Branched Coverings

let $X, Y$ be two $n$-manifolds

a proper map $f: X \to Y$ is a $d$-fold branched covering if

1) $\exists$ a CW-complex $B \subset Y$ (called branch locus)
   of codimension 2
   st. $f^{-1}(B) \subset X$ also a codimension 2 CW-complex

2) $f|_{Y-f^{-1}(B)}: (X-f^{-1}(B)) \to (Y-B)$ is a degree $d$ covering map

Remark: We will frequently odd conditions to $B$ and $f$
(e.g. $B$ a submanifold, $f$ smooth ...)

Thm (Alexander 1920)

Every closed, oriented (PL) $n$-manifold $M$ is a branched cover of $S^n$

Remarks: 1) PL means Piecewise linear this means $M$ is homeomorphic to an $n$-complex
(st. the link of every vertex is a sphere)

2) the branched cover map is PL (linear on simplices, maybe after subdividing)

3) All smooth manifolds are PL (but can map be made smooth ?)

4) Not all topological manifolds are PL (are they branched covers over $S^n$ ?)

but for $n=1, 2, 3$, all manifolds are PL

Sketch of proof:

We do for $n=2$, general case similar idea.

Given a surface $\Sigma$, let $T$ by a simplicial complex as above

let $v_1, \ldots, v_m$ be the vertices of $T$

let $p_1, \ldots, p_n$ be points on $S^2$ (round $S^2$ in $\mathbb{R}^3$)

in "general position" : re. 1) no 2 are antipodes
2) no 3 lie on some great circle

note: any for any $i, j, k$, $p_{1i}, p_j, p_k$ define
a triangle $\Delta_{ijk}$
$S^2 - \Delta_{ijk}$ is also a triangle $\Delta'_{ijk}$.

The points $p_1, \ldots, p_n$ give $S^2$ a triangulation and the $\Delta_{ijk}$ are simplices or unions of simplices in triangulation.

Also if $v_i, v_j, v_k$ are vertices of a simplex $\sigma_{ijk}$ in $T$ then we get "linear" maps

$$f_{\sigma_{ijk}}: \sigma_{ijk} \to \Delta_{ijk}$$

and

$$f'_{\sigma_{ijk}}: \sigma_{ijk} \to \Delta'_{ijk}$$

Orient $\Sigma_g$ and $S^2$.

Given $\sigma$ a 2-dimensional simplex in $T$ choose ordering of vertices $v_i, v_j, v_k$, s.t. or $\sigma$ on $\sigma$ agrees with one induced by ordering.

If or $\sigma$ on $\Delta_{ijk}$ from ordering of vertices agrees with or $\sigma$ on $S^2$ then define

$$f: \sigma \to S^2$$

by $f_i$ if not by $f_2$.

You can check $f$ is a covering map from $\Sigma_g - \{v_3\}$ to $S^2 - \{p_3\}$ i.e. a branch covering map!

Hint: only really need to check edges.

Main Questions:

1) Can we upgrade them to the smooth world with branch set a submanifold?

2) How can we describe branch covers and "see" them?

2 Dimensional Case:

Examples:

1) $f: C \to C: z \to z^n$ is an $n$-fold branched cover with branch locus $0$.
Remark: a) branch locus of will always be collection of points we assume all branch points have neighborhood where they look like \( z \mapsto z^n \) \((n = \text{branching index})\).

Question: is this necessary or can we always arrange this?

b) We will take our branch covering maps smooth.

2) \( S^2 \subset C \times R \)

\[ f: S^2 \to S^2 \]

\((z, x) \mapsto (z^d, x)\)

d-fold cover branched over 2 points

3) \( \text{quotient map} \)

\[ \text{Then: Any oriented closed surface of genus } g \text{ is a } 2\text{-fold cover of } S^2 \text{ branched over } 2g + 2 \text{ points} \]

4) \( \text{2-fold} \)

\( \text{4-fold each point has branching index 2} \)
A branched cover $f : X^n \rightarrow Y^n$ is called **simple** of degree $d$ if $f^{-1}(y)$ always consists of $d$ or $(d-1)$ points. (Note if $d_1$, then every point in $f^{-1}(y)$ is "regular." If $d=1$, the all but one point regular and other point has branching index 2.)

This is a very useful example! For now it has a simple application:

$$
\begin{align*}
A \times A &\rightarrow T^3 \\
\downarrow \text{2-fold} &\downarrow \\
A \times D^2 &\rightarrow S^1 \times S^2 \\
\downarrow \text{2-fold} &\downarrow \\
D^2 \times D^2 &\rightarrow S^3
\end{align*}
$$

So there is a 4-fold branched cover $T^3 \rightarrow S^3$ you can check branch locus is $\bigcirc$
Riemann-Hurwitz Formula

Let $\rho : X \rightarrow Y$ be a branched cover of surfaces
$B = \{y_1, \ldots, y_k \} \subset Y$ the branch set
$\rho^{-1}(B) = \{x_1, \ldots, x_n \} \subset X$, $a_i = |\rho^{-1}(y_i)|$
let $d_i$ = branch index at $x_i$
d = fold of cover

Then

$$\chi(X) = d \chi(Y) - \sum_{i=1}^{n} (d_i - 1)$$
$$= d(\chi(Y) - k) + a_1 + \ldots + a_k$$

