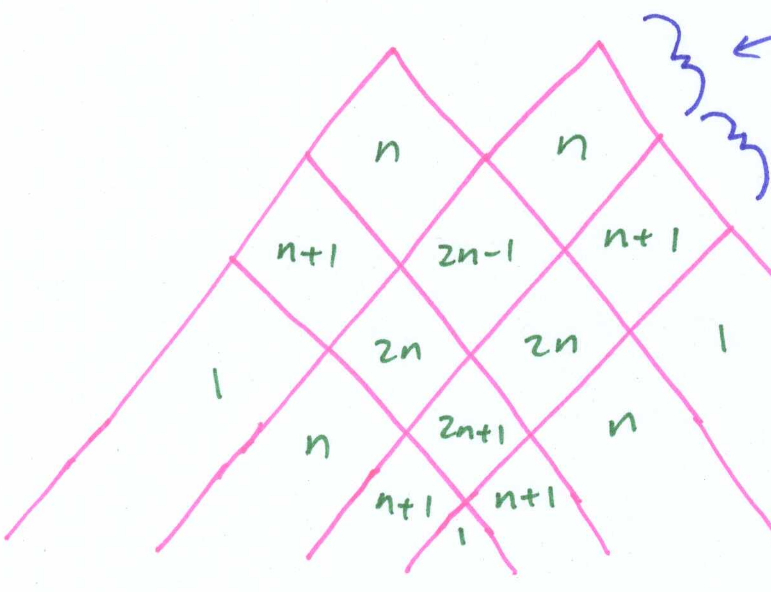
But ...

Thm:

(E-Honda '05) classification of Legendrians in (2,3)-cable of the (2,3) torus knot is

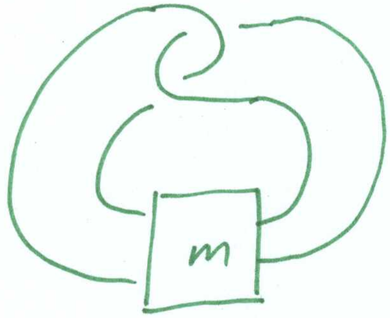


(E-LaFountain-Tosun) classification of Legendrian knots in (p,q)-cable of (2,3) torus knot is



these widths an n depend on (p,q) but can be arbitrarily large.

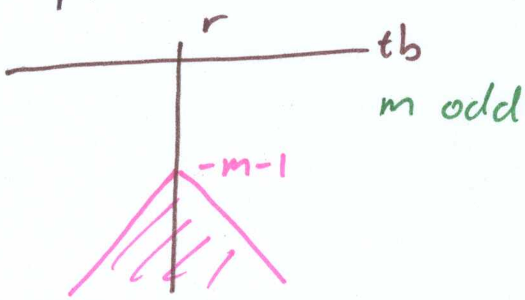
Twist Knots

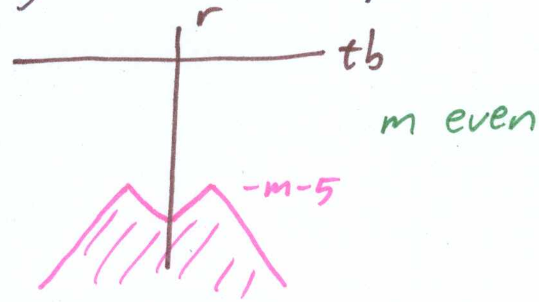


$= K_m$

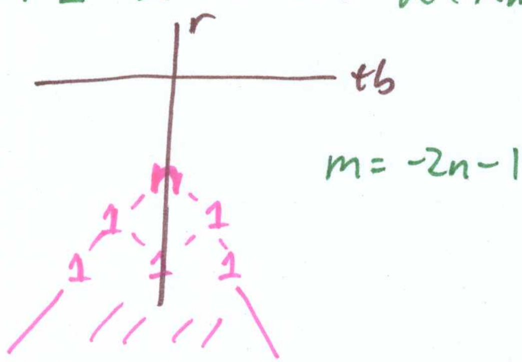
Th$^{\underline{m}}$ (E-Ng-Vértesi)

