Branched Coverings

let X, Y be two n-manifolds

a proper map $f: X \to Y$ is a **d-fold branched covering** if

$f^{-1}(\partial Y) = \partial X$

1) $\exists$ a CW-complex $B \subset Y$ (called **branch locus**) of codimension 2

also called branch locus (or sometimes "branch cover")

s.t. $f^{-1}(B) \subset X$ also a codimension 2 CW-complex

2) $f|_{Y - f^{-1}(B)} : (X - f^{-1}(B)) \to (Y - B)$ is a degree d covering map

Remark: We will frequently add conditions to B and f (e.g. B a submanifold, f smooth ...)

Th^m (Alexander 1920)

> Every closed, oriented (PL) n-manifold M is a branched cover of S^n

Remarks: 1) PL means Piecewise linear this means M is homeomorphic to an n-complex (s.t. the link of every vertex is a sphere)

2) the branched cover map is PL (linear on simplicies, maybe after subdividing)

3) All smooth manifolds are PL (but can map be made smooth?)

4) Not all topological manifolds are PL (are they branched covers over S^n?)

but for $n = 1, 2, 3$, all manifolds are PL

Sketch of proof:

We do for $n=2$, general case similar idea.

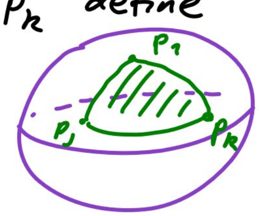
Given a surface Σ_g let $\mathcal{T}$ by a simplicial complex as above

let $v_1, \dots, v_n$ be the verticies of $\mathcal{T}$

let $p_1, \dots, p_n$ be points on S^2 (round S^2 in $\mathbb{R}^3$)

in "general position" : i.e. 1) no 2 are antipodes
2) no 3 lie on some great circle

note: ~~any~~ for any i,j,k, p_i, p_j, p_k define a triangle $\Delta_{i,j,k}$



$\overline{S^2 - \Delta_{ijk}}$ is also a triangle Δ'_{ijk}

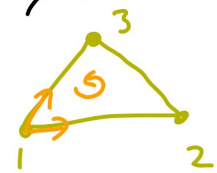
the points $p_1 \dots p_n$ give S^2 a triangulation and the Δ_{ijk} are simplicies or <u>unions</u> of simplicies in triangulation

also if v_i, v_j, v_k are vertices of a simplex σ^2_{ijk} in $\mathcal{T}$ then we get "linear" maps

$$f_{ijk} : \sigma_{ijk} \to \Delta_{ijk} \quad \text{and}$$
$$f'_{ijk} : \sigma_{ijk} \to \Delta'_{ijk}$$

orient Σ_g and S^2

given σ a 2-dim'l simplex in $\mathcal{T}$ choose ordering of vertices v_i, v_j, v_k s.t. or$^{\underline{n}}$ on σ agrees with one induced by ordering



if or$^{\underline{n}}$ on Δ_{ijk} from ordering of vetices agrees with or$^{\underline{n}}$ on S^2 then define $f: \sigma \to S^2$ by f_1

if not by f_2

you can check f is a covering maps from $\Sigma_g - \{v_i\}$ to $S^2 - \{p_i\}$

i.e. a branch covering map!

<u>Hint</u>: only really need to check edges. ▩

Main Questions:

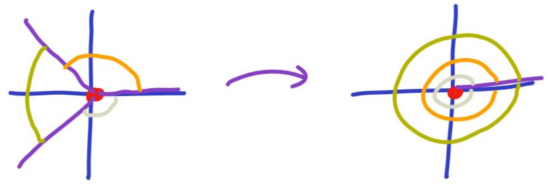
1) Can we upgrade th$^{\underline{m}}$ to the smooth world with branch set a submanifold?

2) How can we describe branch covers and "see" them?

2 Dimensional Case:

<u>examples</u>:

1) $f: \mathbb{C} \to \mathbb{C} : z \mapsto z^n$ is an n-fold branched cover with branch locus 0



Remark: a) branch locus of will always be collection of points
we assume all branch points have neighborhood where they look like $z \mapsto z^n$ ($n =$ branching index)

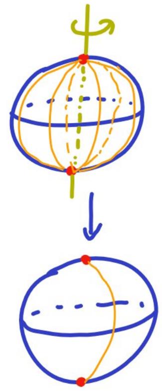
Question: is this necessary or can we alwas arrange this?

b) We will take our branch covering maps smooth.

2) $S^2 \subset \mathbb{C} \times \mathbb{R}$

$f: S^2 \to S^2$
$(z, x) \mapsto (z^d, x)$

d-fold cover branched over 2 points



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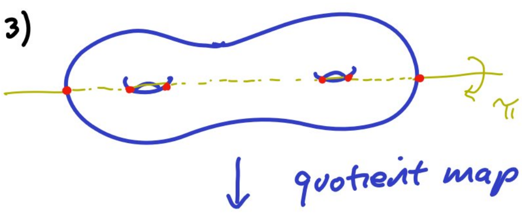
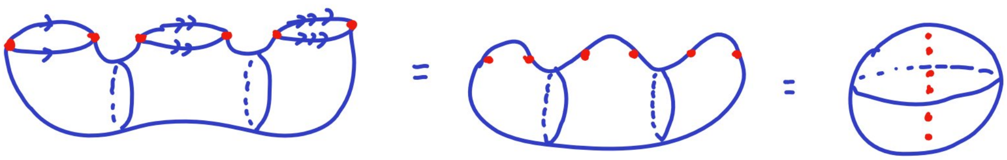


Th^m: Any oriented closed surface of genus g is a 2-fold cover of S^2 branched over $2g+2$ points



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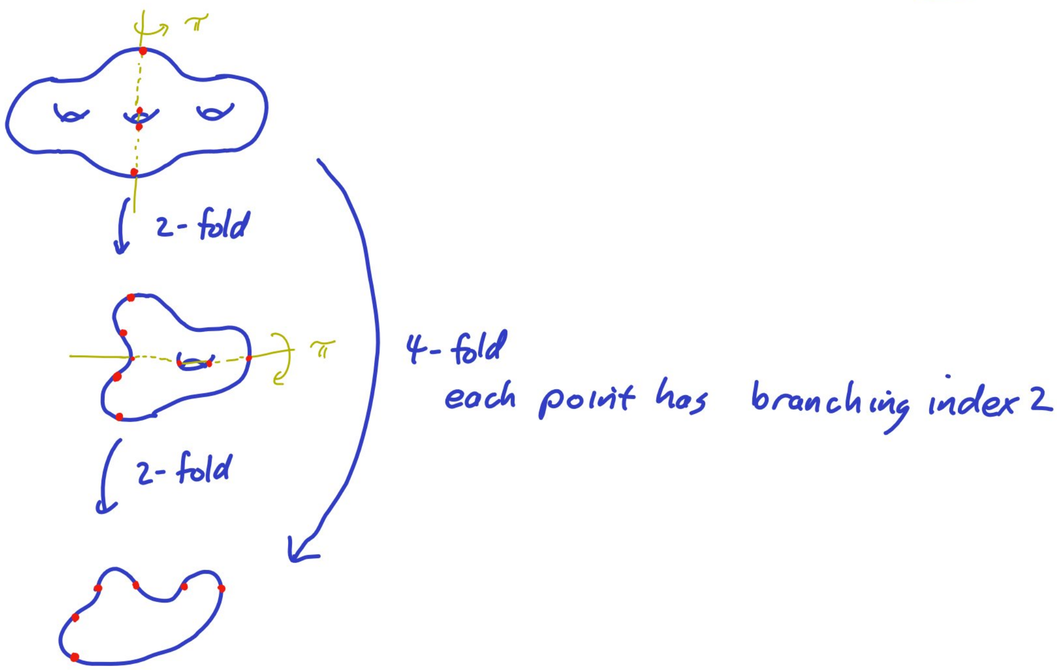

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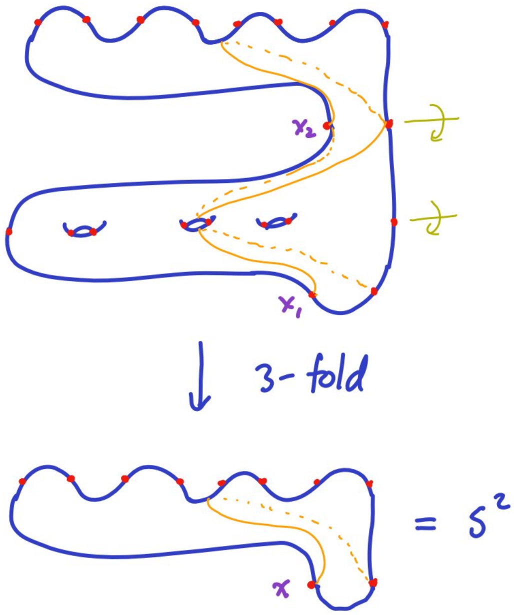


$f^{-1}(x) = \{x_1, x_2\}$

branching index $x_1 = 1$ i.e. <u>no</u> branching

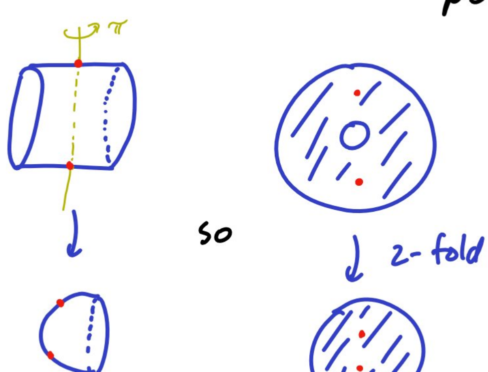
" " $x_2 = 2$

a branched cover $f: X^n \to Y^n$ is called simple of degree d if $f^{-1}(y)$ always consists of d or $(d-1)$ points

(note if d, then every point in $f^{-1}(y)$ is "regular"

if $d-1$, the all but one point regular and other point has branching index 2)

6)



$A = S^1 \times I$

D^2

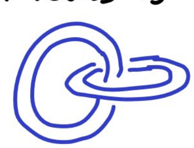
this is a very useful example!

for now it has a simple application:

$$\begin{array}{ccc} A \times A & \xrightarrow{\text{on } \partial} & T^3 \\ \downarrow \text{ 2-fold} & & \downarrow \\ A \times D^2 & \longrightarrow & S^1 \times S^2 \\ \downarrow \text{ 2-fold} & & \downarrow \\ D^2 \times D^2 & \longrightarrow & S^3 \end{array}$$

so $\exists$ 4-fold branched cover $T^3 \to S^3$

you can check branch locus is



and $S^1 \times S^2$ a 2-fold branched cover over S^3
with branch locus 

Riemann-Hurwitz Formula

let $p: X \to Y$ be a branced cover of surfaces

$B = \{y_1, \dots, y_k\} \subset Y$ the branch set

$p^{-1}(B) = \{x_1, \dots, x_n\} \subset X$, $a_i = |p^{-1}(y_i)|$

let d_i = branch index at x_i

d = fold of cover

Then
$$\chi(X) = d\,\chi(Y) - \sum_{i=1}^{n}(d_i - 1)$$
$$= d(\chi(Y) - k) + a_1 + \dots + a_k$$

Proof: note $\chi(Y - B) = \chi(Y) - k$

so $\chi(p^{-1}(Y-B)) = d\,\chi(Y-B) = d(\chi(Y) - k)$

formula for covers

now add back points in $p^{-1}(B)$ there are $a_1 + \dots + a_k$ ✓

for other formula note if $p^{-1}(y_i) = \{x_{i_1}, \dots x_{i_m}\}$ then

$d = d_{i_1} + \dots + d_{i_m}$ N.B.: each b.p. gives a partition of d

since there are k points y_i we see $kd = \sum_{i=1}^{n} d_i$

also $n = a_1 + \dots + a_k$

$\therefore -\sum_{i=1}^{n} d_i - 1 = -kd + n = -kd + \sum a_i$ $\therefore$ done

note: RH formula restricts possible coverings

e.g. if $\Sigma_g \to S^2$ is a branced cover and $g \geq 1$

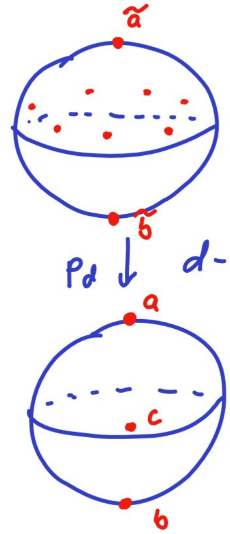
then there are ≥ 3 branch points.