Proof: note $\chi(Y - B) = \chi(Y) - k$

so

$$\chi(\rho^{-1}(Y - B)) = d \chi(Y - B) = d(\chi(Y) - k)$$

for covers

now add back points in $\rho^{-1}(B)$ there are $a_1 + \ldots + a_k$

for other formula note if $\rho^{-1}(y_i) = \{x_1, \ldots, x_{d_i} \}$ then

$$d = d_1 + \ldots + d_m$$

\textbf{N.B.:} each b.p. gives a partition of d

since there are $k$ points $y_i$, we see

$$kd = \sum_{i=1}^{n} d_i$$

also

$$n = a_1 + \ldots + a_k$$

$$\therefore -\sum_{i=1}^{n} (d_i - 1) = -kd + n = -kd + \sum a_i \quad \therefore \text{done}$$

Note: RH formula restricts possible coverings

eg. if $\Sigma_g \rightarrow S^2$ is a branched cover and $g \geq 1$

then there are $\geq 3$ branch points.

(since $d(2-k) + a_1 + \ldots + a_k > 0$ if $k \leq 2$)

Proposition

Any oriented surface is a branched cover of $S^2$ branched over 3 points.
Proof:

\[ P_d \downarrow \text{d-fold branched over } \{a, b\} \]

Note: \( p_d^{-1}(\{a, b, c\}) = \{a', b', c_1, c_2, \ldots c_d\} \)

Now given a surface \( \Sigma \) of genus \( g \), we know there exists a 2-fold cover \( \Sigma \rightarrow S^2 \) branched over \( 2g+2 \) points.

Let these points be \( c_1, \ldots, c_{2g+2} \) in \( p_{2g+2}^{-1}(c) \)

then \( p_{2g+2} \) of \( : \Sigma \rightarrow S^2 \) is desired cover.

Remark: any surface is

1) 2-fold cover over \( S^2 \) branched along some set
2) a cover of \( S^2 \) branched over 3 points (fold of cover not fixed)
3) branch cover with branch locus a submanifold

Questions:

1) Is an \( n \)-manifold an \( n \)-fold cover of \( S^n \)? (with branch locus a submanifold?)
2) Is there a submanifold \( \Sigma \subset S^n \) such that all \( n \)-manifolds are a cover of \( S^n \) branched over \( \Sigma \)?
3) Given \( M \) is there a branched cover over \( S^n \) with branch set a submanifold?

Preview: In dimension 3: answer Yes to all 3!
In dimension 4: answer almost Yes to all 1 and 3 (nothing known about 2 ?)
In higher dimensions answer is sometimes No to all questions
Nice application of RH formula

Example: We can use RH formula to figure out what a given surface is.

\[ \Sigma_n = \{ [x:y:z] \in \mathbb{CP}^n \mid x^n + y^n + z^n = 0 \} \quad \text{Fermat curve} \]

Claim: \( \Sigma_n \) is a surface of genus \( \frac{(n-1)(n-2)}{2} \)

1st \( \Sigma_n \) is a closed surface:

\[ f: \mathbb{C}^3 \to \mathbb{C}; (x,y,z) \mapsto x^n + y^n + z^n \]

\( \forall p \in f^{-1}(0) \) with \( p \neq 0 \) \( df_p \) surjective

so \( f^{-1}(0) - \{0,0,0\} \) a \( n \)-manifold

\( f \) homogeneous \( \implies p \notin f^{-1}(0) \) \( \forall z \neq 0 \)

\[ \Sigma_n = f^{-1}(0) - \{0,0,0\} / \mathbb{C}^* \text{ makes sense} \]

note action of \( \mathbb{C} \) is free and

proper (\( \forall x \in \mathbb{C} \cdot x \cdot (x^{-1}) = 1 \)

proper map)

\[ \therefore \Sigma_n \text{ a real 2-manifold} \]

(complex 1-manifold \( \cong \text{oriented} )

2nd compute genus:

note we get a map \( p: (\mathbb{C}P^2 - \{[0:0:1]\}) \to \mathbb{C}P^1 \cong S^2 \)

\[ [x:y:z] \mapsto [x:y] \]

(if you have not done this before check \( p \) is an \( \mathbb{R}^2 \)-bundle)

\([0:0:1] \in \Sigma_n \quad \therefore \quad p \) induces a map

\[ p|_{\Sigma_n} : \Sigma_n \to S^2 \]

for \([x_0:y_0] \in \mathbb{C}P^1\),

\[ p^{-1}([x_0:y_0]) = \{ [x:y:z] \in \mathbb{C}P^2 \mid z^n = -(x^n + y^n) \} \]

when \( x^n + y^n \neq 0 \) this consists of \( n \) points

(can check \( dp \neq 0 \) at these so covering space)

when \( x^n + y^n = 0 \) only one point

this happens at \([1:1:1] \)
where $\zeta_n = n^{th}$ root of unity
that is at $n$ points

so $p$ is an $n$-fold branched cover
branched over $n$-points
(and each bp has 1 preimage)

\[ \chi(\Sigma_n) = n(\chi(S^2) - 1) + n \]
\[ = 2n - n^2 + n = -n^2 + 3n \]

so $g(\Sigma_n) = \frac{2 - \chi(\Sigma_n)}{2} = \frac{n^2 - 3n + 2}{2} = \frac{(n-1)(n-2)}{2}$

**Question:**

Suppose you are given $\Sigma_g$ and $\Sigma_h$ oriented surfaces $d \in \mathbb{N}$

$k$-partitions of $d: (d_{11}, \ldots, d_{1k}), \ldots, (d_{k1}, \ldots, d_{kk})$
satisfying the RH formula:

\[ \chi(\Sigma_g) = d \chi(\Sigma_h) - \frac{h}{2} \left( \sum_{j=1}^{k} (d_{ij} - 1) \right) \]

then is there a branched cover $\Sigma_g \to \Sigma_h$ realizing this data? How many?

