I. Introduction

For quite some time it has been clear that there are deep connections between the topology of 3-manifolds and Riemannian metrics (i.e. Thurston’s geometrization program). More recently there have also been deep connections between the topology of 3-manifolds and contact geometry. But there seems to be few results relating properties of contact structures (like tightness) and Riemannian geometry. In these talks we will explore such connections among other things we will prove:

**Thm 1 (E-Komendarczyk-Massot):**

Let $(M^3, g, J)$ be a contact metric 3-manifold. If $g$ is a complete metric and $\exists K > 0$ st.

\[ \frac{1}{q} K \leq \sec(g) \leq K \]

then the universal cover of $(M, J)$ is $(S^3, J_{std})$ where $J_{std}$ is the unique tight contact structure on $S^3$.

- Ge-Huang improved $\frac{1}{9}$ to $\frac{1}{4}$
the classical Sphere theorem said “curvature can control topology” here we see it can also control contact topology!

\[ \text{Th}^2 \text{(EKM)}: \]

let \((M,\mathcal{I})\) be a contact 3-manifold weakly compatible with a complete Riemannian metric \(g\)

if \[ \sec(g) \leq -mg^2 \]
then \((M,\mathcal{I})\) is universally tight

here \[ m_g = \sup_{\mathcal{M}} \| \mathcal{A}(\ln \theta') - \theta \ln \rho \| \]

where \(\rho\) is the length of a Reeb vector field
\(\theta'\) is the instantaneous rotation of \(\mathcal{I}\)

one might hope this might be useful in finding tight contact structures on hyperbolic 3-manifolds

another theorem that might help with this is

\[ \text{Th}^3 \text{(EKM)}: \]

let \((M,\mathcal{I})\) be a closed contact manifold

suppose \(M\) admits a complete metric \(g\) such that the sectional curvature of \(g\) is bounded above by \(-k\) for some \(k > 0\)

and \(\mathcal{I}\) a Reeb vector field \(\mathcal{R}\) for \(\mathcal{I}\) such that
\[ N = \mathcal{R}/\|\mathcal{R}\| \quad \text{satisfies} \quad \|\nabla N\| \leq \sqrt{k} \]

then the universal cover of \((M,\mathcal{I})\) is tight

there are several other results and conjectures we will discuss later but first we give some Riemannian
Recall the curvature of a curve: 
Given $\gamma : \mathbb{R} \to \mathbb{R}^3$ unit speed 
$\gamma(0) = p$
then the curvature is 
$\kappa_p = |\gamma''(0)|$

Given a surface $\Sigma \subset \mathbb{R}^3$ a point $p \in \Sigma$ and a unit vector $v \in T_p \Sigma$

Let $\delta$ be the curve $\Sigma \cap \text{span} \{v, N\}$.

Parameterize $\gamma$ so it is unit speed.
The curvature of $\Sigma$ in direction $v$ is

$\kappa_p(v) = \gamma''(0) \cdot N$

Note: $\kappa_p : S^1 \to \mathbb{R}$

So $\kappa_p$ has a max and min: $\kappa_{\text{min}}, \kappa_{\text{max}}$

The Gauss curvature of $\Sigma$ at $p$ is $K = \kappa_{\text{min}} \kappa_{\text{max}}$

Examples:

1) if $K > 0$ then at $p$, $\Sigma$ "locally curves to one side of $T_p \Sigma"$

I.e. if you tried to flatten it on table it would rip
2) If $K < 0$ then at $p$, $\Sigma$ is "locally on both sides of $T_p \Sigma$

i.e. if you try to flatten it would wrinkle

In general, you can define $K$ for any surface with a Riemannian metric (i.e. inner product on tangent vectors)

does not have to be in $\mathbb{R}^3$, but this give intuition

the "curving in on itself" can be made rigorous by saying

if $\Sigma$ a compact oriented surface and $K > 0$
on $\Sigma$ then $\Sigma \cong S^2$

more generally: $M$ an $n$-manifold
$g$ a metric on $M$
$\sigma$ a plane in $T_p M$

more generally: $M$ an $n$-manifold
$g$ a metric on $M$
$\sigma$ a plane in $T_p M$

$\Sigma_\sigma$ gets metric from $M$

define

$$K(\sigma) = \text{Gauss curvature of } \Sigma_\sigma \text{ at } p$$

this is the sectional curvature of $(M,g)$ along $\sigma$

a vast generalization of above observation is

**Sphere Thm (Rauch, Klingenberg, Berger)**

If $M$ is a compact, simply connected, Riemannian $n$-manifold st. $\exists$ a constant $C > 0$ st.
$\frac{1}{4} C < k(\sigma) \leq C$

for all $\sigma$, then $M$ is homeomorphic to $S^n$

- Brendle-Schoen 2007 $\Rightarrow$ diffeo!
- if $<$ changed to $\leq$ then not true! eg $\mathbb{C}P^n$

**Thm (Cartan-Hadamard):**

A simply connected manifold with a complete non positively curved metric is diffeomorphic to $\mathbb{R}^n$

These are two prototypical examples of the interplay between geometry and topology!