(since $d(2-k) + a_1 + \dots + a_k > 0$ if $k \leq 2$)

Proposition

Any oriented surface is a branched cover of S^2 branched over 3 points.

Proof:



$p_d \downarrow$ d-fold branched over $\{a,b\}$

note: $p_d^{-1}(\{a,b,c\}) = \{\tilde{a}, \tilde{b}, c_1, c_2, \dots c_d\}$

now given a surface Σ of genus g

we know $\exists$ a 2-fold cover $\Sigma \xrightarrow{f} S^2$ branched over $2g+2$ points

let these points be $c_1 \dots c_{2g+2}$ in $p_{2g+2}^{-1}(c)$

then $p_{2g+2} \circ f : \Sigma \to S^2$ is desired cover ▦

Remark: any surface is

fix fold of cover

1) 2-fold cover over S^2 branched along some set
2) a cover of S^2 branched over 3 points (fold of cover not fixed)

fix branch locus

3) branch cover with branch locus a submanifold

Questions:

1) Is an n-manifold an n-fold cover of S^n?
(with branch locus a submanifold?)
2) Is there a submanifold $\Sigma \subset S^n$ such that all n-manifolds are a cover of S^n branched over Σ?
3) Given M is there a branched cover over S^n with branch set a submanifold?

Preview: In dimension 3: answer Yes to all 3!

In dimension 4: answer almost Yes to all 1 and 3
(nothing known about 2?)

In higher dimensions answer is sometimes No to all questions

Nice application of RH formula

example: We can use RH formula to figure out what a given surface is.

e.g. $\Sigma_n = \{ [x:y:z] \in \mathbb{C}P^n \mid x^n + y^n + z^n = 0 \}$ Fermat curve

Claim: Σ_n is a surface of genus $\frac{(n-1)(n-2)}{2}$

1^{st} Σ_n is a closed surface:

$f: \mathbb{C}^3 \to \mathbb{C} : (x,y,z) \to x^n + y^n + z^n$

$\forall p \in f^{-1}(0)$ with $p \neq 0$ df_p surjectie

so $f^{-1}(0) - (0,0,0)$ a 4-manifold

f homogeneous $\Rightarrow$ $p \in f^{-1}(0) \Leftrightarrow zp \in f^{-1}(0)$
$\forall z \neq 0$

$\Sigma_n = \overbrace{f^{-1}(0) - (0,0,0)}^{X} / (\mathbb{C}-0) \subset \mathbb{C}P^2$ makes sence

note action of $\mathbb{C}$ is free and

proper (i.e. $X \times (\mathbb{C}-0) \to X \times X$
$(p,z) \mapsto (p, z \cdot p)$
proper map)

$\therefore$ Σ_n a real 2-manifold
(complex 1-manifold $\therefore$ oriented)

2^{nd} compute genus:

note we get a map $p' : (\mathbb{C}P^2 - \{[0:0:1]\}) \longrightarrow \mathbb{C}P^1 \cong S^2$
$[x:y:z] \longmapsto [x:y]$

(if you have not done this before check p is an $\mathbb{R}^2$-bundle)

$[0:0:1] \notin \Sigma_n$ $\therefore$ p' induces a map

$p = p'|_{\Sigma_n} : \Sigma_n \to S^2$

for $[x_0 : y_0] \in \mathbb{C}P^1$,

$p^{-1}([x_0:y_0]) = \{ [x_0 : y_0 : z] \in \mathbb{C}P^2 \mid z^n = -(x_0^n + y_0^n) \}$

when $x_0^n + y_0^n \neq 0$ this consists of n points
(can check $dp \neq 0$ at these so covering space)

when $x_0^n + y_0^n = 0$ only one point
this happens at $[1 : \varepsilon_n]$

where $\varepsilon_n = n^{th}$ root of unity
that is at n points

so p is an n-fold branched cover
branched over n-points
(and each b.p. has 1 preimage)

$$\therefore \chi(\Sigma_n) = n(\chi(S^2) - n) + n$$
$$= 2n - n^2 + n = -n^2 + 3n$$

so $g(\Sigma_n) = \dfrac{2 - \chi(\Sigma_n)}{2} = \dfrac{n^2 - 3n + 2}{2}$
$$= \frac{(n-1)(n-2)}{2}$$

Question:

Suppose you are given Σ_g and Σ_h oriented surfaces
$d \in \mathbb{N}$
k-partitions of d : $(d_{11}, \ldots d_{1\ell_1}), \ldots (d_{k1}, \ldots d_{k\ell_k})$
satisfying the RH formula:

$$\chi(\Sigma_g) = d\,\chi(\Sigma_h) - \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell_i}(d_{ij} - 1)\right)$$

then is there a branched cover $\Sigma_g \to \Sigma_h$ realizing this data? How many?

Answer: Husemoller '62 says Yes if $h \geq 1$ (maybe show later)
if $h = 0$, i.e. $\Sigma_h = S^2$, then sometimes NO
e.g. $\Sigma_g = S^2$, $\Sigma_h = S^2$
$d = 4$
$(2,2), (2,2), (3,1)$ (we check this later)
but $(2,2), (2,2), (2,2)$ OK (example above)

Open Problem:
When is the answer to Question Yes for $\Sigma_h = S^2$?
Is it true if degree prime?

Hurwitz problem

Monodromy and covers:

Start with non-branched covers

given $\begin{array}{c}\tilde{X}\\ \downarrow p\\ X\end{array}$ be an n-fold covering space

we get a homomorphism

$$m: \pi_1(X, x_0) \to S_n$$

where S_n = symmetric group of n letters

as follows: let $p^{-1}(x_0) = \{x_1, \dots, x_n\}$

given any loop $\gamma: S^1 \to X$ based at x_0

let $\gamma_i: [0,1] \to \tilde{X}$ be the lift of γ based at x_i

define $\sigma_\gamma: \{1, \dots n\} \to \{1, \dots n\}$

$i \longmapsto$ index on $\gamma_i(1)$

easy to check σ_γ only depends on homotopy class of γ, $[\gamma]$

define $m_p([\gamma]) = \sigma_\gamma$

note: if we relabel points in $p^{-1}(x_0)$ then m_p is conjugated

exercise: m_p is transitive

i.e. $x, y \in p^{-1}(x_0)$, then $\exists\, \gamma$ st. lift of γ based at x has end pts $\{x, y\}$

now given any homomorphism

$$f: \pi_1(X, x_0) \to S_n$$

let $H = \{g \in \pi_1(X, x_0) : f(g)(1) = 1\}$ this is a subgroup of $\pi_1(X, x_0)$

if f is trasitive (i.e. i,j then $\exists [\gamma]$ s.t. $f([\gamma])(i) = j$)

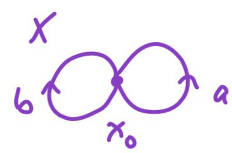
then H is index n

$\exists$ a covering space $\tilde{X}$ associated to H

and $\begin{array}{c} \tilde{X} \\ \downarrow p \\ X \end{array}$ is n-fold and its monodromy

is (conjugate) to f

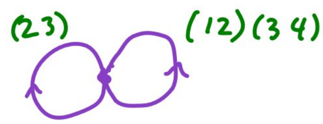
example:



$\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$

so $\pi_1(X, x_0) \longrightarrow S_n$ determined by image of generators

i.e. just label edges with elements of S_n



build cover as follows:

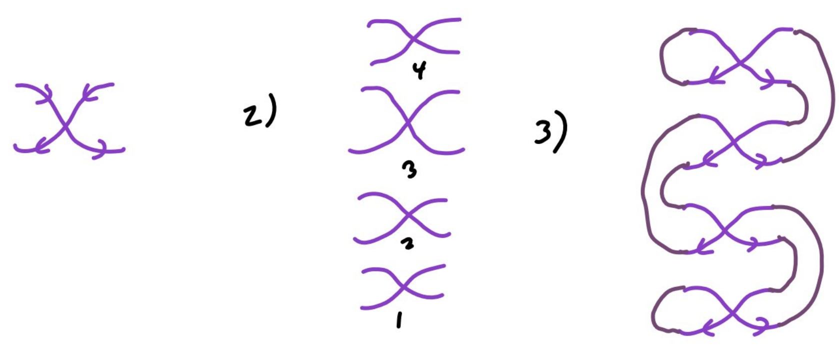
1) cut each edge at a point (branch cut)

2) take n copies

3) glue copies according to data

e.g.

1) 2) 3)



now for branched covers.

lemma

$\boxed{\begin{array}{l} B \subset \Sigma^2 \text{ any finite set} \\ \text{any cover of } \Sigma - B \text{ extends to} \\ \text{a branced cover over } \Sigma \end{array}}$

exercise: prove this (cone)

so branched covers are determined by the monodromy of their associated covers

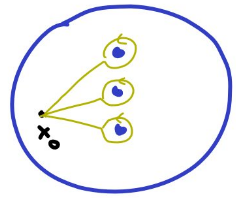
example:

consider D^2

if $B = \{x_1, \dots x_n\}$, then $\pi_1(D^2 - B) \cong F_n$ free group

so homomorphism $\pi_1(D^2 - B) \to S_n$ determined by image of generators

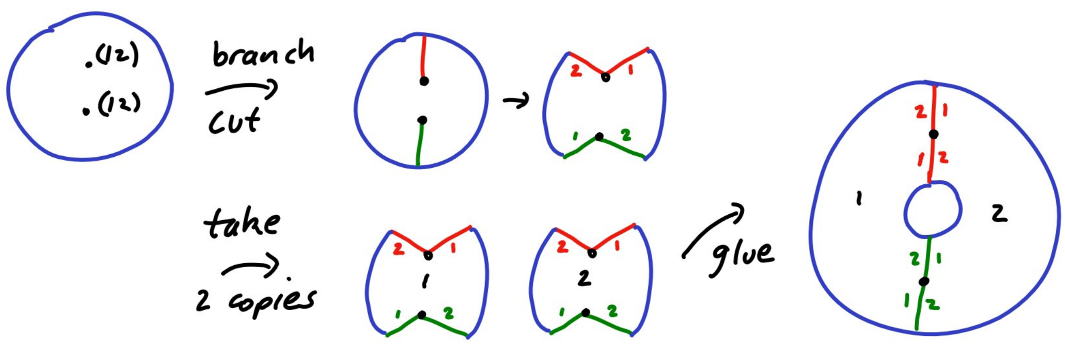
i.e. just label points with elements of S_n