**Answer:** Husemoller '62 says Yes if $h \geq 1$ (maybe show later)

if $h=0$, i.e. $\Sigma_h = S^2$, then sometimes NO

e.g. $\Sigma_g = S^2$, $\Sigma_h = S^2$

$d = 4$

$(2,2), (2,2), (3,1)$ (we check this later)

but $(2,2), (2,2), (2,2)$ OK (example above)

**Open Problem:**

When is the answer to question Yes for $\Sigma_h = S^2$?

Is it true if degree prime?

Hurwitz problem
Monodromy and covers:

Start with non-branched covers

\[ \tilde{X} \xrightarrow{\text{p}} X \]

given be an n-fold covering space

we get a homomorphism

\[ m : \pi_1(X, x_0) \to S_n \]

where \( S_n \) = symmetric group of n letters

as follows:

1. let \( p^{-1}(x_0) = \{x_1, \ldots, x_n\} \)
2. given any loop \( \gamma : S^1 \to X \) based at \( x_0 \)
3. let \( \tilde{\gamma}_i : [0, 1] \to \tilde{X} \) be the lift of \( \gamma \) based at \( x_i \)
4. define \( \sigma_\gamma : \{1, \ldots, n\} \to \{1, \ldots, n\} \)
   \[ i \mapsto \text{index on } \tilde{\gamma}_i(1) \]

   easy to check \( \sigma_\gamma \) only depends on homotopy class of \( \gamma \)

   define \( m_p([\gamma]) = \sigma_\gamma \)

**note:** if we relabel points in \( p^{-1}(x_0) \)
then \( m_p \) is conjugated

**exercise:** \( m_p \) is transitive

\( \text{r.e. } x, y \in p^{-1}(x_0), \text{ then } \exists \delta \)

\( \delta. \text{lift of } \gamma \) based at \( x \)

has end pts \( \{x_1, y\} \)

now given any homomorphism

\[ f : \pi_1(X, x_0) \to S_n \]

let \( H = \{ g \in \pi_1(X, x_0) : f(g)(1) = 1 \} \) this is a

subgroup of \( \pi_1(X, x_0) \)
if \( f \) is transitive (i.e., \( f \) is a one-to-one correspondence), then \( \exists \) \( O \) s.t. 
\[ f(\{y\})(x) = y \]
then \( H \) is index \( n \)

\( \exists \) a covering space \( X \) associated to \( H \)

and \( \downarrow \) is \( n \)-fold and its monodromy

is (conjugate) to \( f \)

**Example:**

\[ \pi_1(X, x_0) = \mathbb{Z} \times \mathbb{Z} \]

So \( \pi_1(X, x_0) \rightarrow S_n \) determined by

image of generators

i.e. just label edges with

elements of \( S_n \)

build cover as follows:

1) cut each edge at a point
   (branch cut)

2) take \( n \) copies

3) glue copies according to data

e.g.

1) \[ \begin{array}{c}
 x \\
 1
\end{array} \]

2) \[ \begin{array}{c}
 x \\
 1
\end{array} \]

3) \[ \begin{array}{c}
 x \\
 1
\end{array} \]

now for branched covers.

**Lemma:**

\( \Sigma \subset \Sigma' \) any finite set

any cover of \( \Sigma - \beta \) extends to

a branched cover over \( \Sigma \)

**Exercise:** prove this
so branched covers are determined by the monodromy of their associated covers.

**Example:**

Consider $D^2$ if $B = \{x_1, \ldots, x_n\}$, then $\pi_1(D^2 - B) \cong F_n$, free group so homomorphism $\pi_1(D^2 - B) \to S_n$ determined by image of generators.

i.e. just label points with elements of $S_n$, called Hurwitz system.

E.g. branch cut take 2 copies glue

Exercise show cover is

back to $S^2$ 4-fold cover with data (2,2), (2,2), (3,1) can't be realized.

$\pi_1(S^2 - \text{3 points}) = \mathbb{Z} \times \mathbb{Z}$

need $a \mapsto \text{product 2 transpositions}$ $(12)(34)$ can assume this by relabeling $b \mapsto \text{product 2 transpositions}$

$(12)(34)$ or $(13)(24)$ or $(14)(23)$

Can't be $(12)(34)$ or image only $(11)(34)$ if $(13)(24)$ then $c$ goes to $(14)(23)$.
2 branched covers
\[ f_0, f_1 : X \rightarrow Y \]
are equivalent if there exist homeomorphisms/diffeomorphisms
\[ g : X \rightarrow X \]
\[ h : Y \rightarrow Y \]
such that
\[ X \xrightarrow{g} X \]
\[ f_0 \circ f_1 \xrightarrow{h} f_1 \circ f_0 \]