More types of curvature:

- **Ricci curvature** is a "average" of sectional curvature:

  Given unit vector $v \in T_pM$
  let $v_1, \ldots, v_n \in T_pM$ st.
  $v_1, v_2, \ldots, v_n$ is an orthonormal basis

  $Ric_v(v) = \sum_{i=1}^{n-1} K(\text{span}\{v_i, v_i\})$

Scalar curvature is an "average" of Ricci curvature:

if $v_1, \ldots, v_n$ an orthonormal basis for $T_pM$
then

$s_p = \sum_{i \neq j} K(\text{span}\{v_i, v_j\})$
recall a geodesic in a Riemannian manifold \((M,g)\)

is a path \(\gamma\) that is locally length-minimizing

Fact: given \(v \in T_p M\) there is a unique geodesic

\[
\gamma : (-\varepsilon, \varepsilon) \to M \text{ s.t. } \gamma(0) = p \\
\gamma'(0) = v
\]

we say \(g\) is complete if each geodesic can be extended to

a geodesic defined on \(\mathbb{R}\) as

\[
\gamma : \mathbb{R} \to M
\]

e.g. \(\mathbb{R}^2\{-0\}\) with standard “flat” metric is not complete

we can define a map

\[
\exp_p : T_p M \to M
\]

by sending \(v \in T_p M \ (v \neq 0)\) to \(\gamma_v(1)\)

(and 0 to \(p\))

it is known \(\exp_p\) is a diffeomorphism from

a nbhd of 0 \(\in T_p M\) to a nbhd of \(p \in M\)

we define the injectivity radius at \(p\) to be

\[
\text{inj}_p = \sup \{ r \text{ s.t. } \exp_p |_{B_p(r)} \text{ is a diffeomorphism onto its image} \}
\]

ball of radius \(r\) about \(0 \in T_p M\)
one last thing we will need (for now) is the Hodge star operator

let $V$ be a vector space with inner product
and $e_1, \ldots, e_n$ an oriented orthonormal basis
and $e^1, \ldots, e^n$ the dual basis for $V^*$

\[
* : \Lambda^k V^* \to \Lambda^{n-k} V^*
\]

is defined by sending the basis element $e^{i_1} \ldots e^{i_k}$ to $e^{j_1} \ldots e^{j_{n-k}}$

where $e_1, \ldots, e_n, e_1, \ldots, e_{n-k}$ is an oriented basis for $V$

**exercise:** 1) $*1 = e^1 \ldots e^n$

so $* : \Lambda^0 V \to \Lambda^n V : \alpha \mapsto r e^{i_1} \ldots e^{i_n}$
2) \( *e^i = (-1)^{i-1} e^1 \wedge \cdots \wedge \hat{e}^i \wedge \cdots \wedge e^n \)

3) \( ** : \Lambda^p V^* \rightarrow \Lambda^p V^* \) is multiplication by \( (1)^{p(n-p)} \)

4) \( \langle v, w \rangle = \ast (\omega \wedge v \wedge w) = \ast (v \wedge \ast w) \)

Exercise:

Given a metric \( g \) we get an isomorphism

\[ \phi_g : TM \rightarrow T^*(M) : v \mapsto g(v, \cdot) \]

If we let \( \mathfrak{X}(M) = \) vector fields on \( M \)
and \( C^\infty(M) = \) functions on \( M \)
then for a 3-manifold we have

\[
\begin{align*}
C^\infty(M) \xrightarrow{D_3} \mathfrak{X}(M) & \xrightarrow{D_2} \mathfrak{X}(M) \xrightarrow{D_1} C^\infty(M) \\
\downarrow_{\text{id}} & \downarrow_{\phi_g} & \downarrow_{** \phi_g} & \downarrow_{\text{id}} \\
\Omega^0(M) & \xrightarrow{d} \Omega^1(M) & \xrightarrow{d} \Omega^2(M) & \xrightarrow{\text{grad}} \Omega^3(M)
\end{align*}
\]

The vertical arrows are isomorphisms define \( D_i \) using isomorphisms and \( d \)
Show for \( \mathbb{R}^3 \) with standard metric

\( D_i = \text{gradient} \)
$D_3 = 	ext{divergence}$

### III Contact Geometry

A contact structure on a 3-manifold $M$ is a plane field $\xi^2 \subset TM$ such that

$$\forall \xi \in M$$

that is non-integrable (i.e., not tangent to a surface along an open set in the surface).

One can show $\xi$ is contact $\iff$ (locally) a 1-form $\alpha$ s.t.

$$\xi = \ker \alpha$$

$$\alpha \wedge d\alpha > 0$$

(we always assume $\alpha$ can be defined globally)

**Examples:**

1) $\mathbb{R}^3$  $\xi_{\text{std}} = \ker (dz - r^2d\theta) = \text{span} \left\{ \frac{2}{\partial r}, \frac{r^2}{\partial z} + \frac{2}{\partial \theta} \right\}$

2) $S^3 = \text{unit sphere in } \mathbb{C}^2$
\[ \mathfrak{g}_{\text{std}} = \text{complex tangencies to } S^3 \]
\[ = \ker \left( x_i dy_i - y_i dx_i + x_i d\gamma_i - y_i d\tau_i \right) \]
\[ = \text{orthogonal planes to Hopf fibration} \]

3) \[ \mathbb{R}^3 \quad \mathfrak{g}_{0+} = \ker \left( \cos \varphi d\varphi + r \sin \varphi d\theta \right) \]

Note: \[ D = \{(r, \varphi, z) \mid z = 0, r \leq \pi\} \]

has \( \partial D \) tangent to \( \mathfrak{g}_{0+} \)

Such a disk is called an overtwisted disk

If a contact structure has such a disk, it is called overtwisted

Otherwise called tight

**Facts:**

1) (Darboux 1882) every contact structure is locally equivalent to \((\mathbb{R}^3, \mathfrak{g}_{\text{std}})\)

2) (Lutz, Martinet 1970) every closed oriented 3-manifold admits a contact structure
3) (Bennequin 1982)

\((S^3, \xi_{\text{std}})\) and \((\mathbb{R}^3, \xi_{\text{std}})\) are tight

(Birth of contact topology!)