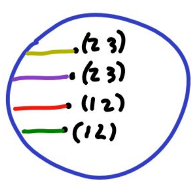


generators

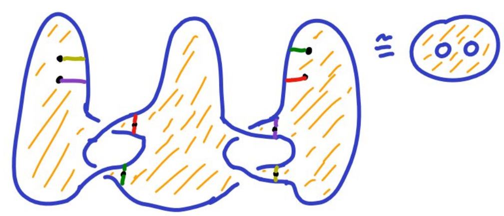
called Hurwitz system

eg



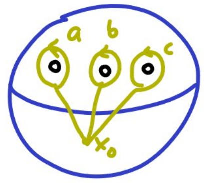


exercise show cover is



back to S^2 4-fold cover with data $(2,2), (2,2), (3,1)$ can't be realized

$$\pi_1(S^2 - (3\text{points})) = \underset{a}{\mathbb{Z}} * \underset{b}{\mathbb{Z}}$$



$c = ab$

need $a \mapsto$ product 2 transpositions
$(12)(34)$ (can assume this by relabling)

$b \mapsto$ product 2 transpositions

$(12)(34)$
or $(13)(24)$
or $(14)(23)$

can't be $(12)(34)$ or image only $(12)(34)$

if $(13)(24)$ then c goes to $(14)(23)$

but need $c \mapsto (a\,b\,c)$
same for $(14)(23)$

2 branched covers
$$f_0, f_1 : X \to Y$$
are equivalent if $\exists$ homeomorphisms / diffeomorphisms
$$g : X \to X$$
$$h : Y \to Y$$

such that
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "f_1"] \\ Y \arrow[r, "h"'] & Y \end{tikzcd}$$

Th$^{\underline{m}}$ (Hurwitz)

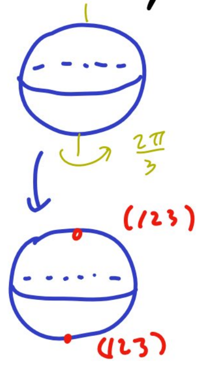
2 branched covers $f_0, f_1 : \Sigma \to \Sigma'$ of surfaces are equivalent	$\Leftrightarrow$	$\exists$ homeomorphism $h : (\Sigma', B_{f_0}, *) \to (\Sigma', B_{f_1}, *)$ (branch locus, base point) such that $m_0 = m_1 \circ h_*$ mod conjugation where $m_i = m_{f_i\|_{\Sigma' - B_i}}$ monodromy and $h_* : \pi_1(\Sigma' - B_0, *) \to \pi_1(\Sigma' - B_1, *)$

exercise prove this (just fact about covers and monodromy)

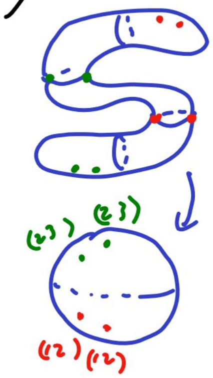
a branched cover $X \xrightarrow{f} Y$ of degree d is called simple if
$$\forall y \in Y, \ |f^{-1}(y)| = d \text{ or } d-1$$
i.e. if $b \in Y$ a branch point then $f^{-1}(b) = \{x_1, \dots x_{d-1}\}$ with
x_1 having ramification 2
x_i "regular points" $i \geq 2$

for a surface this corresponds to having a Hurwitz system with only transpositions

e.g.



not simple



we call 2 branched coverings

$$f_0, f_1 : X \to Y$$

b-homotopic if $\exists$ a homotopy $f_t : X \to Y$ $t \in [0,1]$ through branched coverings

compact open topology
given $C \subset X$ compact
$U \subset Y$ open
let $W(C,U) = \{f \in C^0(X,Y) \mid f(C) \subset U\}$
$\{W(C,U)\}_{C,U}$ sub-basis for topology on $C^0(X,Y)$

Th$^{\underline{m}}$

stable — 1) let $f: X \to Y$ be a simple branched cover, Y compact
$\exists$ an open set U in compact-open topology on $C^0(X,Y)$ s.t. any branched cover in U is simple

generic — 2) any branched covering of a surface is b-homotopic to a simple branched cover

Proof:

1) let $\{B_\alpha\}$ be balls in Y such that $f^{-1}(B_\alpha) =$ union of n or $(n-1)$ disjoint balls in X.

for each α choose smaller ball $A_\alpha \subset \text{int } B_1$

Y compact so $\exists$ finite #, $A_1, \dots, A_k$ s.t. $Y = \bigcup \text{int } A_i$

$\exists$ open set in compact-open topology on $C^0(X,Y)$
s.t. g in open set $\Rightarrow$ for each A component $f^{-1}(A_i)$
$g(A) \subset \text{int } B_i$

$\therefore |g^{-1}(x)| \geq n-1 \quad \forall x \in Y$

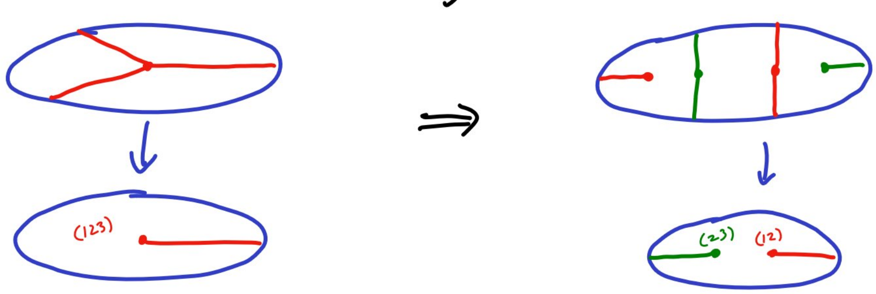
also $\deg g = \deg f = n$ (since if f, g close enough they are homotopic)

so g an n-fold branched cover $\therefore$ simple

2) note just need to perturb cover near branch points

near branch point $z \mapsto z^n$

change to $z \mapsto z^n + \varepsilon$ now path to rest of cover

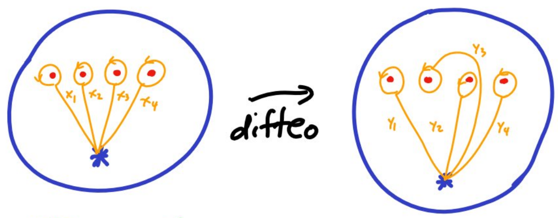


key to 3D branched covers

Th^{m}

> If Σ a connected surface then any two simple branched covers over S^2 of the same degree are equivalent and b-homotopic

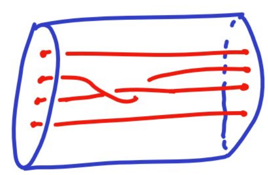
for proof need Hurwitz moves



$x_1 = y_1$
$x_2 = y_2 y_3 y_2^{-1}$
$x_3 = y_2$
$x_4 = y_4$

$$\left.\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right\} \longmapsto \begin{matrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{matrix} \in S_n \qquad\qquad \left.\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right\} \longmapsto \begin{matrix} \alpha_1 \\ \alpha_2 \alpha_3 \alpha_2^{-1} \\ \alpha_2 \\ \alpha_4 \end{matrix}$$

so cover corresponding to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}, \alpha_2, \alpha_4)$ are equivalent and b-homotopic:



if $\Sigma \to S^2$ a simple branched cover then its Hurewitz data is

$$\alpha_1, \alpha_2, \dots, \alpha_k$$

s.t. (a) $\alpha_1 \alpha_2 \dots \alpha_k = $ identity in S_n
(b) α_i transpositions
(c) $\{\alpha_1 .. \alpha_k\}$ generate a transitive subgroup of S_n

from above we know we can make change

$$(*) \quad \alpha_i, \alpha_{i+1} \longrightarrow \alpha_i \alpha_{i+1} \alpha_i, \alpha_i \qquad (\text{or} \quad \alpha_{i+1}, \alpha_{i+1}\alpha_i\alpha_{i+1})$$

also if x, y transpositions can make following changes

$$(\dagger) \quad x, y, y, \longmapsto y, y, x \qquad \text{obvious}$$

$$(\ddagger) \quad x, x, y, y \longmapsto yxy, yxy, y, y$$

only interesting case

$$(\underset{x}{(ab)}, \underset{x}{(ab)}, \underset{y}{(bc)}, \underset{y}{(bc)}) \mapsto (ab), \overbrace{(ab)(bc)(ab)}^{(ac)}, (ab)(bc)$$
$$\mapsto (ac), (bc), (ab), (bc)$$
$$\mapsto \underset{yxy}{(ac)}, \underset{yxy}{(ac)}, \underset{y}{(bc)}, \underset{y}{(bc)})$$

proof of th^{m} follows from

lemma

> any sequence $(\alpha_1, \dots, \alpha_k)$ satisfiying (a)-(c) can be

put in the form $(12)(12)\dots(12)(12)(23)(23)(34)(34)\dots(n-1\ n)(n-1\ n)$

by a sequence of $\circledast$ moves

Proof:

since $\alpha_1 \dots \alpha_k =$ identity we know $k = 2m$ even

Claim: can arrange $\alpha_{2i-1} = \alpha_i$ $\forall i = 1, \dots m$

proof: suppose $\alpha_1 = (x_1\ x_2)$

must be another term with x_2 choose one closest to α_1

$\alpha_i = (x_2\ x_3)$

can conjugate so $i = 2$, if $x_3 = x_1$ we reduce m by 1 done by induction

if $x_3 \neq x_2$, then term (x_3, x_4) closest to α_2 can assume α_3

if $x_4 = x_1$, then

$(x_1\ x_2)(x_2\ x_3)(x_3\ x_1) \rightsquigarrow (x_1\ x_2)(x_1\ x_2)(x_2\ x_3)$

& done by induction

if $x_4 = x_2$, then

$(x_1\ x_2)(x_2\ x_3)(x_3\ x_2) \overset{(\dagger)}{\rightsquigarrow} (x_2\ x_3)(x_2\ x_3)(x_1\ x_2)$

& done by induction

if $x_4 \neq x_1$ or x_2, then continue to get

$(x_1\ x_2)(x_2\ x_3) \dots (x_{r-1}, x_r)$

since finite number of x_i's eventually $x_r = x_j$ some $j < r$ and done as above.

now $(\dagger)$ and $(\ddagger)$ imply can replace any pair by a conjugate of any pair

since $\alpha_1 \dots \alpha_k$ generate transitive group can find

$(1\ x_1)(1\ x_1) \dots (x_1\ x_2)(x_1\ x_2) \dots (x_2 x_3)(x_2 x_3) \dots (x_{r-1}\ 2)(x_{r-1}\ 2)$

so get pair $(12)(12)$ at front

remove pair

can assume remaining terms transitive on $2, \dots, n$

indeed if $x \in \{3, \dots n\}$ and can't send $2 \mapsto x$ then can send 1 to x by transitivity of original α_i's

so as above can use $\circledast$ to get $(1x)(1x)$

move to front: $(12)(12)(1x)(1x) \overset{(\ddagger)}{\longrightarrow} (12)(12)(2x)(2x)$

now if 1 in remaining terms then they are transitive on $\{1, 2, \dots n\}$

and can remove another $(12)(12)$ pair

keep going untill no 1's left

now can pull off some number of (23)(23) pairs

if more than one then we see

$$(12)(12)(23)(23)(23)(23) \to (12)(12)(13)(13)(23)(23)$$
$$\to (12)(12)(12)(12)(23)(23)$$

continuing gives desired result!

Thm:

let $\Sigma \xrightarrow{\phi} S^2$ be a simple cover of degree at least 3 and $f:\Sigma\to\Sigma$ any homomorphism then $\exists$ a homeomorphism $h: S^2\to S^2$ and $\tilde{h}:\Sigma\to\Sigma$ such that

$$\begin{tikzcd} \Sigma \arrow[r, "\tilde{h}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circ"] & \Sigma \arrow[d, "\phi"] \\ S^2 \arrow[r, "h"] & S^2 \end{tikzcd}$$

and $\tilde{h}$ isotopic to f

Moreover $\phi\circ\tilde{h}$ is b-homotopic to ϕ.