**Thm (Hurwitz)**

2 branched covers
\[ f_0, f_1 : \Sigma \rightarrow \Sigma' \]
of surfaces are equivalent
\[ \iff \exists \text{homeomorphism} \]
\[ h : (\Sigma', B_{f_0}, x) \rightarrow (\Sigma', B_{f_1}, x) \]
such that
\[ m_0 = m_1 \circ h \text{ mod conjugation} \]
where
\[ m_i = f_{i,|\Sigma'-B_i} \text{ monodromy} \]
and
\[ h_* : \pi_1(\Sigma' - B_0, x) \rightarrow \pi_1(\Sigma' - B_1, x) \]

**Exercise** prove this (just fact about covers and monodromy)

A branched cover \[ X \xrightarrow{f} Y \] of degree \( d \) is called simple if
\[ \forall y \in Y, |f^{-1}(y)| = d \text{ or } d-1 \]

**Ex.** if \( b \in Y \) a branch point then
\[ f^{-1}(b) = \{x_1, \ldots, x_{d-1}\} \]
with
\[ x_i \text{ having ramification 2} \]
\[ x_i \text{ "regular points"} \]

for a surface this corresponds to having a Hurwitz system with only transpositions

**e.g.**

- Not simple
- Simple (2,3)
we call 2 branched coverings \( f_0, f_1 : X \to Y \)

**b-homotopic** if \( \exists \) a homotopy \( f_t : X \to Y \) \( t \in [0, 1] \)

through branched coverings

**Th**

1) let \( f : X \to Y \) be a simple branched cover, \( Y \) compact

\( \exists \) an open set \( U \) in compact-open topology on \( C^0(X, Y) \)

st. any branched cover in \( U \) is simple

2) any branched covering of a surface is

\( b \)-homotopic to a simple branched cover

**Proof:**

1) let \( \{B_i\} \) be balls in \( Y \) such that \( f^{-1}(B_i) = \text{union of } n \text{ or } (n-1) \text{ disjoint balls in } X \).

for each \( i \) choose smaller ball \( A_i \subset \text{int } B_i \)

\( Y \) compact so \( \exists \) finite \( A_1, \ldots, A_k \) s.t. \( Y = \bigcup \text{int } A_i \)

\( \exists \) open set in compact-open topology on \( C^0(X, Y) \)

st. \( g \) in open set \( \Rightarrow \) for each \( A \) component \( f^{-1}(A_i) \)

\( g(A) \subset \text{int } B_i \)

\( \therefore |g^{-1}(x)| \geq n-1 \) \( \forall x \in Y \)

also \( \text{deg } g = \text{deg } f = n \) (since if \( f, g \) close enough they are homotopic)

so \( g \) an \( n \)-fold branched cover \( \therefore \) simple

2) note just need to perturb cover near branch points

near branch point \( z \to z^n \)

change to \( z \to z^n + \epsilon \) now path to rest of cover

![Diagram](image-url)
If $\Sigma$ a connected surface then any two simple branched covers over $S^2$ of the same degree are equivalent and $b$-homotopic.

For proof need Hurwitz moves

So, cover corresponding to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4^{-1}, \alpha_2, \alpha_4)$ are equivalent and $b$-homotopic.

If $\Sigma \to S^2$ a simple branched cover then its Hurwitz data is $\alpha_1, \alpha_2, \ldots, \alpha_k$

St.
(a) $\alpha_1 \alpha_2 \cdots \alpha_k$ = identity in $S_n$
(b) $\alpha_i$ transpositions
(c) $\{\alpha_1, \ldots, \alpha_k\}$ generate a transitive subgroup of $S_n$

From above, we know we can make change

$$(\star) \quad \alpha_1 \alpha_4 \alpha_1 \rightarrow \alpha_1 \alpha_4 \alpha_1 \alpha_4 \alpha_1 \text{ (or } \alpha_1 \alpha_4 \alpha_1 \alpha_4 \alpha_1)$$

Also if $x, y$ transpositions can make following changes

(†) $x, y, y, y \rightarrow y, y, x$ obvious

(‡) $x, x, y, y \rightarrow y, y, y, x$

Proof of theorem follows from

Lemma
Any sequence $(\alpha_1, \ldots, \alpha_k)$ satisfying (a)-(c) can be...
put in the form 
\[(12)(12) \ldots (n)(12)(23)(23)(34)(34) \ldots (n-1\ n) (n-1\ n)\]
by a sequence of \(\circ\) moves

**Proof:**

Since \(x_1 \ldots x_n = \text{identity}\) we know \(n=2m\) even.

**Claim:** can arrange \(x_{2i-1} = x_i\) \(\forall i = 1 \ldots m\)

**Proof:** suppose \(x_i = (x, y)\)

must be another term with \(x_2\) choose one closest to \(x_1\)

\(x_1 = (x, y)\)

can conjugate so \(i = 2\), if \(x_3 = x_1\) we reduce \(m\) by 1 done by induction

if \(x_3 \neq x_2\), then term \((x_3, y)\) closest to \(x_2\) can assume \(x_3\)

if \(x_4 = x_1\), then

\[\begin{align*}
(x_1, x_2) (x_3, x_3) (x_4, x_4) & \Rightarrow (x_1, x_2) (x_3, x_3) (x_4, x_4) \\
& \text{done by induction}
\end{align*}\]

if \(x_4 = x_2\), then

\[\begin{align*}
(x_1, x_1) (x_2, x_3) (x_4, x_4) & \Rightarrow (x_1, x_1) (x_2, x_3) (x_4, x_4) \\
& \text{done by induction}
\end{align*}\]

if \(x_4 \neq x_1\) or \(x_2\), then continue to get

\[\begin{align*}
(x_1, x_1) (x_2, x_3) \ldots (x_{r-1}, x_r)
\end{align*}\]

since finite number of \(x_i\)'s eventually \(x_r = x_j\) some \(j < r\)

and done as above.