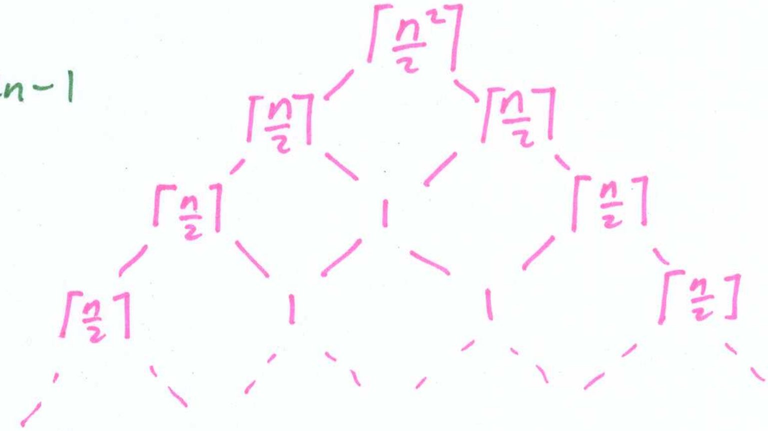
m positive then K_m Legendrian simple





$m \leq -2$ then $\mathcal{L}(K_m)$ classified by





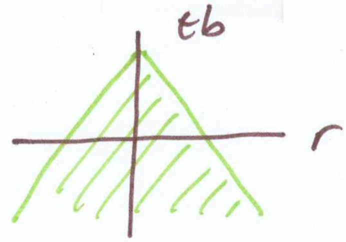
What We Know

Botany:

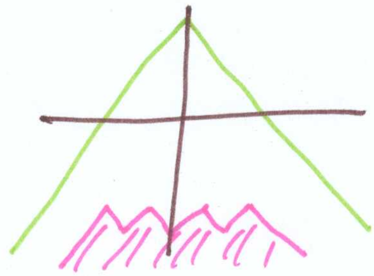
1. In a tight contact structure $\Phi^{-1}(r,t)$ can have only finitely many elements (Colin-Giroux-Honda)
2. But $\Phi^{-1}(r,t)$ can be arbitrarily large (E-Honda, E-Ng-Vertési, E-LaFountain-Tosun)
3. Elements in $\Phi^{-1}(r,t)$ become isotopic after sufficiently many positive and negative stabilizations (Fuchs-Tabachnikov)
4. There are elements in $\Phi^{-1}(r,t)$ that can
   a) be stabilized infinitely often in one direction and stay distinct (EH, ENV, ELT)
   b) require arbitrarily many stabilizations in both directions (EH, ELT)

Geography:

1. Image Φ is bounded above. (Bennequin)

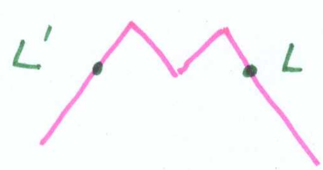


2. Upper bound can be arbitrarily negative (Kanda, Fuchs-Tabachnikov ...)



3. Arbitrarily many peaks and arbitrarily low valleys (EH, ELT)

4. There are L with $tb(L)$ non-maximal that do not destabilize



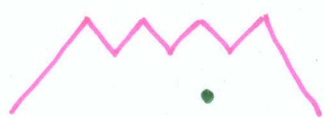
Main Qualitative Questions:

1. Can you have non-maximal peaks?

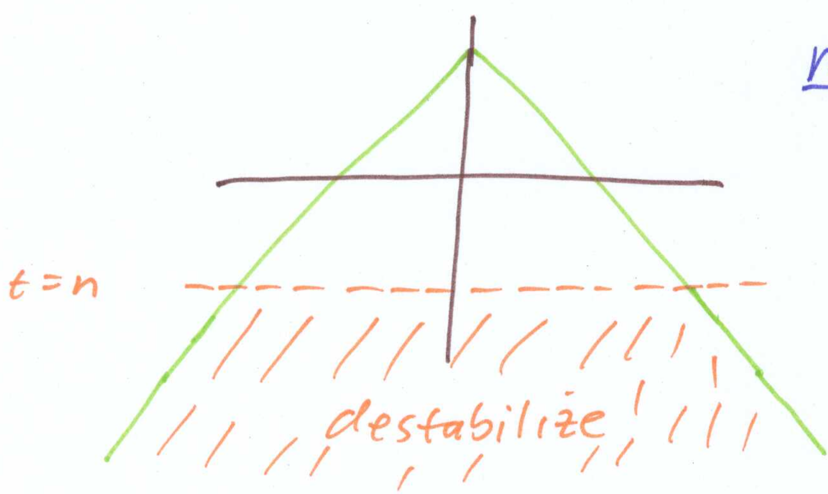
$\mathrm{Im}\,\Phi =$  ?