4) (Eliashberg 1992)

classified overtwisted contact structures

\[
\{ \text{of structures up to isotopy} \} \leftrightarrow \{ \text{plane fields up to homotopy} \}
\]

can understand via algebraic topology

5) (Eliashberg 1992)

there is a unique tight contact structure on \(S^3\)

6) (Etnyre-Honda 2001)

not all closed orientable 3-manifolds have tight contact structures

\[ J^+ \text{ doesn't } \]

(Poincaré homology sphere with opposite orientation)

later Lisca-Stipsicz: all Seifert fibered spaces have tight-contact structures except \(\mathbb{S}^{2n-1}\)
7) tight contact structures are important
   in CR-geometry
   as boundaries of symplectic manifolds
   in fluid mechanics
   in knot theory
   in 3-manifold topology
   and they have a rich and subtle structure

Major open question

Do hyperbolic manifolds admit tight contact structures

IV Metrics and Contact Structures

let $\mathcal{I}$ be a plane field on a 3-manifold $M$

exercise: the Frobenius theorem says $\mathcal{I}$ is integrable
   iff the flow of a non-zero vector field tangent to $\mathcal{I}$ preserves $\mathcal{I}$

so if $\mathcal{I}$ contact then it must twist as you flow
   along a vector field tangent to $\mathcal{I}$

let's see how to measure this with a Riemannian metric

let $g$ be a metric on $M$ and
   $\mathcal{I}$ be a plane field

fix an orthonormal basis $u,v$ for $\mathcal{I}$ and
   let $n$ = oriented unit normal to $\mathcal{I}$
we want to measure how much $v$ twists as we flow along $\tau$

let $\phi_t$ be the flow of $u$

$$g((\phi_t)_* v, n)$$

says how much $v$ twists but to normalize we scale and define

$$\Theta(t) = \cos^{-1} \left( - \frac{g((\phi_t)_* v, n)}{\|((\phi_t)_* v)\|} \right)$$

for $\theta$ to be a (positive) contact structure $\Theta(t)$ must be increasing.

we call $\theta' = \Theta'(0)$ the instantaneous rotation of $\theta$

$\theta'$ is a function on $M$ and $\theta$ is contact $\iff \theta' > 0$

**note:** $\theta'$ might depend on $v,u$

to see this is not the case we can rewrite (1) as

$$\cos \Theta(t) = - \frac{g((\phi_t)_* v, n)}{\|((\phi_t)_* v)\|}$$

differentiate with respect to $t$ to get

$$\theta'(t) \sin(\Theta(t)) = \frac{d}{dt} \frac{g((\phi_t)_* v, n)}{\|((\phi_t)_* v)\|}$$

$$= \frac{d}{dt} \left( \frac{1}{\|((\phi_t)_* v)\|} \right) g((\phi_t)_* v, n) + \frac{1}{\|((\phi_t)_* v)\|} \frac{d}{dt} g((\phi_t)_* v, n)$$

at $t=0$

$$\theta' = (\theta) g(u,v) + \frac{1}{\theta} g((\phi_t)_* v, n)$$

**only consider one point so no derivative of $g$**
\[ \theta' = g(n, [u,v]) \]

if \( \alpha \) is any contact form for \( \mathcal{I} \) then \( m \alpha (\cdot) = g(n, \cdot) \)

for some function \( m \)

so \[ \theta' = m \alpha ([u,v]) \]

\[ = m \left( u \cdot \alpha (v) - v \cdot \alpha (u) + \alpha ([u,v]) \right) \]

\[ = m \, d\alpha (u,v) \]

now if \( u', v' \) are other oriented orthonormal vector fields spanning \( \mathcal{I} \) then

\[ u' = au + bv \quad \text{for functions } a,b,c,d \]

\[ v' = cu + dv \quad \text{st. } ac-bd = 1 \]

now \( m \, d\alpha (u', v') = m \, d\alpha (au+bv, cu+dv) \)

\[ = m ((ad-bc) d\alpha (u,v)) \]

\[ = m \, d\alpha (u,v) \]

so \( \theta' \) only depends on \( \mathcal{I} \) and \( g \)

**definition:**

we say a metric \( g \) on \( M \) is **weakly compatible** with a contact structure \( \mathcal{I} \) if there is a Reeb vector field \( R \) for \( \mathcal{I} \) such that \( R \perp g \)

(recall \( R \) is a Reeb vector field for \( \mathcal{I} \) if \( R \) is transversal to \( \mathcal{I} \) and flow of \( R \) preserves \( \mathcal{I} \)

given a contact form \( \alpha \) for \( \mathcal{I} \), \( \exists \), a unique Reeb field \( R_\alpha \) satisfying \( \alpha (R_\alpha) = 1 \) and \( (R_\alpha \cdot \alpha = 0) \)
Proposition 1:

Let \( \alpha \) be a contact form on \( M \)

\( g \) a Riemannian metric

\( R_\alpha \) the Reeb field of \( \alpha \)

Then the following are equivalent:

1) \( R_\alpha \perp g \) \( \) (i.e. \( g \) weakly compat. with \( \alpha \))

2) \( *d\alpha = \Theta' \alpha \)

3) \( g(u,v) = \frac{1}{2} d\alpha(u,\phi(v)) + \rho^2 \alpha(u) \alpha(v) \)

where \( \rho = \| R_\alpha \| \)

\( J \) is complex structure on \( \alpha \)

given by rotation by \( \frac{\pi}{2} \)

\( \phi: TM \to \alpha \) is projection to

\( \alpha \) followed by \( J \)