Now (†) and (‡) imply can replace any pair by a conjugate of any pair.

Since \(x_1 \ldots x_n\) generate transitive group can find

\[\begin{align*}
(1 \ 1) (1 \ 1) \ldots (x, y) (x, y) \ldots (x, y) (x, y) \ldots (x, y) (x, y)
\end{align*}\]

so get pair \((12)(12)\) at front

remove pair

can assume remaining terms transitive on \(2, \ldots, n\)

Indeed if \(x \in \{3, \ldots, n\}\) and can't send \(2 \rightarrow x\) then can send \(1 \rightarrow x\) by transitivity of original \(x_i\)'s

So as above can use \(\circ\) to get \((1x)(1x)\)

move to front: \((12)(12)(1x)(1x) \Rightarrow (12)(12)(1x)(1x)\)

now if \(1\) in remaining terms then they are transitive on \([1, 2, \ldots, n]\)

and can remove another \((12)(12)\) pair.
keep going until no 1's left
now can pull off some number of (23)(23) pairs
if more than one then we see
\[\to (12)(12)(12)(23)(23)(23)\]
continuing gives desired result!

**Thm.**

Let \( \Sigma \to S^2 \) be a simple cover of degree at least 3
and \( f: \Sigma \to \Sigma \) any homomorphism
then \( \exists \) a homeomorphism \( h: S^2 \to S^2 \) and \( \tilde{h}: \Sigma \to \Sigma \)
such that \( \phi \circ \tilde{h} \circ \phi^{-1} \)
\[ S^2 \to S^2 \]
and \( \tilde{h} \) isotopic to \( f \)
Moreover \( \phi \circ \tilde{h} \) is \( b \)-homotopic to \( \phi \).

**Proof:** Start with degree 3
recall a Dehn twist \( \tau_\gamma \) along a curve \( \gamma \) in a surface \( \Sigma \) is defined as
follows: let \( N \) be a nbhd of \( \gamma \) and \( f: [1/2, 1/2] \times S^1 \to N \) a diffeomorphism
where
\[ \tau_\gamma: \Sigma \to \Sigma, \quad \rho \mapsto \begin{cases} f(\rho) & \rho \in N \\ \rho & \rho \in \Sigma-N \end{cases} \]

**Thm.** (Humphries '79)

Let \( \Sigma \) be a closed oriented surface then any diffeomorphism is
isotopic to a composition of
Dehn twists about \( \gamma_1, \ldots, \gamma_{2g+1} \)

**Exerc.** Up to isotopy \( \tau_\gamma \)
only depends on the isotopy class of \( \gamma \)
So to prove our theorem we just need to see can realize all \( T_\alpha \) in a branched cover

let a \( c \Sigma \) be an arc

a has a nbhd \( N \) diffeomorphic to \( D^2 \) = unit disk in \( \mathbb{R}^2 \)

a arc twist is the diffeomorphism

\[
tw_\alpha : \Sigma \to \Sigma : \rho \mapsto \begin{cases} +1 & \rho \in N \\ \rho & \rho \in \Sigma - N \end{cases}
\]

where

\[
tw : D^2 \to D^2 \\
(\rho, \theta) \mapsto (\rho, \theta + tw(\rho))
\]

\[0 \quad \frac{1}{2} \quad 1\]

\[2\pi\]

exercise: Show isotopy type (rel \( \partial a \)) of \( tw_\alpha \) only depends on the isotopy class (rel \( \partial a \)) of a

**lemma:**

let \( \phi : [-1,1] \times S^1 \to D^2 \) be the cyclic 2-fold cover branched over \( \{(0, \pm \frac{\pi}{2})\} \)

the twist map \( tw_\alpha \) where \( a = \{(0, \pi) | |1| \leq \frac{\pi}{2}\} \)

is covered by the Dehn twist \( T_\alpha \) where \( \alpha = \{0\} \times S^1 \)

\[
\begin{array}{c}
\phi \\
\downarrow \\
\rho \\
\downarrow \\
\phi \\
\end{array}
\]

\[
\begin{array}{c}
T_\alpha \\
\downarrow \\
\tau_\alpha \\
\downarrow \\
T_\alpha \\
\end{array}
\]

**Proof:** Note \( tw_\alpha = \text{identity near } \partial \)

rigid \( \pi \) rotation near a twist on rest
and $\gamma_y = \text{identity near } \partial$ 
rigid $\pi$ rotation near $\gamma$ 
and twist on rest 

on green $\phi$ is $2 \times \text{"identity"}$ so $\gamma$ is clearly related 
on purple $\phi$ is where branching occurs 
but clearly rigid rotation related by $\phi$

on yellow $\phi$ is $2 \times \text{"identity"}$ and half twists lift

\textbf{Aside}: Relations in the mapping class group

\textbf{Thm.}: 