(Ng has potential examples)

2. Are there non-destabilizable L with $\Phi(L)$ in interior of $\mathrm{Im}\,\Phi$?

 ?

3. Given K is there an n such that $L \in \mathcal{L}(K)$ with $tb(L) < n$ implies L destabilizes?



note: If true then $\mathcal{L}(K)$ "finitely generated."

(i.e. finitely many ~~Kot~~ knots that generate all others by stabilization)

4. Non-symmetric $\mathrm{Im}\,\Phi$?

Applications To Contact Geometry

Th$^{\underline{m}}$ (E - Van Horn-Morris):

For any closed oriented M there is a collection of knots $\mathcal{C}$ that "classify contact structures on M".

That is $\xi_1 \cong \xi_2$ on M if and only if

$$\left|\mathcal{L}_{(r,t)}(K,\xi_1)\right| = \left|\mathcal{L}_{(r,t)}(K,\xi_2)\right|$$

for all $K \in \mathcal{C}$ and $(r,t) \in \mathbb{Z} \times \mathbb{Z}$

examples:

1. Any one knot can determine if a contact structure ξ on M is tight or overtwisted.
2. There are 3 knots in T^3 that classify tight contact structures.

Question: Given M can you find a simple effective $\mathcal{C}$? (like for T^3)

Given a Legendrian L in (M, ξ)

let M' be obtained from M

by $tb(L) - 1$ surgery on L

there is a "natural" ξ' on M'

We say (M', ξ') is obtained from (M, L)

by Legendrian surgery

Main ingredient in

Th^{m} (Eliashberg '04, E '04):

$$\boxed{\text{A weak filling of a contact 3-manifold can be embedded in a closed symplectic manifold.}}$$

This, in turn, is a main ingredient of many applications of contact geometry to topology.

Big Question: Does Legendrian surgery on a tight contact manifold always yield a tight contact manifold?

(No, but ...)

Overtwisted Contact Structures

A Legendrian knot L in an overtwisted contact manifold (M,ξ) is called **loose** if $\xi|_{M-L}$ is overtwisted

otherwise it is **non-loose**.

(from here on we say $L = L'$ if there is a contactomorphism of ξ sending L to L'.)

Th$^{\underline{m}}$ (folk):

$\boxed{\text{for any } (n,m) \in \mathbb{Z}\times\mathbb{Z} \text{ there is a unique loose knot in } \Phi^{-1}(n,m) \text{ (of course need } n+m \text{ odd)}}$

So geography and botany not interesting for loose knots.

from now ~~one~~ on we consider Φ restricted to $\mathcal{L}^{nl}(K)$

non-loose subset of $\mathcal{L}(K)$

Fact: On S^3 the overtwisted contact structures are in one-one correspondence with $\mathbb{Z}$

$$\{\ldots \xi_{-1}, \xi_0, \xi_1, \xi_2 \ldots\}$$

Th^m (Eliashberg - Fraser '09):

(S^3, ξ_n) for $n \neq 1$ contains no non-loose unknots

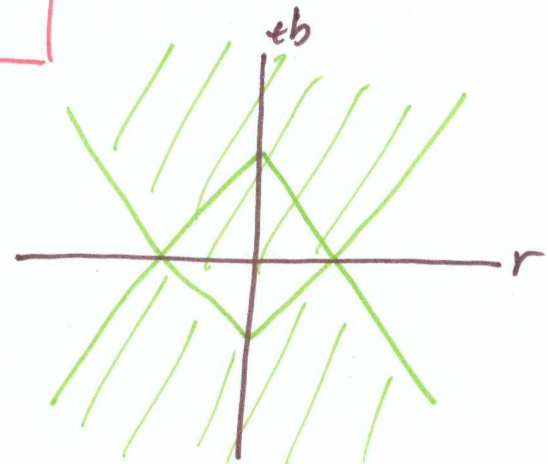
Non-loose unknots in (S^3, ξ_1) are

Th^m (Swiathowski, Dymara):