Note: given any contact form \( \alpha \) with Reeb field \( R_\alpha \)

any positive functions \( \rho, \Theta': M \to \mathbb{R} \)

and any complex structure \( J: \alpha \to \alpha \)

such that \( d\alpha(v, Jv) > 0 \) for \( v \neq 0 \)

and \( d\alpha(Jv, Jv) = \alpha(v,v) \)

\( \alpha \) is said to be compatible with \( d\alpha \)

(lots of these)

Define \( TM = \mathbb{R} + \text{span}\{R_\alpha\} \overset{\rho \phi J}{\longrightarrow} \alpha \overset{J}{\longrightarrow} \alpha \)

then \( g(u,v) = \frac{1}{2} d\alpha(u, \phi(v)) + \rho^2 \alpha(u) \alpha(v) \)

is weakly compatible with \( \alpha \)

so every \( \alpha \) has lots of weakly compatible metrics!

Proof: 1) \( \Rightarrow \) 2)

\( \rho = \| R_\alpha \| \)

unit orthogonal to \( \alpha \) is \( n = \frac{R_\alpha}{\rho} \)
so \( p \alpha(u) = g(n, v) \)

from above computation we have

\[ \theta' = p \, d\alpha(u, v) \]

for any oriented orthonormal basis \( u, v \) for \( \mathfrak{g} \) (might only exist locally)

\( \{ e, e_1, e_3 \} = \{ u, v, n \} \) is an orthonormal basis for \( TM \)

let \( \{ e', e_1', e_3' \} \) be the dual basis for \( T^*M \)

so \( e_3' = p \alpha \) and \( \alpha = \frac{1}{p} e_3' \)

write \( d\alpha = a \, e_1' \wedge e_2 + b \, e_1' \wedge e_3' + c \, e_2' \wedge e_3' \)

\[ \text{note: } l_{e_3} d\alpha = l_{R^p x} d\alpha = \frac{1}{p} l_{R^p u} d\alpha = 0 \]

\[ \therefore b = c = 0 \]

\[ a = d\alpha \langle e, e_2 \rangle = d\alpha (u, v) = \theta'/p \]

\[ d\alpha = \frac{\theta'}{p} e_1' \wedge e_2 \]

**Exercise:** \( * e_1' \wedge e_2 = e_3 \)

so \( *d\alpha = \frac{\theta'}{p} e_3 = \theta' \alpha \)

2) \( \Rightarrow \) 1) let \( u, v \) be an oriented orthonormal basis for \( \mathfrak{g} \)

\( n \) the unit normal to \( \mathfrak{g} \)

\( u, v, n \) an orthonormal basis for \( TM \)

denote it \( e_1, e_2, e_3 \)

let \( e_1', e_2', e_3' \) be the dual basis

\[ *d\alpha = \theta' \alpha = \theta' m e_3 \]

some \( m \)

\[ \therefore d\alpha = * * d\alpha = * \theta' \alpha = \theta' m e_1 \wedge e_2 \]
so \( n \cdot \alpha = 0 \) \( \iff \) \( n \) parallel to \( X \alpha \)

1c. \( X \alpha \) orthogonal to \( \beta \)

3) \( \implies \) 1) is obvious so we are left to show

1) \( \implies \) 3) \( \left\{ \beta \in \Lambda^1 : \beta \cdot (R \alpha) = 0 \right\} = \{ \}

**Exercise:** \( \Phi_{\alpha} : \Lambda^1 \rightarrow \Lambda^1 : \beta \mapsto \alpha(\beta \cdot \cdot) \)

\[ \Phi_{\alpha} : \Lambda^1 \rightarrow \Lambda^1 : \sigma \mapsto g(\sigma \cdot \cdot) \]

are both isomorphisms

set \( A = \Phi_{\alpha} \circ \Phi_{\alpha} \)

**Claim:** \( A^2 = -\frac{1}{m^2} \text{id} \) for some positive \( m : M \rightarrow \mathbb{R} \)

indeed at a point pick a symplectic basis for \( \alpha \)

1x. \( e_1, e_2 \) s.t. \( \alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

let \( f^1, f^2 \) be the algebraic dual basis

\[ \Phi_{\alpha} : e_1 \mapsto f^2 \]

\[ \Phi_{\alpha} : e_2 \mapsto -f^1 \]

\( g \) is represented by some positive definite \( (a, b) \)

\[ (a, b)^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \]

so \( \Phi_{g} : e_1 \mapsto a f^1 + b f^2 \)

\( e_2 \mapsto b f^1 + c f^2 \)

\( \Phi_{g}^{-1} : f^1 \mapsto \frac{1}{ac-b^2} (ce_1 - be_2) \)

\( f^2 \mapsto \frac{1}{ac-b^2} (-be_1 + ae_2) \)

and \( \Phi_{g}^{-1} \circ \Phi_{\alpha} : e_1 \mapsto \frac{1}{ac-b^2} (-be_1 + ae_2) \)