If surface is closed then 
$(\gamma_1 \cdots \gamma_g)^{4g+2} = \text{id}$

\begin{equation}
(\gamma_{1_{\gamma}} \cdots \gamma_{g_{\gamma}})^{4g+2} = \gamma_b
\end{equation}

\text{isotopic}

\begin{equation}
(\gamma_{1_{\gamma}} \cdots \gamma_{g_{\gamma}})^{4g+2} = \gamma_{b_{\gamma}}
\end{equation}

\text{isotopic}

\textbf{Proof}:

Consider $D^2$ 2-fold branched cover is genus $g$

let $\alpha_1 = \text{core } \gamma_1 + \gamma_{g+1}$

note $\alpha_1$ lifts to $\gamma_i$

so $\gamma_{\alpha_1}$ covered by $\gamma_{2\gamma}$

also note if $\hat{b}$ is

then $\gamma_{\alpha_1}$ lifts to $\gamma_{b_1} \circ \gamma_{b_2}$ since $\text{a union of } \delta \text{ lifts to a union of } b_1 \cup b_2$
note: $\Sigma_b$ is isotopic to $(tw_{a_1} \circ \ldots \circ tw_{a_{2g+1}})^{2g+2}$

to see this recall isotopy classes of diffeos $(D^2, \{x_i\}) \to (D^2, \{x_i\})$
correspond to "braids"; given $f : (D^2, \{x_i\}) \to (D^2, \{x_i\})$

let $\phi_t : D^2 \to D^2$ be isotopy of $f$ to $\text{id}_{D^2}$

"graph of $\phi_t$" is

if two maps $f, g : (D^2, \{x_i\}) \to (D^2, \{x_i\})$ give
same braid then $f, g$ isotopic rel $\{x_i\}$

now braid for $\Sigma_b$ is

so clearly $\Sigma_b = (tw_{a_1} \circ tw_{a_2} \circ \ldots \circ tw_{a_{2g+1}})^{2g+2}$

lifting to 2-fold branched cover gives

$\Sigma_b, \Sigma_{b_2} = (\Sigma_{b_1} \circ \ldots \Sigma_{b_{2g+1}})^{2g+2}$

other case similar except for $\Sigma_b$ does not lift but $\Sigma_{b_2}$ lifts to $\Sigma_b$

now return to proof of "any diffeo is lift of diffeo for simple cover of degree $\geq 3$"

recall we consider degree 3 first

by previous theorem any degree 3 simple cover of $\Sigma_g$ over $S^2$
is equivalent to

\[ \text{3 fold} \]

let \( a_i \) be the arcs

\[ a_1, a_2, a_3, \ldots, a_{2g+1} \]

and \( o_{2g+2} \)

\[ o_1, o_2, o_3, \ldots, o_{2g+2} \]

Note \( t_{\gamma_i} \circ \gamma_{g+1} \) lifts to \( t_{\gamma_i} \circ \gamma_{g+1} \)

Example:

\[ \gamma_i \]

\[ o \]

\[ \text{3 fold} \]
and $\tau \circ \tau_{b_{2g+1}}$ lifts to $\tau_{b_{2g+1}} \circ \tau_{b_{2g+1}} = \tau_{b_{2g+1}}$

so all the Humphries generators are lifts of diffeos of base

\[ \therefore \text{all diffeos, up to isotopy, are} \]

And can see "b-homotopy" as a "braid" as above.

**Exercise:** check for higher fold simple covers

### 3 Dimensional case

Recall 3 general questions we have about branched covers in a given dimension are

1) Can $M^n$ be realized as a cover of $S^n$ branched over a submanifold?

2) If we fix the degree of the cover, is the answer yes?

3) Is there a "universal branch locus" $B \subset S^n$ that is a submanifold?

(ie every $M^n$ is a branched cover of $S^n$ branched over $B$)

For 2-manifolds we said:

Yes to 1)

Yes to 2) with degree 2

Yes to 3) with $B = 3$ pts
We will prove

**Th** \(^{m}\) (Hilden '76, Hirsch '74, Montesinos '76)

Every closed orientable 3-manifold is a 3-fold simple branched cover of \(S^3\) branched over a link.

**Remark:** We can take the branch locus in the theorem to be a knot.

**Th** \(^{m}\) (Thurston 82?)

There is a link \(L\) in \(S^3\) such that any closed orientable 3-manifold is a branched cover of \(S^3\) branched over \(L\).

**Remarks:**

1) Hilden-Lozano-Montesinos '83 showed can take \(L\) to be a knot (much harder than in the above)

2) Universal links:

- Borromean rings
- Whitehead link
- Figure 8 knot

Any 2-bridge link that is not a torus knot or link is universal.

3) Torus knots and iterated cables of torus knots are not universal.
4) There are cables of some knots that are universal, there are Whitehead doubles of some knots that are universal. 

Eg. 

Open problems:

1) Are all hyperbolic knots/links universal?
2) Is a knot universal \( \iff \) a term in its JSJ decomposition is hyperbolic?