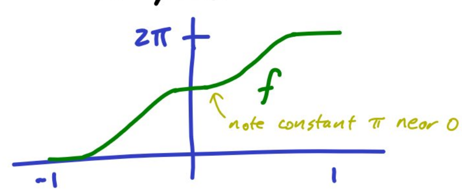
Proof: Start with degree 3

recall a Dehn twist τ_γ along a curve γ in a surface Σ is defined as follows: let N be a nbhd of γ and $f:[-1,1]\times S^1 \to N$ a diffeomorphism

$\gamma = \{0\}\times S^1$ or$^{=}$ pres

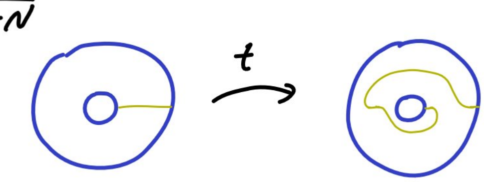
set $t:[-1,1]\times S^1 \to [-1,1]\times S^1$

$(t,\theta)\mapsto (t,\theta - f(t))$ where



and

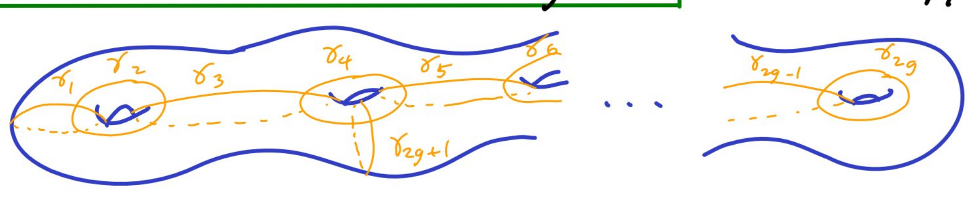
$$\tau_\gamma:\Sigma\to\Sigma: p\mapsto \begin{cases} f\circ t\circ f^{-1}(p) & p\in N \\ p & p\in \overline{\Sigma - N}\end{cases}$$



Thm (Humphries '79)

let Σ be a closed oriented surface then any diffeomorphism is isotopic to a composition of Dehn twists about $\gamma_1,\dots\gamma_{2g+1}$

exercise: Up to isotopy τ_γ only depends on the isotopy class of γ



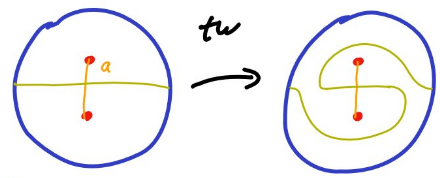
So to prove our theorem we just need to see can realize all τ_{γ_i} in a branched cover

let $a \subset \Sigma$ be an arc

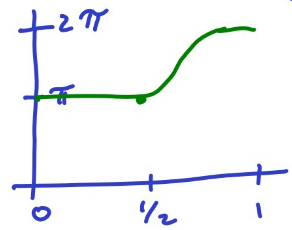
a has a nbhd N diffeomorphic to D^2 = unit disk in $\mathbb{R}^2$ with $a = \{(0,t) \mid |t| \leq 1/2\}$

a **arc twist** is the diffeomorhism

$$tw_a : \Sigma \to \Sigma : p \mapsto \begin{cases} tw(p) & p \in N \cong D^2 \\ p & p \in \overline{\Sigma - N} \end{cases}$$



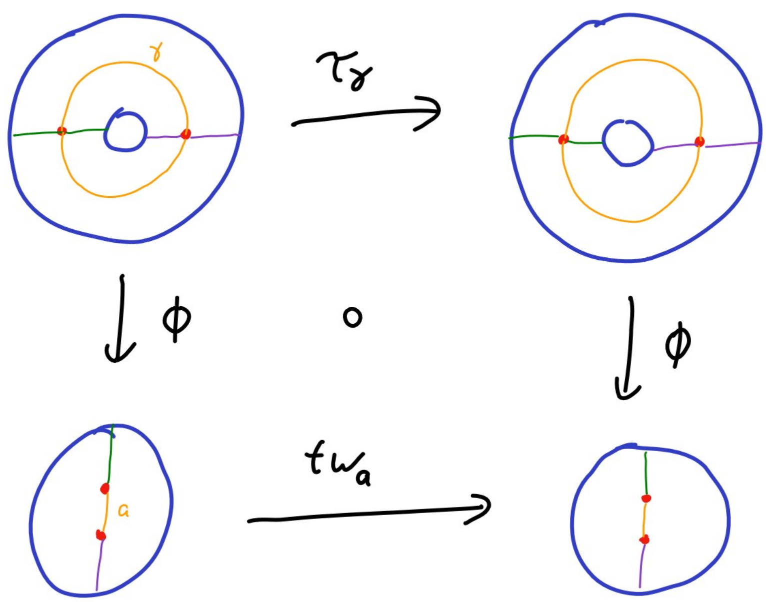
where $tw : D^2 \to D^2$
$(r,\theta) \mapsto (r, \theta + f(r))$ where



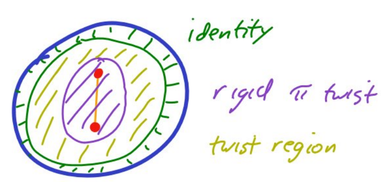
exercise: Show isotopy type (rel ∂a) of tw_a only depends on the isotopy class (rel ∂a) of a

lemma:

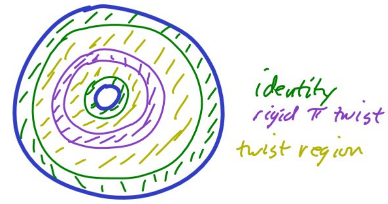
let $\phi : [-1,1] \times S^1 \to D^2$ be the cyclic 2-fold cover branched over $\{(0, \pm 1/2)\}$
the twist map tw_a where $a = \{(0,t) \mid |t| \leq 1/2\}$ is covered by the Dehn twist τ_α where $\alpha = \{0\} \times S^1$



Proof: Note tw_a = identity near ∂
rigid π rotation near a
twist on rest



and τ_γ = identity near ∂
rigid π rotation near γ
twist on rest



on green ϕ is $2\times$ "identity" so τ & tw clearly related

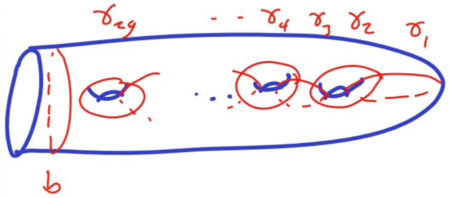
on purple ϕ is where branching occures
but clearly rigid rotation related by ϕ

on yellow ϕ is $2\times$ "identity" and half twists lift

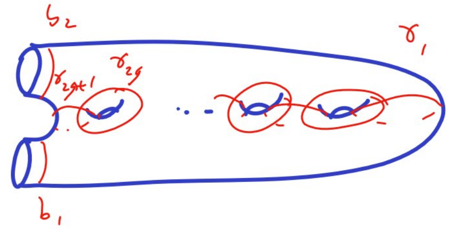
Asside: Relations in the mapping class group

Th$^{\underline{m}}$:

if surface is closed then $(\tau_{\gamma_1}\circ\dots\circ\tau_{2g})^{4g+2}\simeq id$



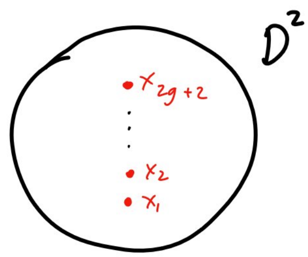
$(\tau_{\gamma_1}\circ\dots\circ\tau_{2g})^{4g+2} = \tau_b$ ← isotopic



$(\tau_{\gamma_1}\circ\dots\circ\tau_{2g+1})^{2g+2} = \tau_{b_1}\circ\tau_{b_2}$ ← isotopic

Proof:

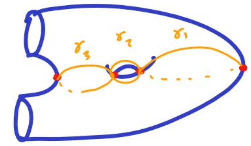
Consider

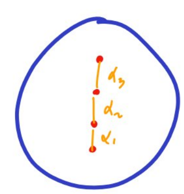


2-fold branched cover is

let α_i = arc x_i + x_{i+1}

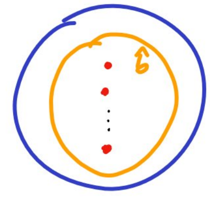
note α_i lifts to γ_i
e.g.





so tw_{α_i} covered by τ_{γ_i}.

also note if $\hat{b}$ is



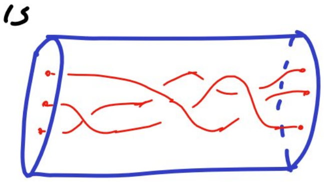
then $\tau_{\hat{b}}$ lifts to $\tau_{b_1}\circ\tau_{b_2}$ since a nbhd of $\hat{b}$ lifts to a nbhd of $b_1\cup b_2$

note: $\tau_{\hat{b}}$ isotopic to $\left(tw_{\alpha_1} \circ \dots \circ tw_{\alpha_{2g+1}}\right)^{2g+2}$

to see this recall isotopy classes of diffeos $(D^2, \{x_i\}) \to (D^2, \{x_i\})$ correspond to "braids": given $f: (D^2, \{x_i\}) \to (D^2, \{x_i\})$

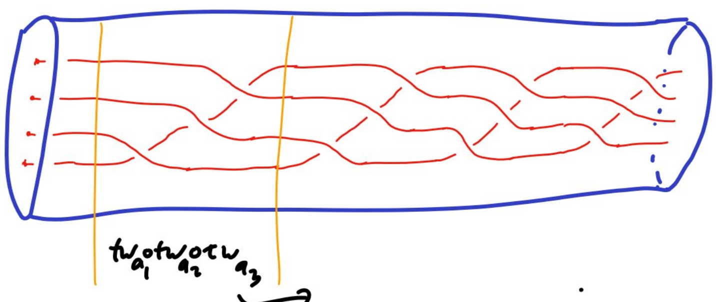
let $\phi_t : D^2 \to D^2$ be isotopy of f to id_{D^2}

"graph of ϕ_t" is



if two maps $f, g : (D^2, \{x_i\}) \to (D^2, \{x_i\})$ give same braid then f, g isotopic rel $\{x_i\}$

now braid for $\tau_{\hat{b}}$ is

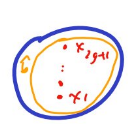


repeat 3 more times

so clearly $\tau_{\hat{b}} = \left(tw_{a_1} \circ tw_{a_2} \circ \dots \circ tw_{2g+1}\right)^{2g+2}$

lifting to 2-fold branched cover gives

$$\tau_{b_1} \tau_{b_2} = \left(\tau_{\gamma_1} \circ \dots \tau_{\gamma_{2g+1}}\right)^{2g+2}$$

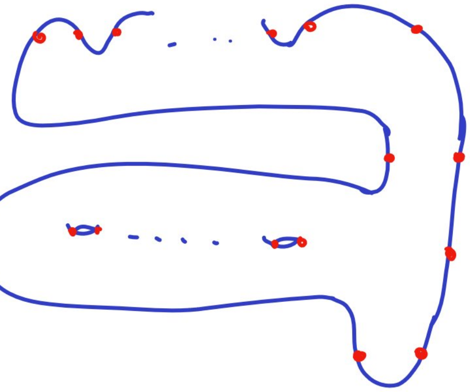
other case similar except for  $\tau_{\hat{b}}$ does not lift but $\tau_{\hat{b}}^2$ lifts to τ_b

now returne to proof of "any diffeo is lift of diffeo for simple cover of degree ≥ 3"