\( e_2 \mapsto \frac{1}{ac-b^2} (ce_1 + be_2) \)
ie. its matrix is \[ A = \frac{1}{ac-b^2} \begin{pmatrix} -b & -c \\ a & b \end{pmatrix} \]

\[ \therefore A^2 = \frac{1}{(ac+b^2)^2} \begin{pmatrix} -b & -c \\ a & b \end{pmatrix} \begin{pmatrix} -b & -c \\ a & b \end{pmatrix} = \frac{1}{(ac+b^2)^2} \begin{pmatrix} b-a & bc-bc \\ -ba+ba & b^2-ac \end{pmatrix} = -\frac{1}{ac+b} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

so if \( m=\sqrt{ac-b^2} \), \( A^2 = -\frac{1}{m^2} \text{id} \)

set \( J = mA \)

note \( J^2 = m^2 A^2 = -\text{id} \)

so \( J \) is a complex structure on \( \mathbb{R}^2 \)

for \( v,w \in \mathbb{R}^2 \)

\[ g(Av,w) = \phi_g(Av)(w) = \phi_{dv}(w)(v) = dv(w,v) = -g(Av,w) \]

\[ \therefore g(Jv,w) = -g(v,Jw) \]

\[ g(Jv,Jw) = -g(v,J^2w) = g(v,w) \]

\[ g(v,w) = -g(v,J^2w) = -m^2 g(v,A^2w) = -m^2 dv(Aw,v) = -m dv(Jw,v) = mdv(v,Jw) \]

let \( u,v \) be orthonormal basis for \( \mathbb{R}^2 \), \( n \) unit normal to \( \mathbb{R}^2 \)

a general vector is

\[ w = au + bv + cn \]
we know \( L_n \alpha = L_{R_n} \alpha = 0 \)

so \( d\alpha(au+bv+cn, V) \)

\[
= d\alpha(au+bv, V) \\
= d\alpha(w^1, V)
\]

any vector

projection of \( w \) to \( \mathbb{R}^3 \)

normal component

now

\[
g(U, V) = g(U^1, V^1) + g(U^2, V^2) \\
= m d\alpha(U^1, V^1) + g(g(U, n)n, g(V, n)n) \\
= m d\alpha(U, f(V)) + g(U, n)g(V, n) \\
= m d\alpha(U, f(V)) + \rho^2 \alpha(U)\alpha(V)
\]

\( (3) \)

\( \text{(since } \|R_n\|\alpha = L_n\gamma) \)

we need to determine \( m \)

for this note for \( v \in \mathbb{R}^2 \)

\[
g(v, JV) = mg(v, Av) = m d\alpha(v, r) = 0
\]

so \( JV \) orthogonal to \( v \) (so \( J \) rotation by \( \pm \pi/2 \))

\( (4) \)

\[
\|v\| = 1 \text{ then } d\alpha(v, JV) = m d\alpha(v, Av) \\
= mg(Av, Av) \\
= \frac{1}{m} g(v, v) > 0
\]

\( (5) \)

so \( v, JV \) oriented orthonormal

basis for \( \mathbb{R}^3 \) (\( J \) rotation by \( \pi/2 \))

let \( \{e_1, e_2\} = \{v, JV\} \) and \( e_3 = n = R_\alpha / \rho \)

and \( e_1, e_2, e_3 \) be dual basis
as in proof of 1) ⇒ 2) we see
\[ \theta' = \frac{\theta}{\rho} e^\rho \neq e^2 \]

\[ \theta' = \frac{\theta}{\rho} \]
\[ = \frac{1}{m} g(v, v) = \frac{1}{m} \]
\[ (5) \]

so \( m = \frac{\theta}{\theta'} \)

plug into (4) to get
\[ g(U, V) = \frac{\theta}{\theta'} \cdot d\alpha(U, \phi(U)) + \rho^2 \alpha(U)\alpha(V) \]

**Definition:**

A contact structure \( \xi \) and a metric \( g \) are compatible if there is a contact form \( \alpha \) for \( \xi \) such that

\[ \| \alpha \| = 1 \quad \text{and} \]
\[ \alpha \cdot d\alpha = \theta' \alpha \]

for some constant \( \theta' \)

(this is equivalent to saying the unit orthogonal to \( \xi \) is a Reeb field and the instantaneous rotation is constant)

**Remark:**

This is the same as Chern and Hamilton's definition from 1984 if \( \theta' = 2 \)

This form of compatibility has been extensively studied from a Riemannian geometry perspective

(see book of David Blair and below)
we note that given a contact structure \( \xi \) there is a projection

\[
\{ \text{all metrics} \} \to \{ \text{compatible metrics $\theta' = 1$} \}
\]

to define \( \Pi_\xi \) note that if \( g_3 \) is an inner product on \( \xi \) then we can get a compatible metric as follows:

- \( g_3 \) gives area form \( \omega \) on \( \xi \)
- take any contact form \( \omega_0 \) for \( \xi \)

\[
\text{note } \omega_0 \text{ an area form on } \xi
\]
and for any \( f > 0 \),

\[
d(f \omega_0) = f \omega_0
\]
so \( \exists ! \) function \( f_0 \) s.t.

\[
d(f_0 \omega_0) = \omega
\]
set \( \omega = f_0 \omega_0 \)
- define \( g \) to be \( g_3 \) on \( \xi \) and \( R_\omega \) to be orthogonal to \( \xi \) and of unit length

now define \( \Pi_\xi(g) = \) extension of \( g|_\xi \) to a compatible metric

exercise: if \( g \) is compatible with \( \xi \) and \( \theta' = 1 \) then \( \Pi_\xi(g) = g \)

Questions:

1) how do geometric quantities, like various curvatures, of \( g \) and \( \Pi_\xi(g) \) compare? what if \( \Pi_\xi(g) \) is “close” to \( g \)?
2) if \( \xi_t \) is a family of contact structures how do \( \Pi_\xi(g) \) compare?
3) what can you say about image \( (\Pi_\xi) \)
4) if you fix \( g \) s.t. \( g \in \text{int } \Pi_\xi \cap \text{int } \Pi_{\xi'} \), what can you say about \( \xi \) and \( \xi' \)?
recall if $E \to M$ is a bundle then a connection on $E$ is a way to differentiating sections of $E$

more specifically if $\Gamma(E)$ = sections of $E$ and $\mathcal{X}(M) = \Gamma(TM)$ = vector fields on $M$

then a connection is a map

$$\nabla: \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$$

$$(\sigma, \sigma) \mapsto \nabla_\sigma \sigma$$

satisfying

1) $\nabla_\sigma \sigma$ is linear in $\mathcal{X}(M)$ as a $C^\infty(M)$ module

\[ \nabla_{f\sigma + g\psi} \sigma = f \nabla_\sigma \sigma + g \nabla_\psi \sigma \]