Proof of "3-fold cover" theorem:

Note: branched cover of disk

\[
\mathcal{D} = \{x_1, ..., x_{2g+2}\}
\]

is the surface

so cover \( \mathcal{B}^3 = (\mathcal{B}^2 \times [0,1], \mathcal{B} \times [0,1]) \)

is \( (\Sigma_g - (\mathcal{B}^3 \cup \mathcal{B}^3)) \times [0,1] \cong \) handlebody of genus \( 2g+1 \)

so we have

"doubling" (take 2 copies and glue 3's by identity i.e. \( 2(M \times I) \))
\[ \#_{2g+1} S^1 \times S^2 \to S^3 \text{ a 3-fold cover branched over unlinked with } 2g+2 \text{ components} \]

**Fact:** Every closed oriented 3-manifold has a Heegaard splitting: \( M^3 = V_g \cup \phi V_g \) where \( V_g \) a handlebody of genus \( g \)

and \( \phi: \partial V_g \to \partial V_g \) a diffeomorphism.

**Remarks:**
1) diffeo type of \( M \) only depends on isotopy class of \( \phi \)
2) Can prove this using either
   a) Morse theory
   ![Morse theory diagram]
   b) Triangulation of \( M \):
      \[ V^1 = \text{nbhd 1-skeleton} \]
      \[ V^2 = \overline{M-V^1} \]
3) If \( M \) has splitting of genus \( g \) then it has splitting of any genus \( \geq g \) by stabilization
   ![Stabilization diagram]
   remove \( N \) from \( V^1 \) and add it to \( V^1+1 \)
4) from above we see a Heegaard splitting of
   \[ \#_{2g+1} S^1 \times S^2 \text{ along a surface } S_{2g+1} \]

So given \( M \) and a Heegaard splitting of genus \( 2n+1 \) (can assume odd)
the there is some diffeo \( \phi: S_{2n+1} \to S_{2n+1} \) such that
we can split $\#_{n+1} S^1 \times S^2$ along $S_{2n+1}$ and reglue by $\phi$ to get $M$.

Since any $\phi: S_{2n+1} \to S_{2n+1}$ can be realized by a $b$-homotopy of the $3$-fold cover $S_{2n+1} \to S^2$ we can realize $M$ as a branched cover over $S^3$ branched over

![Diagram](image)

**Monodromy**

let $\hat{\phi}: \hat{M} \to Y$ be a covering map of degree $d$ (recall, determined by map $\Pi_1(\hat{Y},x) \to S_d$)

where $\hat{Y} = Y - \text{neighborhood}\ L$ for some link $L$

**Lemma:** $\hat{\phi}$ extends over $\hat{M}$ to give a branched cover $\phi: M \to Y$

**Proof:** let $S^1 \times D^2$ be a component of $(\text{neighborhood}\ L)$

$\phi^{-1}(\partial (S^1 \times D^2))$ is a union of tori let $T$ be one of them

so $\phi: T \to \partial (S^1 \times D^2)$ is a covering map

$\exists$ some smallest $k$ such that $k((1,p) \times D^2)$ lifts to an embedded loop $m$ in $T$

pick a diffeomorphism $\psi: S^1 \times S^1 \to T$ st.

$(p) \times S^1 \to m$

so $\phi \circ \psi: S^1 \times S^1 \to S^1 \times S^1$ is isotopic to

$(\phi, \theta) \mapsto (a \phi + b \theta, k \theta)$ for $a, b \in \mathbb{Z}$
extends to $S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2$

$(\phi, (r, \theta)) \mapsto (a \phi + b \theta, (r, \theta))$

So branched covers $M \to Y$ with branch set $B \subset Y$ a link are determined by their monodromy

$\Pi_1(Y - B, x) \to S_n$

if $L \subset S^3$ is a link with diagram

then the Wirtinger presentation of $\Pi_1(S^3 - L)$ is

$\langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$

where we cyclically label the arcs $a_1, \ldots, a_n$ in the diagram

and $x_i$ is the meridian to $a_i$;

$r_i$ is the following relation at the end point of $a_i$:

$$a_k a_i a_k = a_{i+1}$$

eg.

$$\langle x_1, x_2, x_3 | a_3 a_1 a_3^{-1} = a_2, a_1 a_2 a_1^{-1} = a_3 \rangle$$

so a branched cover of $S^3$ branched over $L$ is determined by labelling arcs in diagram by elts of $S_n$ so they satisfy relations at crossings similarly for a tangle in $B^3$

example: what is following branched cover?
to see, recall in 2D we have

\[ D^2 \]

\[ \downarrow \]

\[ D^2 \]

so we see

\[ B^3 \]

\[ \downarrow \]

\[ B^3 \]

\[ \text{similarly} \]

\[ B^3 \]

\[ \downarrow \]

\[ B^3 \]

so if you see a labeled link diagram with

\[
\begin{array}{c}
(27) \\
(12) \\
\end{array}
\]

this:

and replace it with

\[
\begin{array}{c}
(23) \\
(27) \\
(31) \\
(32) \\
(22) \\
\end{array}
\]

then in branched cover you are removing a ball (if 3-fold cover, otherwise d-2 balls in d-fold cover) and gluing back in a ball.

\[ \therefore \] manifold in cover did not change!

**Cor:**

every closed oriented 3-manifold is a 3-fold branched cover of \( S^3 \) branched over a knot.
Proof: from above we know $M$ is a 3-fold simple cover of $S^3$ branched over a link $L$ (assume $L$ has 2 or more components).