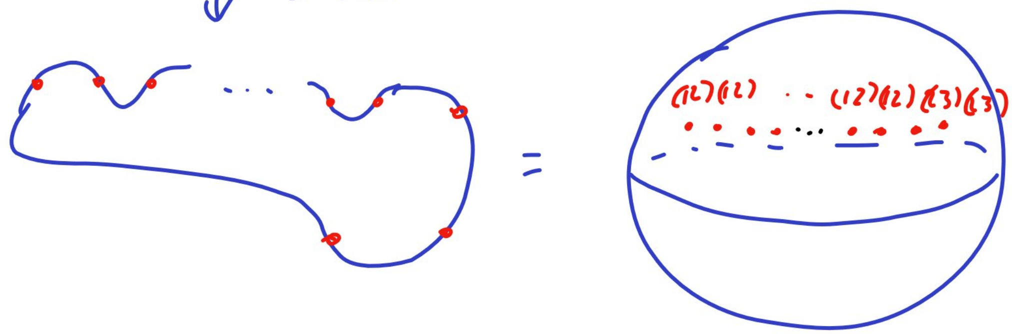
recall we consider degree 3 first

by previous theorem any degree 3 simple cover of Σ_g over S_2

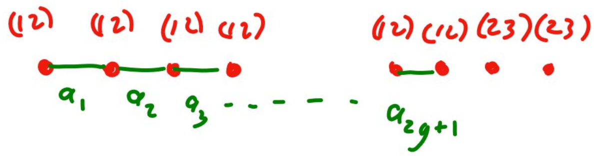
is equivalent to



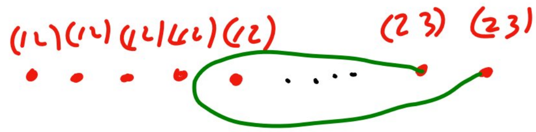
3 fold



let a_i be the arcs

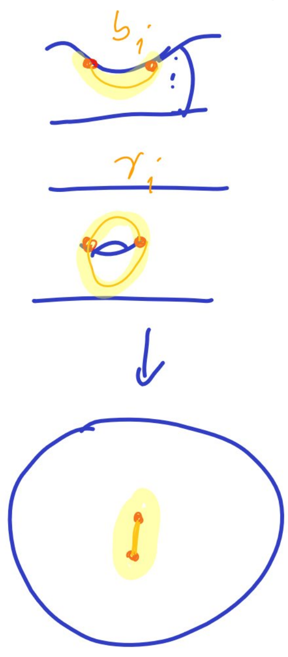


and a_{2g+2}

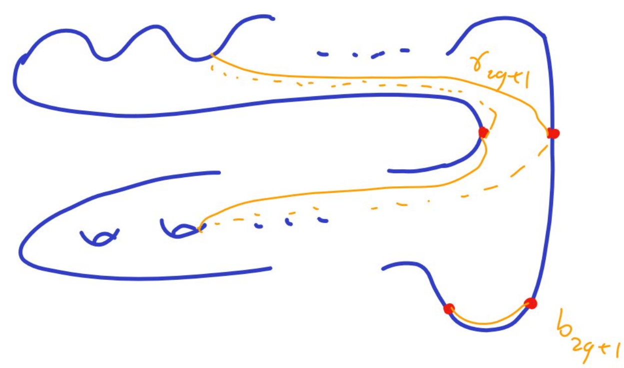


note tw_{a_i} $i \le 2g+1$ lifts to $tw_{b_i} \circ \tau_{\gamma_i} \simeq \tau_{\gamma_i}$

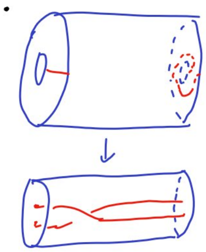
eg



and $\tau w_{a_{2g+2}}$ lifts to $\tau_{\gamma_{2g+1}} \circ tw_{b_{2g+1}} \simeq \tau_{\gamma_{2g+1}}$



so all the Humphries generators are lifts of diffeos of base
∴ all diffeos, up to isotopy, are
And can see "b-homotopy" as a "braid" as above.



<u>exercise</u>: check for higer fold simple covers

3 Dimensional case

Recall 3 general questions we have about branced covers in a given dimension are

1) Can M^n be realized as a cover of S^n branched over a submanifold?
2) If we fix the degree of the cover, is the answer yes?
3) Is there a "universal branch locus" $B \subset S^n$ that is a submanifold?
(i.e. every M^n is a branched cover of S^n branched over B)

for 2-manifolds we said:

Yes to 1)
Yes to 2) with degree 2
Yes to 3) with $B = 3$ pts

We will prove

Th^{m} (Hilden '76, Hirsch '74, Montesinos '76)

> Every closed orientable 3-manifold is a 3-fold simple branched cover of S^3 branched over a link

Remark: We can take the branch locus in the theorem to be a <u>knot</u>.

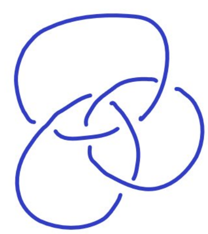
Th^{m} (Thurston 82?)

> There is a link L in S^3 such that any closed orientable 3-manifold is a branched cover of S^3 branched over L

Remarks:

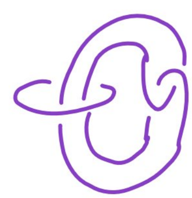
1) Hilden-Lozano-Montesinos '83 showed can take L to be a <u>knot</u> (much harder than in th^{m} above)

2) Universal links:



Borromean rings

whitehead link



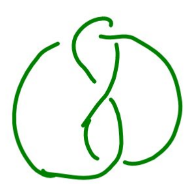


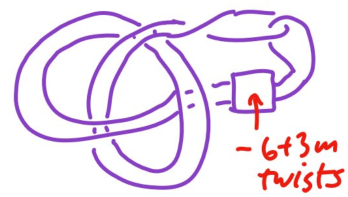
figure 8 knot

Any <u>2-bridge link</u> that is <u>not</u> a torus knot or link is universal

3) Torus knots and iterated cables of torus knots are <u>not</u> universal

4) There are cables of some knots that are universal
there are Whitehead doubles of some knots that are universal

e.g.



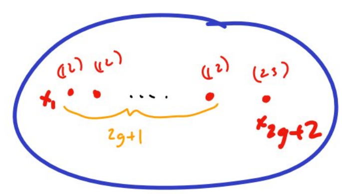
So there are universal satellite knots

Open problems:

1) Are all hyperbolic knots/links universal?
2) Is a knot universal $\Leftrightarrow$ a term in its JSJ decomposition is hyperbolic?

Proof of "3-fold cover" Th$^{\underline{m}}$:

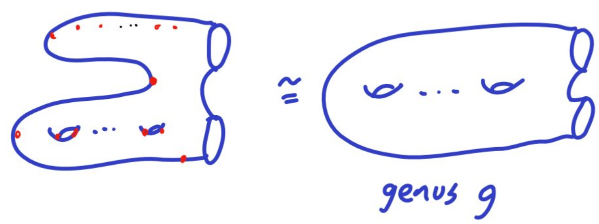
<u>note</u> branched cover of disk



D^2

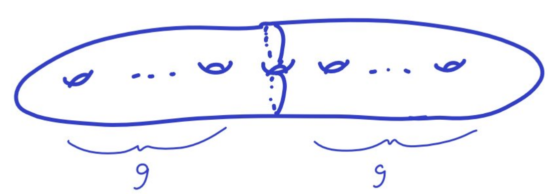
$B = \{x_1, \dots, x_{2g+2}\}$

is the surface

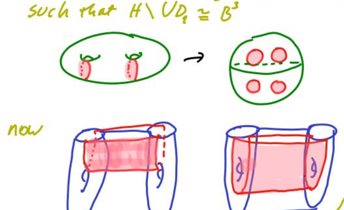


so cover $B^3 = (B^2 \times [0,1], B \times [0,1])$

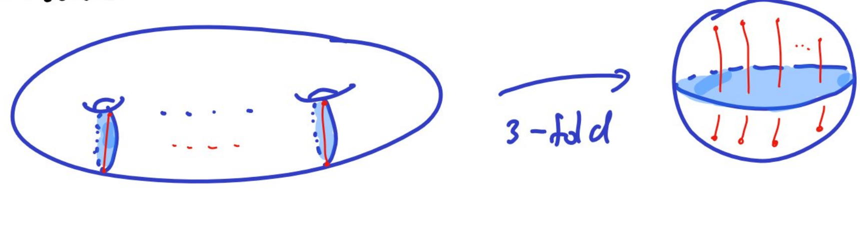
is $(\Sigma_g - (B^2 \cup B^2)) \times [0,1] \cong$ handlebody of genus $2g+1$



H a handlebody of genus g
$\Leftrightarrow$
$\exists$ disjoint proper embedding of g disks D_i such that $H \setminus \cup D_i \cong B^3$



so we have



"doubling" (take 2 copies and glue ∂'s by identity
i.e. $\partial(M \times I)$)

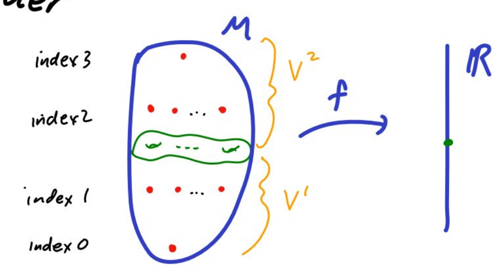
gives $\#_{2g+1} S^1 \times S^2 \longrightarrow S^3$ a 3-fold cover branced over unlink with $2g+2$ components

Fact: Every closed oriented 3-manifold has a Heegaard splitting: $M^3 = V_g^1 \cup_\phi V_g^2$ where V_g^i a handle body of genus g and $\phi: \partial V_g^2 \to \partial V_g^2$ a diffeomorphism

Remarks: 1) diffeo type of M only depends on isotopy class of ϕ

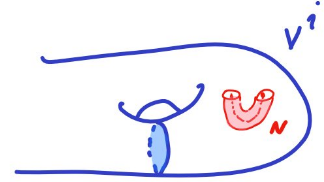
2) Can prove this using either

a) Morse theory



b) Triangulation of M: $V^1 =$ nbhd 1-skeleton, $V^2 = \overline{M - V^1}$

3) If M has splitting of genus g then it has splitting of any genus $\geq g$ by stabilization

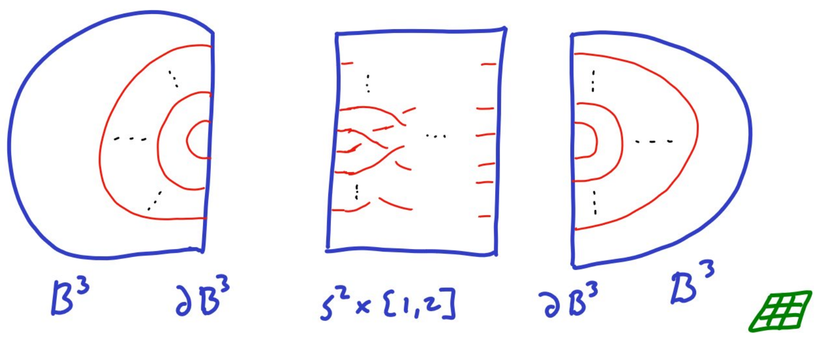


remove N from V^i and add it to V^{i+1}

4) from above we see a Heegaard splitting of $\#_{2g+1} S^1 \times S^2$ along a surface S_{2g+1}

So given M and a Heegaard splitting of genus $2n+1$ (can assume odd) the there is some diffeo $\phi: S_{2n+1} \to S_{2n+1}$ such that we can split $\#_{2n+1} S^1 \times S^2$ along S_{2n+1} and reglue by ϕ to get M.