2) $\nabla_\sigma \sigma$ is linear in $\Gamma(E)$ as an $\mathbb{R}$-vector space

\[ \nabla_\sigma a\sigma + b\eta = a \nabla_\sigma \sigma + b \nabla_\sigma \eta \]

for $a,b \in \mathbb{R}$

3) $\nabla_\sigma \sigma$ satisfies a product rule

$$\nabla_\sigma (f \sigma) = f \nabla_\sigma \sigma + (\sigma \cdot f) \sigma$$

for $f \in C^\infty(M)$

_a linear connection is a connection on $\mathcal{X}(M) = \Gamma(TM)$_

**Facts:** 1) connections exist for any $E$
2) given a linear connection there exist unique connections on $\mathbf{T}M\otimes\ldots\otimes\mathbf{T}M\otimes\mathbf{T}^*M\otimes\ldots\otimes\mathbf{T}^*M$ such that

1) on $C^\infty(M) (=$ 0th tensor power of $\mathbf{T}M$)
\[ \nabla_v f = v.f = df(v) \]

2) \[ \nabla_v (\sigma \otimes \eta) = (\nabla_v \sigma) \otimes \eta + \sigma \otimes \nabla_v \eta \]

3) \[ \nabla_v (\text{tr} \ \sigma) = \text{tr} (\nabla_v \sigma) \]

3) given a metric $g$ there is a unique linear connection satisfying

1) \[ \nabla_v g(u, w) = g(\nabla_v u, w) + g(u, \nabla_v w) \] (compatible)

2) \[ \nabla_v w - \nabla_w v = [v, w] \] (symmetric)

this is called the Levi-Civita connection of $g$

we will always use this connection

\underline{Lemma 2:}

If $\mathbf{v}$ and $g$ are weakly compatible and $n$ is the unit vector field normal to $\mathbf{v}$, then
\[ \nabla_n n = - (\nabla n)^\flat \]
where $\nabla n$ is the component of $n$ in $\mathbf{v}$

and $\mathbf{n} = \llbracket R \rrbracket$ where $R$ is the Reeb vector field showing weak compatibility.
\textbf{Proof:} for any vector field $u$

\[ 0 = \nabla_u g(n,n) = 2g(\nabla u n, n) \]

so $\nabla n n$ is tangent to $\mathbb{P} = n^\perp$ now for $v \in \mathbb{P}$

\[ g(\nabla n n, v) = -g(n, \nabla n v) \quad (g(n,v) = 0 \quad \text{so } g(\nabla n r, n) + g(n, \nabla r n) = 0) \]

\[ = -g(n, \nabla n v - \nabla r n) \quad (\text{from above } g(n, \nabla n n) = 0) \]

\[ = -g(n, [n, v]) \quad (\text{recall } g(n, n) = \rho < n >) \]

\[ = \rho \alpha ([n, v]) \quad (\text{as above}) \]

\[ = \rho \alpha ([n, v]) = \rho \left( \frac{d}{dr} \alpha (r) \right) \]

\[ = \rho \left( \frac{d}{dr} \alpha (r) \right) = -\rho \left( \frac{d}{dr} \alpha (r) \right) \]

\[ = -g(\nabla \ln \rho, v) \quad (\text{def. gradient}) \]

so $\nabla n n$ is in $\mathbb{P}$ and pairs with all vectors in $\mathbb{P}$ the same as

\[ -\nabla \ln \rho, \quad \nabla n n = -(\nabla \ln \rho) \]

\textbf{Remark:}

1) note that is $\mathbb{P}$ and $g$ are compatible then the flow of the associated Reeb vector field is tangent to geodesics

2) Zeghib showed that no closed hyperbolic manifold can have a non-singular vector field whose flow traces out geodesics

this is one of the major motivations for introducing weakly compatible metrics!
**Convexity**

let $S$ be a hypersurface in a Riemannian manifold $(M, g)$ that bounds a domain $U$

$U$ is **geodesically convex** if for any geodesic $\gamma$ tangent to $S$ at a point $p$ we have

$$\forall \gamma U = \{p\}$$

locally a neighborhood of $p$ in $\gamma$ only intersects $U$ at $p$


**Lemma 1:**

if $f : M \to \mathbb{R}$ s.t. $c \in \mathbb{R}$ is a regular value and

$$S = f^{-1}(c)$$

$$U = f^{-1}(-\infty, c]$$

then $U$ is geodesically convex

$$\iff \nabla^2 f(\tau, \nu) > 0 \quad \forall \nu \in TS$$

here $\nabla^2 f(\tau, \nu)$ is the **Hessian of $f$** and is defined by

$$\nabla^2 f(u, v) = (\nabla_u df)(v)$$

$$= \nabla_u \nabla_v f - \nabla_{[u, v]} f = \frac{1}{2} \sum_{i,j} g^{ij} (u_i v_j f)$$

**Idea:** compose $f$ with a geodesic to get map $(-\epsilon, \epsilon) \to \mathbb{R}$

its first derivative is 0 since tangent to $S$

its second derivative is positive ...
let \( \text{conv}(g) = \sup \{ B_r(p) \text{ is geodesically convex} \} \)