Braid $L$ look here

we see each strand is labeled $(12), (23)$, or $(13)$ since this is a 3-fold cover at least 2 of the labels are used

so we see if these correspond to different components of $L$ then add 3 twists as above to reduce # of components.

if some component then look for closest strand of different component

$(a,b)$ different from $(c,j)$ or $(j,k)$

so can move appropriate strand up

and add 3 twists as above.

Another example: recall we had

so we get

$S^1 \times I$ $\rightarrow$ $D^2$

$S^1 \times D^2$ $\rightarrow$ $B^3$
note cover of

also a solid torus, but meridion is different

so if you see \( \square \) in the data for a branch

cover and replace it with \( \square \) then

in the cover you remove \( \square \) and replace

with \( \square \)

i.e., if \( a \) is the arc \( \square \)

then \( a \) lifts to a knot \( K_a \) in cover \( M \)

so changing branch set by twist
changes the cover by

\( \pm 1 \) Dehn surgery on \( K_a \)

Cor:

another proof that any closed oriented 3-manifold is a
simple 3-fold cover over \( S^3 \)
branched along a link
Proof: recall we have

\[ \begin{array}{c}
\text{3-fold} \\
\end{array} \]

Glue 2 copies together to get

\[ s^3 \]

Note: let $B^3$ be the ball shown

In cover you see

so changing the branch set by

will change cover by ± Dehn surgery on solid torus above depending on the twist done

we need

\[ \text{Theorem: Any 3-manifold is obtained from } s^3 \text{ by surgery on some link} \]
now given $K u \ldots v K_n$ framings $n_1 \ldots n_k$ such that $M$ is obtained from $S^3$ by surgery on $K_i$ with framing $n_i$

Then we construct our cover as follows

unfix $K$ in cover

now fix framing

We now consider the existence of a universal link

This follows directly from

**Lemma 1:**

Any compact oriented 3-manifold is a simple branched cover over the link $L_{m n}$ on right for some $n$ and $m$
Lemma B: There is a branched cover $S^3 \to S^3$ branched over the Borromean rings $B$ such that $L_{n,m}$ from lemma A is a subset of $p^{-1}(B)$.

Note: It clearly follows that $B$ is universal.

Proof of $B$:

Let $T$ be the Heegaard torus for $S^3$ and $H$ be the Hopf link st. $H$ is the core of Heegaard tori.

Note: The branched cover $H$ is defined as $S^3 = S^1 \times D^2 \cup S^1 \times D^2$ with a fold cover.

\[ \{ \text{fold cover} \} \cup \{ \text{branch core} \} \]

\[ S^1 \times D^2 \cup S^1 \times D^2 = S^3 \]

denote this $p_n : S^3 \to S^3$

and $p_n^{-1}(H) = H$.

Similarly $q_m : S^3 \to S^3$ given by

and $q_m^{-1}(H) = H$.

$q_m \circ p_n : S^3 \to S^3$ is a branched cover over $S^3$ with branch set $H$.

Moreover $T \subset S^3$ (and hence a neighborhood of $T$) just gets $m \cdot n$ fold covered by unwrapping "longitudes" $n$ times and "meridians" $m$ times.
Now in a nbhd of $T$ consider link $L'$ note $(\varphi_m \circ \varphi_n)^{-1}(L') = L_{n,m}$ from lemma A

so

$\Rightarrow$

is universal!

Now we isotop

$\Rightarrow$

now note if $B$ is

$\Rightarrow$

then 2 fold cover of $S^3$ branched over $U$ is $S^3$ and $B$ lifts to $L$

so $B$ is universal!

To prove lemma A we need

**Observation**: Following changes to branching data don't affect covers
Proof of Lemma A:

let \( M \to S^3 \) be a 3-fold simple cover with branch set \( L \)

we can braid \( L \)

Step 1: Can assume label at a crossing are different

Proof: if you see \( (12) \) \( (13) \)

then above or below see \( (23) \) see \( (13) \) or all labels in link same and not 3-fold cover

find closest strand with different label, now

\[ \text{isotope} \]

\[ \text{reduced number of "monochromatic" crossings} \]
Step 2: Make all crossings positive

If not positive replace

\((1h)\) with \((1j)\) (type IV) move

\((7j)\)

Step 3: Add strands to trivialize braid

\(\ldots\)

\(\ldots\)

\(\ldots\)

\(\ldots\)

isotopy

isotopy

when we do this at all crossings we get
In box we see lots of $\downarrow \quad \text{or} \quad \downarrow$

For each:

- if labels different $\rightarrow (j) \quad \rightarrow (k) \quad \rightarrow (j) \quad \rightarrow (k)$
- if cover is 2-1 fold $\rightarrow \circ \rightarrow \circ \rightarrow \circ$

If labels same

- $d$-fold cover $\rightarrow (j) \rightarrow (j)$

Since there are crossings like $\downarrow \downarrow$ we now have a $d$-fold cover with $d \geq 3$

and all labels on $\downarrow \downarrow$ have $\ell, j, k \in \{1, 2, 3\}$

So for each $\downarrow \downarrow$ do

- $\rightarrow (o) \quad \rightarrow (d, o) \quad \rightarrow (d, o)$
- $\rightarrow (d, o) \quad \rightarrow (d, o)$

This gives link as in lemma

(maybe all crossings swapped but that's ok could have arranged other.)