Since any $\phi : S_{2n+1} \to S_{2n+1}$ can be realized by a b-homotopy of the 3-fold cover $S_{2n+1} \to S^2$ we can realize M as a branched cover over S^3 branched over



Monodromy

let $\mathring{M} \xrightarrow{\mathring{\phi}} \mathring{Y}$ be a covering map of degree d

(recall, determined by map $\pi_1(\mathring{Y}, *) \to S_d$) symmetric group

where $\mathring{Y} = Y - (\text{nbhd } L)$ for some link L

Lemma: $\boxed{\mathring{\phi} \text{ extends over } \mathring{M} \text{ to give a branched cover } \phi : M \to Y}$

Proof: let $S^1 \times D^2$ be a component of $(\text{nbhd } L)$

$\phi^{-1}(\partial(S^1 \times D^2))$ is a union of tori let T be one of them

so $\phi : T \to \partial(S^1 \times D^2)$ is a covering map

$\exists$ some smallest k such that $k(\{p\} \times \partial D^2)$ lifts to an embedded loop m in T

$\in S^1$

pick a diffeomorphism $\psi : S^1 \times S^1 \to T$ st.

$$\{p\} \times S^1 \to m$$

so $\phi \circ \psi : S^1 \times S^1 \to S^1 \times S^1$ is isotopic to

$$(\phi, \theta) \mapsto (a\phi + b\theta, k\theta) \quad \text{for } a, b \in \mathbb{Z}$$

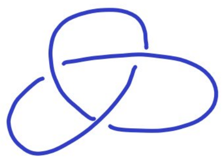
extends to $S^1 \times D^2 \to S^1 \times D^2$

$(\phi, (r, \theta)) \mapsto (a\phi + b\theta, (r, k\theta))$

So branched covers $M \to Y$ with branch set $B \subset Y$ a link are determined by their monodromy

$$\pi_1(Y - B, *) \to S_n$$

if $L \subset S^3$ is a link with diagram



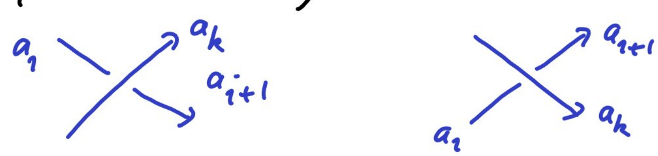
then the Wirtinger presentation of $\pi_1(S^3 - L)$ is

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$$

where we cyclically lable the arcs $a_1, \dots, a_n$ in the diagram (pick some orientation on components of L)

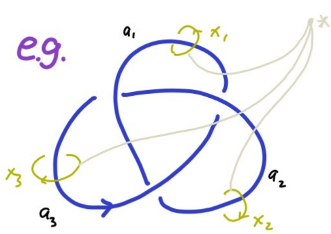
and x_i is the meridian to a_i

r_i is the following relation at the end point of a_i



$a_k a_i a_k^{-1} = a_{i+1}$ $\qquad$ $a_k^{-1} a_i a_k = a_{i+1}$

e.g.

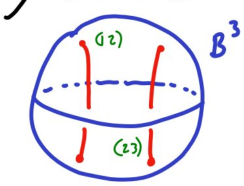


$\langle x_1, x_2, x_3 \mid a_3 a_1 a_3^{-1} = a_2,\ a_1 a_2 a_1^{-1} = a_3 \rangle$

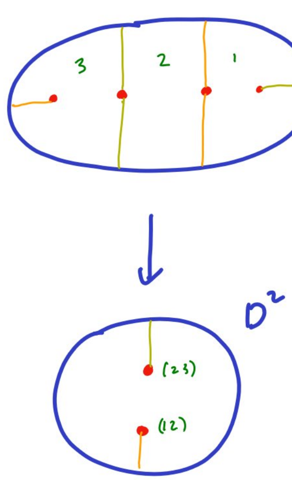
so a branched cover of S^3 branched over L is determined by labelling arcs in diagram by elts of S_n so they satisfy relations at crossings

similarly for a tangle in B^3

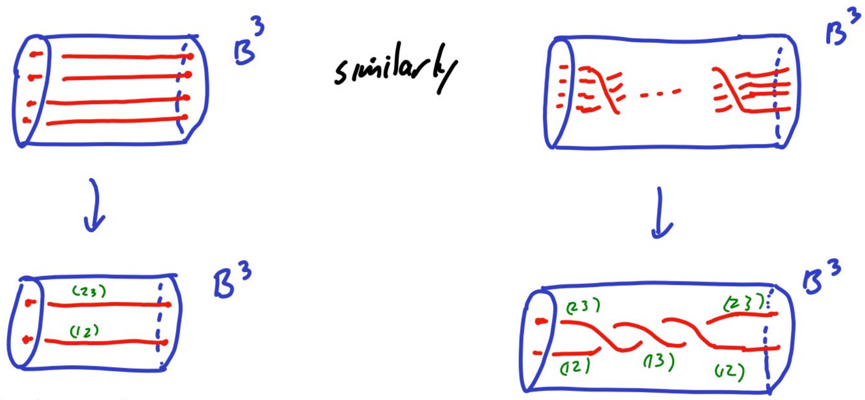
example: what is following branched cover?



to see, recall in 2D we have

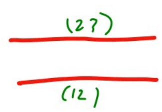


so we see



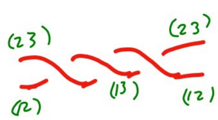
so if you see a lableled link diagram with

this:



in it

and replace it with



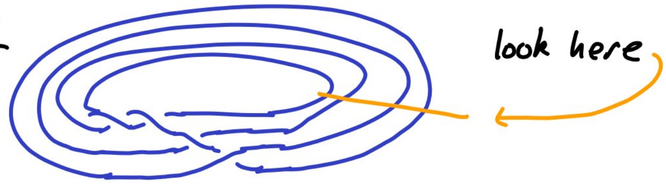
then in branched cover you are removing a ball (if 3-fold cover, otherwise $d-2$ balls in d-fold cover) and gluing back in a ball

$\therefore$ manifold in cover did not change!

Cor: every closed oriented 3-manifold is a 3-fold branched cover of S^3 branched over a <u>knot</u>.

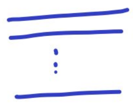
Proof: from above we know M is a 3-fold simple cover of S^3 branched over a link L (assume L has 2 or more components)

Braid L



look here

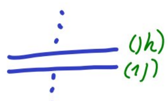
we see



each strand is labled (12), (23), or (13)

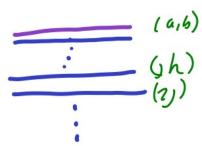
since this is a 3-fold cover at least 2 of the lables are used

so we see

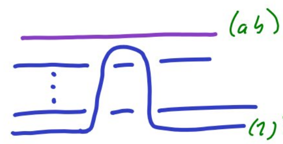


if these correspond to different components of L then add 3 twists as above to reduce # of components.

if same component then look for closest strand of different component



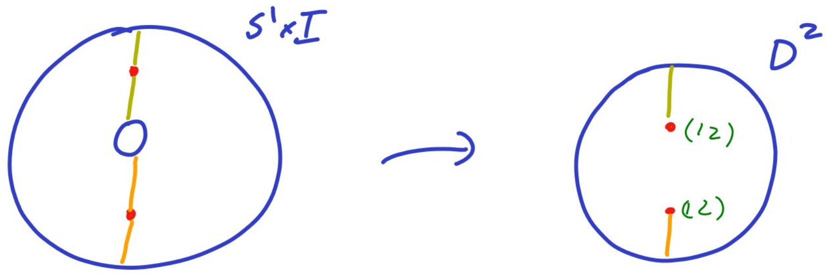
(a,b) different from (1j) or (jh) so can move appropriate strand up



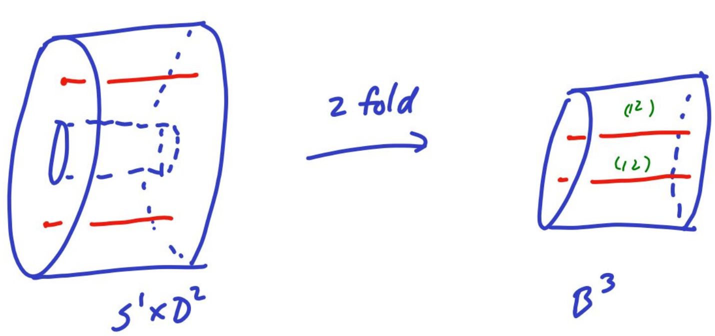
and add 3 twists as above

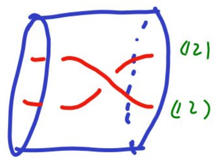
Another example:

recall we had

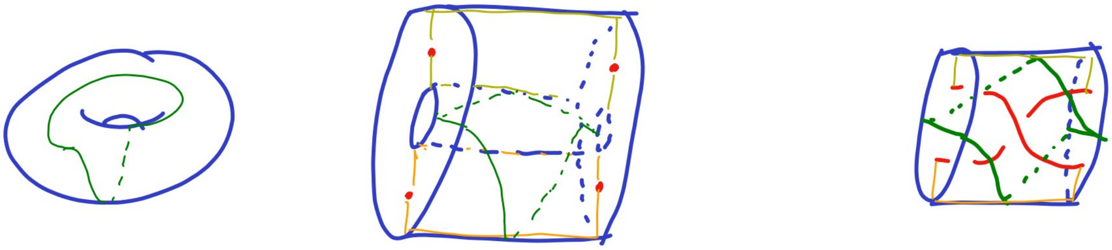


so we get

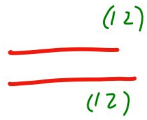


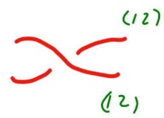
note cover of 

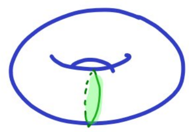
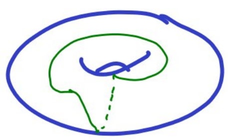
also a solid torus, but meridian is different

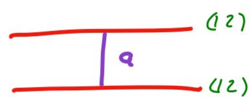


← bounds disk disjoint from branch locus

so if you see 

in the data for a branch cover and replace it with 

then in the cover you remove  and replace with 

i.e. if a is the arc 

then a lifts to a knot K_a in cover M

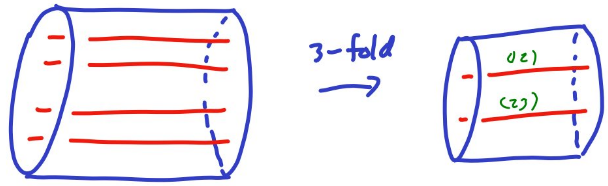
(and some arcs if fold of cover >2)

so changing branch set by twist changes the cover by ± 1 Dehn surgery on K_a

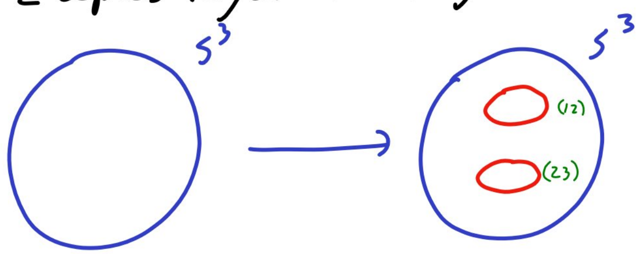
Cor:

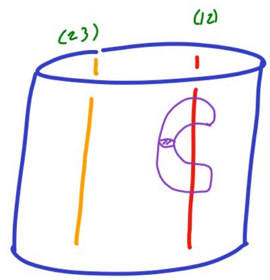
$$\boxed{\text{another proof that any closed oriented 3-manifold is a simple 3-fold cover over } S^3 \text{ branched along a link}}$$