**example:** unit \( S^2 \subset \mathbb{R}^3 \) with induced metric has

\[
\text{conv} = \frac{\pi}{2}
\]

**Thm 2:**

if \( K > 0 \) and \( \sec(g) \leq K \) then

\[
\text{conv}(g) \geq \min \{ \text{inj}(g), \frac{\pi}{2\sqrt{K}} \}
\]

where \( \text{inj}(g) \) is the injectivity radius

if \( \sec(g) \leq 0 \), then \( \text{conv}(g) = \text{inj}(g) \)

now for symplectic convexity

let \((W, J)\) be an almost complex manifold

\( \Sigma \) a domain in \( W \) bounded by \( \Sigma \)

let \( C \subset T\Sigma \) be the complex tangencies to \( \Sigma \)

\( \text{i.e., } C = T\Sigma \cap J(T\Sigma) \)

we say \( \Sigma \) (or \( S^2 \)) is (strongly) pseudo convex if \( C \) is

a positive contact structure (and \( \Sigma \) oriented as \( \partial \Sigma \))

if \( f: W \to \mathbb{R} \) is a function and \( c \) a regular value s.t.

\[
\Sigma = f^{-1}(c)
\]

\[
\mathcal{S}_\Sigma = f^{-1}((-\infty, c])
\]

then

\[
C = \ker (-df \circ J)
\]
so $C$ a contact structure $\iff$ $L(u,v) > 0$ for $v \in \mathcal{C}$

where $L(u,v) = -d(df \circ J)(u,Jv)$

is the Levi form

Why do we care about pseudoconvex hypersurfaces?

**answer:** control holomorphic curves

given a Riemannian surface $(F,i)$ and an almost complex manifold $(X,J)$ a map $u: F \rightarrow X$ is called **holomorphic** if

$du \circ J = J du$ (du preserves respects almost complex str)

if $\Sigma$ is pseudoconvex surface bounding $\mathcal{L}$ and $u(F) \subset \mathcal{L}$ then $u(F)$ can't be tangent to $\Sigma$ (if $\Sigma = f^{-1}(c)$ as above $ou$ satisfies a "maximum principal")

where do we use holomorphic curves?

**Thm (Hofer):**

if $M$ is closed and $\mathcal{L}$ is an overtwisted contact str on $M$, then any Reeb vector field for $\mathcal{L}$ has a close orbit

**Sketch of proof:**

consider $W = (-\infty,0] \times M$
If $\alpha$ is a contact form for $\mathbb{R}$, then $\omega = d(e^{\alpha})$ a symplectic structure on $W$ and $J$ an almost complex str. $J$ on $W$ that sends $R_\alpha$ to $\frac{\partial}{\partial t}$ and preserves $\mathbb{R}$.

you can easily check $\{t\} \times M$ pseudoconver $\forall t$

let $D$ be an overtwisted disk in $(M, J)$

its characteristic foliation is

- Bishop proved there are holomorphic disks
  
  $u_t : (D^2, \partial D^2) \rightarrow (W, \{0\} \times D) \quad t \in [0, e)$
  
  such that
  
  $u_0$ constantly $p \in D$

  $\bigcup u_t(\partial D^2)$ fill a nbhd of $p \in D$

**Fact:** $u(\partial D^2)$ must be transverse to leaves of $D$, (if not constant)

- "Standard" functional analysis says

  if you extend the family of $u_t$ above they always
  
  fill out an open subset of $D$

  (this is because the holomorphic curve equations are elliptic)

- What happens if we have a Cauchy sequence of holomorphic disks $u_n : (D^2, \partial D^2) \rightarrow (W, \{0\} \times D)$

  if $u_n$ stay in $[a, a] \times W$ for some $a$,

  then Arzelà-Ascoli says they will
  
  converge to another holomorphic disk
unless the $\nabla u_n$ blows up
but in this situation Gromov says
that can't happen ("no bubbling" since all $\mathcal{E}_t \times M$ convex)

so if $u_n$ don't converge to a holomorphic
disk, then image of $u_n$ must
"go to $-\infty$"

Hofer says if this is the case then there
must be a periodic orbit in $R^x$
specifically, limit of $u_n$ will be asymptotic
to $(-\infty, 0] \times R$ for some periodic orbit

so if no periodic orbits any $u_n$ converge
to another holomorphic disk

$\therefore$ subset of $D$ filled by boundaries of
holomorphic disks is closed!

$\bullet$ in this case subset of $D$ filled by such
boundaries is open and closed $\therefore$ all of $D$!

but first boundary of holomorphic disk to touch $2D$
will be tangent to $\partial D = \text{leaf of } D_i$

this contradicts Fact above

\[ \therefore \text{must have closed orbit in flow of } R_a \]

\textbf{note:} if $M$ not closed, but $(-\infty, 0] \times 2M$ pseudo-convex

then same argument says must be periodic orbit!