If L is non-loose then

$$-|tb| + |r| \leq -\chi$$

$\mathrm{Im}\,\Phi|_{\mathcal{L}^{ne}} \subseteq$



Th$^{\underline{m}}$ (E '10):

for any overtwisted ξ on ξ there is a knot type K and n,m such that

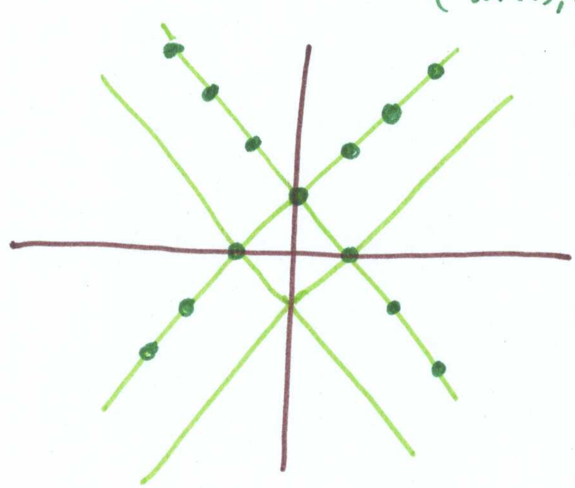
$$\left| \Phi^{-1}(n,m) \cap \mathcal{L}^{ne} \right| = \infty$$

Th$^{\underline{m}}$ (E '10):

In (S^3, ξ_0) let K be 

then for all $n \neq -1$

$$\mathcal{L}_{(-(n+1),n)}(K) = \{ L^{\pm}_{n,i} \}_{i \in \mathbb{N}}$$



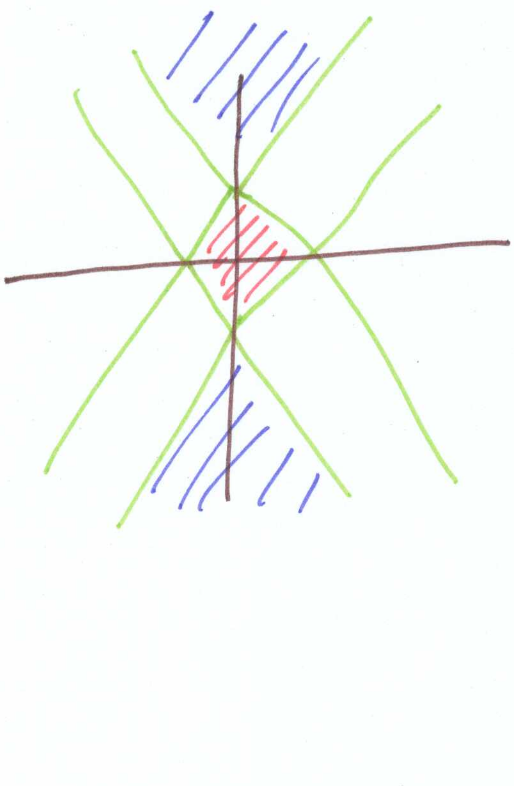
- $S_{\pm}(L^{\pm}_{n,i}) = L^{\pm}_{n-1,i}$.
- $S_{\mp}(L^{\pm}_{n,i})$ all the same.
- all Heegaard-Floer invariants of $L^{\pm}_{n,i}$ are 0.

Questions:

1. Given M and K are there only finitely many overtwisted structures on M that can have non-loose knots in $\mathcal{L}(K)$?

   (Kaloti: yes for positive torus knots)

2. Given M and K are there only finitely many overtwisted contact structures such that $\Phi^{-1}(m,n)$ can be infinite? (maybe ≤ 2?)

3. 

   a) Any non-loose in /// ? /// ?

   b) Finite botany off of the green lines? How about for /// ?