Proof: recall we have



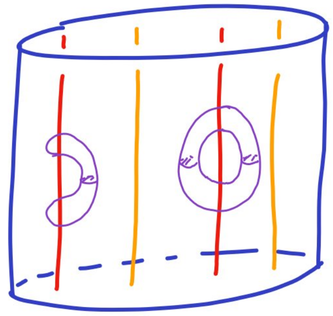
glue 2 copies together to get



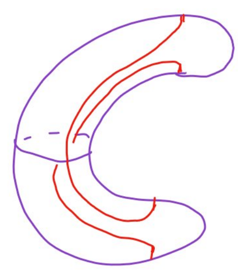


<u>note:</u> let B^3 be the ball shown

in cover you see



so changing the branch set by



will change cover by $\pm$ Dehn surgery on solid torus above depending on the twist done

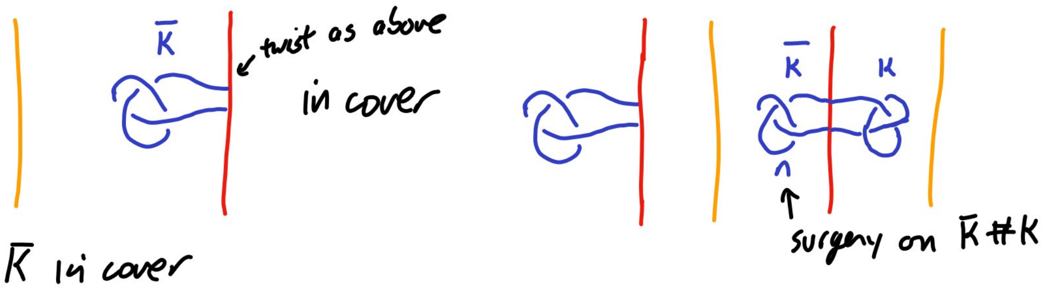
we need

Thm:

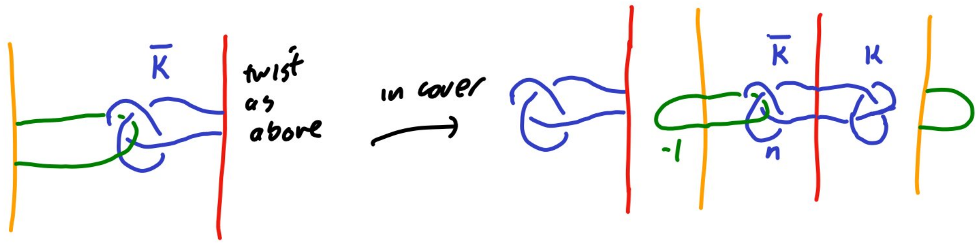
$$\boxed{\text{Any 3-manifold is obtained from } S^3 \text{ by surgery on some link}}$$

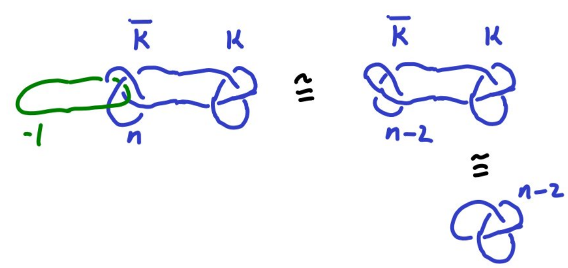
now given $K_1 \cup \dots \cup K_k$ framings $n_1 \dots n_k$ such that M is obtained from S^3 by surgery on K_i with framing n_i

Then we construct our cover as follows

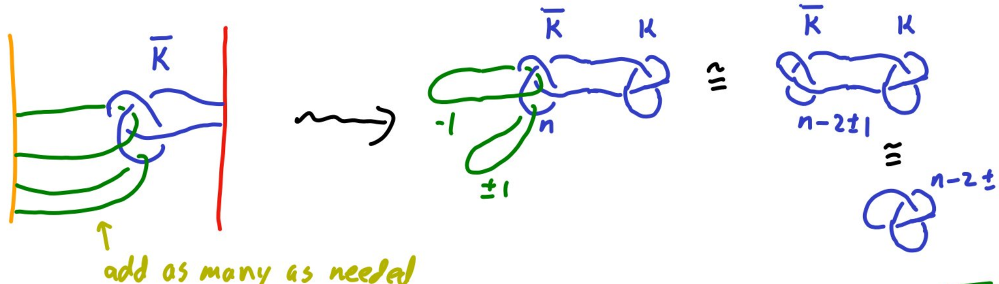


untie $\bar{K}$ in cover





now fix framing

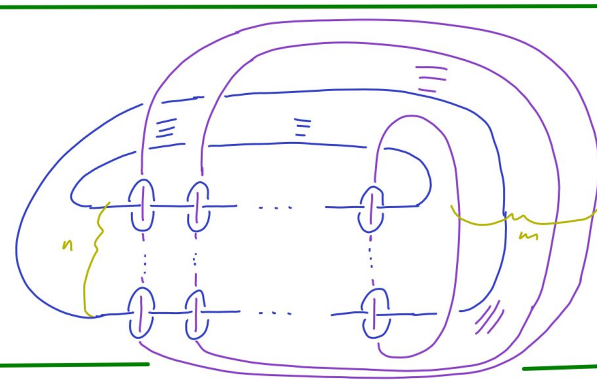


∎

We now consider the existance of a universal link

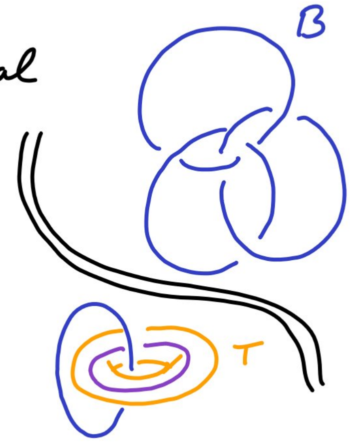
This follows directly from

lemma A: Any compact oriented 3-manifold is a simple branched cover over the link $L_{n,m}$ on right for some n and m



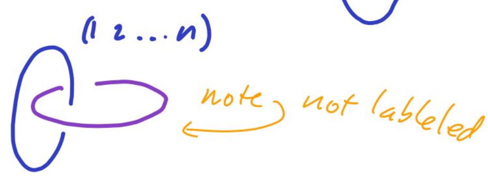
lemma B: there is a branched cover $S^3 \xrightarrow{p} S^3$ branched overe the Borromean rings B such that $L_{n,m}$ from lemma A is a subset of $p^{-1}(B)$

note: It cleanly follows that B is universal



Proof of B: let T be Heegaard torus for S^3 and H be the Hopf link st. H is the core of Heegaard tori

note: The branched cover

H 

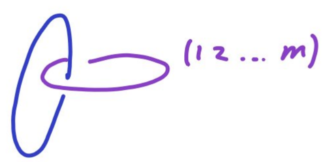
is cleanly S^3 ($S^3 = S^1 \times D^2 \cup S^1 \times D^2$

n-fold cover　　branch over core

$S^1 \times D^2 \;\cup\; S^1 \times D^2 = S^3$)

denote this $p_n : S^3 \to S^3$

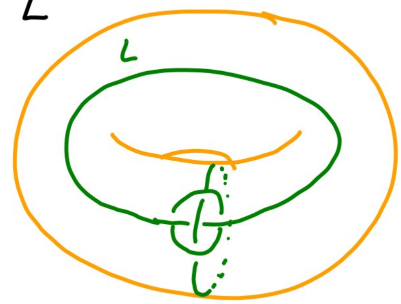
and $p_n^{-1}(H) = H$

similarly $q_m : S^3 \to S^3$ given by 

and $q_n^{-1}(H) = H$

$q_m \circ p_n : S^3 \to S^3$ is a branched cover over S^3 with branch set H

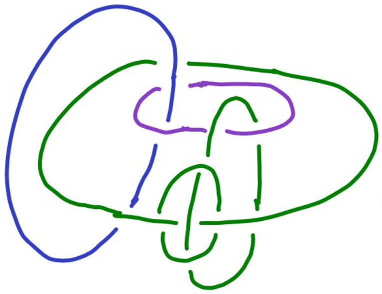
moreover $T \subset S^3$ (and hence a nbhd of T) just gets $m \cdot n$ fold covered by unwrapping "longitude" n times and "meridian" m times now in a nbhd of T consider link L'



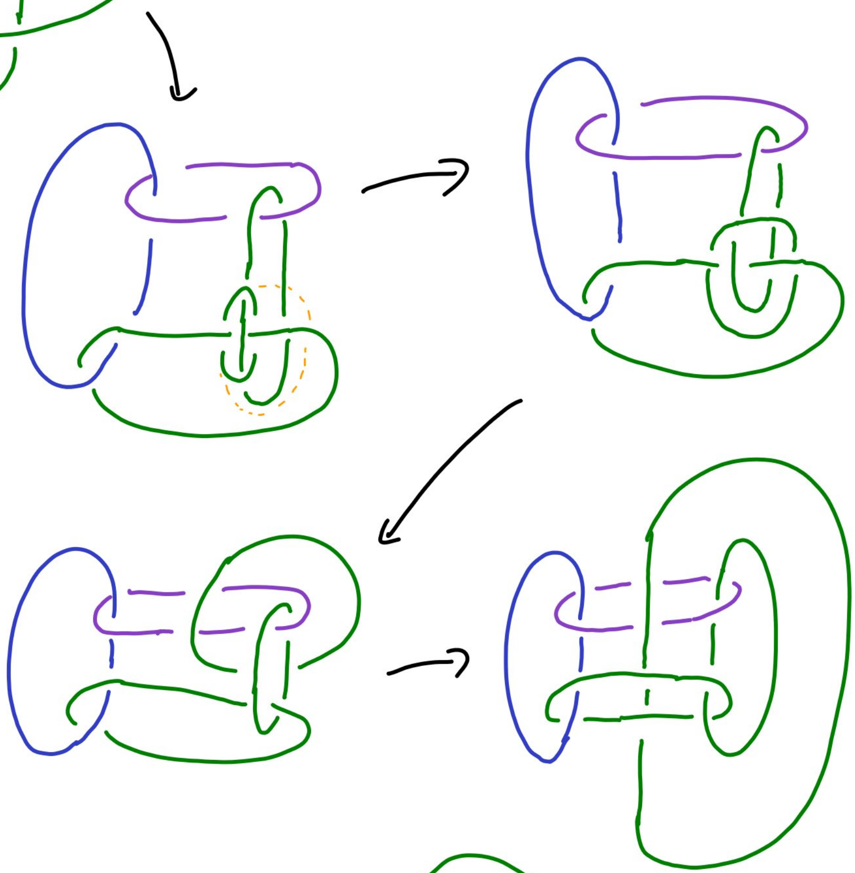
note $(q_m \circ p_n)^{-1}(L') = L_{n,m}$

from lemma A

so

L  is universal!

now we isotop



now note if B is

 U

then 2 fold cover of S^3 branched over U

is S^3 and B lifts to L

so B is universal!

to prove lemma A we need

Observation: Following changes to branching data don't affect covers

I)

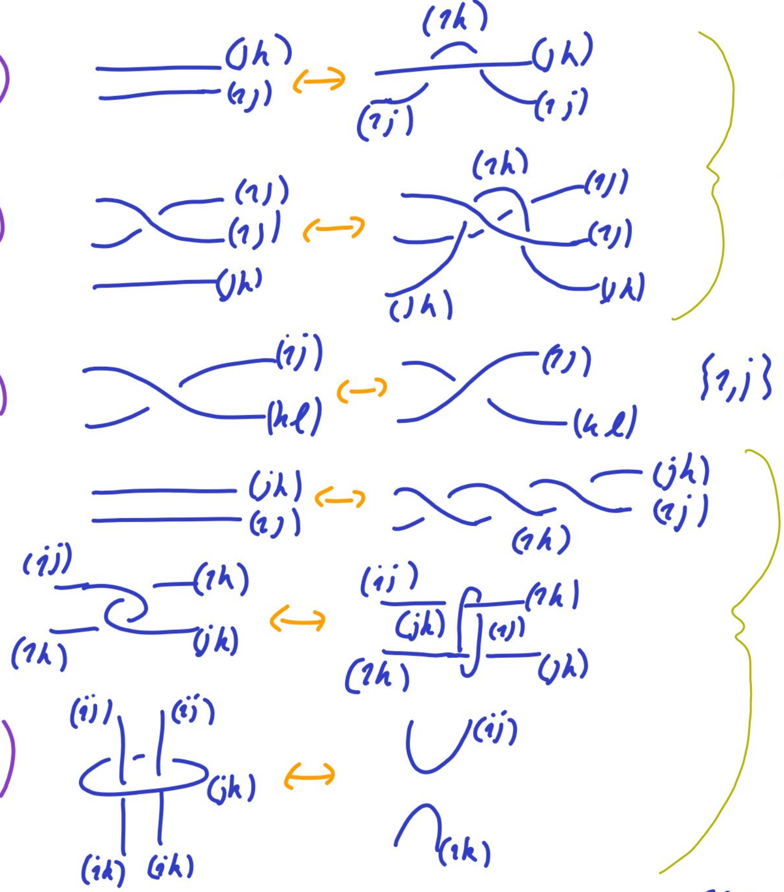
II)

III)

IV)

V)

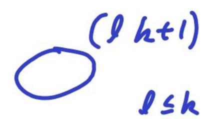
VI)

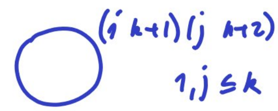


isotopy
(not doing anything)

$\{i,j\} \cap \{k,l\} = \emptyset$ sheets independent

preimage union of balls

VII) to a k-fold cover add 

or 

n k-sheeted cover

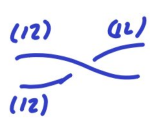
cut open on branch
cut disk & glue in
S^3 cut along disk

Proof of lemma A:

let $M \to S^3$ be a 3-fold simple cover with branch set L

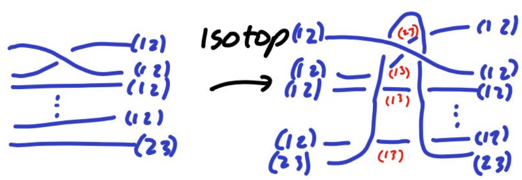
we can braid L 

Step I: Can assume label at a crossing are different

Proof: if you see 

then above or below see (23) see (13) or all labels in link same and not 3-fold cover

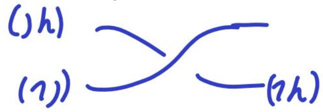
find closest strand with different label, now



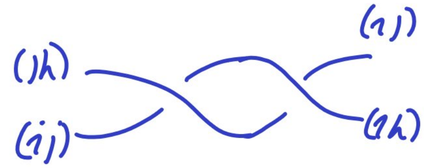
reduced number of "monocromatic" crossings

Step 2: Make all crossings positive

if not positive replace

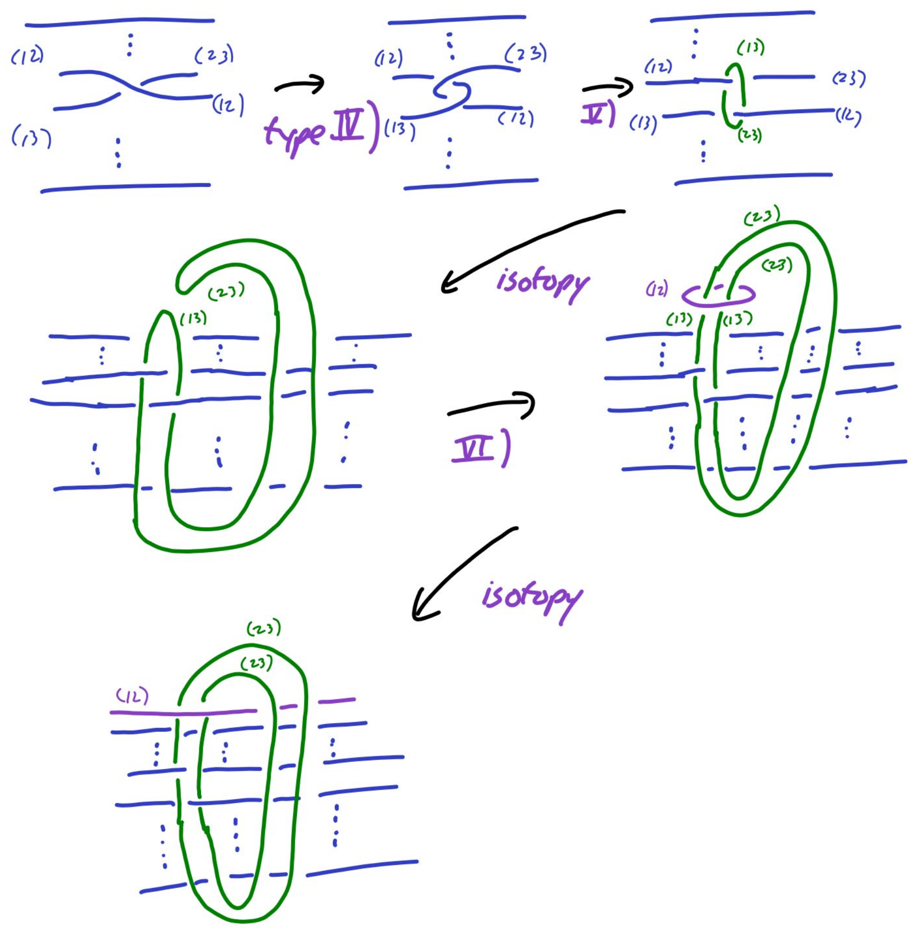


with

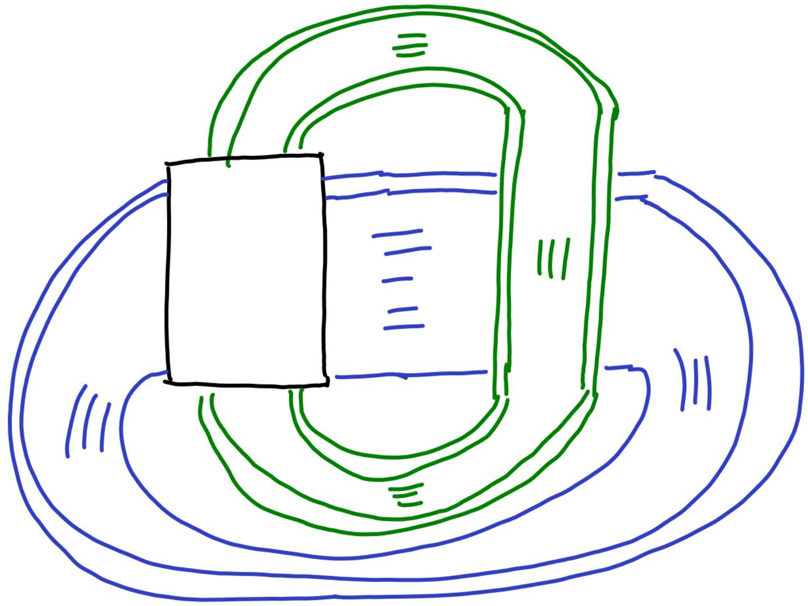


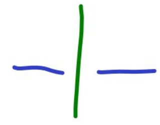
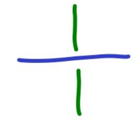
type IV) move

Step 3: Add strands to trivialize braid

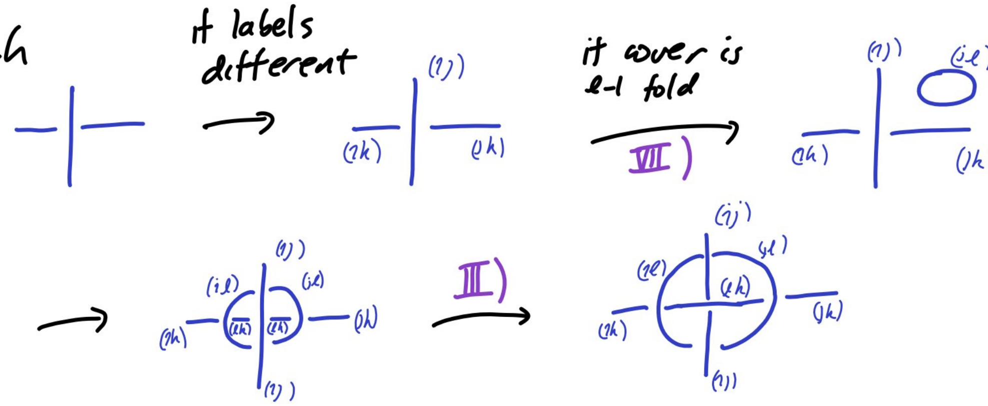


when we do this at all crossings we get

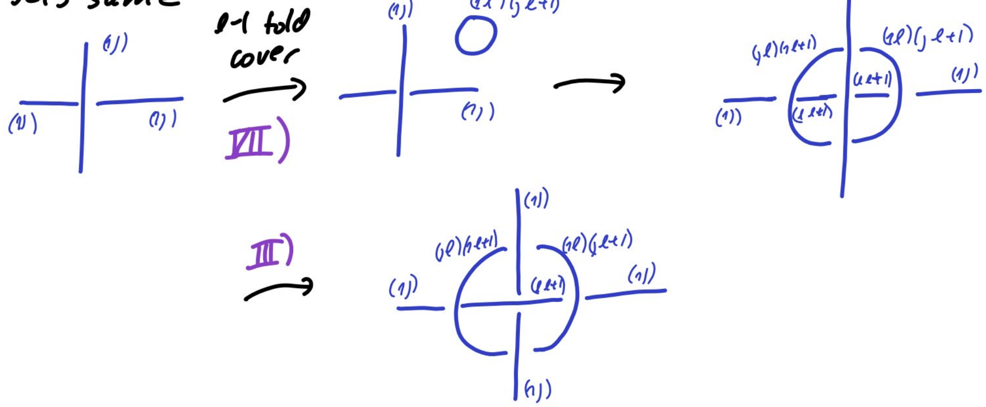


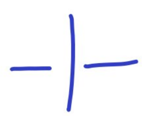
in box we see lots of  or 

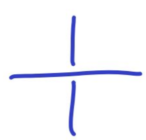
for each



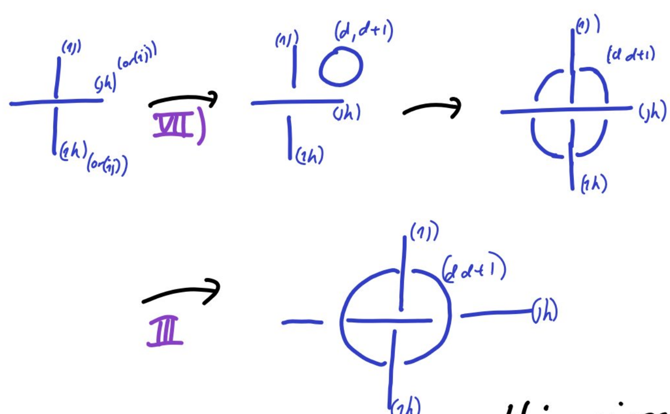
if labels same



since there are crossings like  we now have a d-fold cover with d > 3

and all labels on  have i, j, k in {1, 2, 3}

so for each  do



this gives link as in lemma

(maybe all crossings swapped but that's ok could have arranged other.)