\textbf{Putting 2 convexities together}

\textbf{Thm 3 (E-Komendarczyk-Massot)}:

Let $g$ be weakly compatible with $(M, \jmath)$

$S$ a surface in $M$ cut out by $f$ and

$U$ the sublevel set

$\Sigma = R \times S \subseteq R \times M$

$\Sigma = R \times U \subseteq R \times M$

Let $R$ is the Reeb field for $\jmath$ showing weak compatibility with $g$

$\jmath$ be an almost complex structure on $R \times M$ that

preserves $\jmath$ and sends $R$ to $\frac{\partial}{\partial t}$

for any $v \in C\subseteq \text{complex tangencies to } \Sigma$

we have

\[ L(\sigma, v) = \nabla^2 f(v, \sigma) + \nabla^2 f(Jv, J\sigma) \| R \|^2 \]

\[ - \| v \|^2 g(\nabla \ln \rho - (\nabla \ln \theta)^T, \nabla f) \]

\textbf{the proof is a long computation}

we can use this to prove a \textbf{Darboux theorem with estimates}

given $(M, \jmath)$ a contact manifold and

$g$ a metric on $M$

we define
\( \gamma(g) = \sup \{ I \text{ restricted to } B_r(p) \mid I \text{ is tight for all } p \in M \} \)

**Theorem (EKM):**

If \( g \) is a metric compatible with \((M, J)\), then \( \gamma(g) \geq \text{conv}(g) \)

**Note:** If \( M \) compact it is easy to use Darboux + Lebesgue number to prove \( \gamma(g) \) bounded by positive number, but not possible on non-compact manifold and computing a lower bound in compact case would be hard.

**Proof:**

Fix a point \( p \in M \) for all \( r < \text{conv}(g) \) we know \( B_r(p) \) is geodesically convex.

Let \( C_r \) = complex tangencies to \( \mathcal{E}(R \times B_r(p)) \)

then \( \nabla^2 f(v,v), \nabla^2 f(Jv,Jv) \geq 0 \) and one must be positive.

\( L(v,v) > 0 \)

and \( C_r \) pseudoconvex for \( r < \text{conv}(g) \)

we can adapt a theorem of Hofer (see note above) to see that if \( \exists \) \( B_r(p) \) is overtwisted then there is a close Reeb orbit \( \delta \) in \( B_r(p) \)

Recall \( \delta \) is also a geodesic.
but now

let \( r' \) be the largest radius \( \sigma \)

\( \partial B_{r'}(p) \cap \sigma \)

must have \( \partial B_r(p) \) tangent to \( \sigma \)

and \( \gamma \subset B_{r'}(p) \) \& convexity

so \( \partial B_r(p) \) tight!

\[ \text{Thm 5 (EKM):} \]

let \((M,\mathfrak{h})\) be a contact 3-manifold weakly compatible

with a complete Riemannian metric \( g \)

if

\[ \sec(g) \leq -m_g^2 \]

then \((M,\mathfrak{h})\) is universally tight

here

\[ m_g = \sup_{M} \left\| \nabla \ln p - \nabla (\ln \Theta') \right\| \]

where \( p \) is the length of a Reeb vector field

\( \Theta' \) is the instantaneous rotation of \( \partial \)

Proof:

pull everything back to the universal cover \( \tilde{M} \cong \mathbb{R}^3 \)

let \( B_r(p) \) be ball of radius \( r \) about \( p \)

if \( \sec(g) \leq -K \) for some \( K > 0 \), then

below we see for \( c_{K}(r) > m_{g} \)

\( \mathcal{D}\left( \mathbb{R} \times B_{r}(p) \right) \) is pseudo-convex

where

\( c_{K}(r) = \sqrt{K} \coth \left( \sqrt{K} r \right) \)
but we are assuming $\sqrt{K} \geq \sqrt{-\text{sec}(g)} > mg$

so $\exists (R \times B_r(p))$ is pseudo-convex for all $r$

now arguing as in last proof if $(R^3, \beta)$ is overtwisted

$\exists$ a closed Reeb orbit in $B_r(p)$ for some $r$

note: $R \times \Sigma$ is holomorphic in $R \times M$ (with $J$ used above)

Start shrinking $B_r(p)$ to first $r_0$ where $\partial B_{r_0}(p) \cap \Sigma \neq \emptyset$

there $R \times \Sigma$ will be tangent to $R \times \partial B_{r_0}(p)$

but this contradicts pseudo-convexity

$\therefore M = R^3$ is tight

now for the above claim about pseudoconvexity

fix $p$ and let

$q_p : M \to R : x \mapsto d(p, x)$

if $K > 0$ and $\text{sec}(g) \leq -K$ then it is known that

$\nabla^2 g \geq c \epsilon_k(n) g$

now for $v \in C_p$ we can write it as

$v = v^1 + x R + b \frac{d}{dt}$

recall $J$ is an isometry on $\Sigma$ so
\[
g(v,v) = g(v;v^3) + a^2 + b^2 = g(Jv;Jv^3) + a^2 + b^2 = g(Jv,Jv)
\]

so

\[
\mathcal{L}(v,v) = \nabla^2 \rho (v,v) + \nabla^2 \rho (Jv,Jv) - g(D_g,\nabla \rho) \| \sigma \|^2 \\
\uparrow \text{Theo.3}
\]

\[
\geq 2 \, c_t \, (r) \| \sigma \|^2 - \| \nabla \rho \| \| g(D_g,\rho) \| \| \sigma \|^2 \\
\geq 2 \, c_t \, (r) \| \sigma \|^2 - \| D_g \| \| \sigma \|^2 \\
\geq 2 \, c_t \, (r) \| \sigma \|^2 - m_g \| \sigma \|^2 \\
\geq (c_t \, (r) - m_g) \| \sigma \|^2
\]

so \( \mathcal{L}(R \times B \rho(v)) \) is convex if \( c_t \, (r) \geq m_g \).
VI Seeing Overtwisted Disks and the Contact Sphere Thm

Thm 6 (EKM):

let \((M, \sigma)\) be a contact manifold compatible with \(g\)
if \(r < \text{inj}(g)\) and \(\sigma\) is overtwisted on \(B_r(\rho)\)
then \(\partial B_r(\rho) = S_r(\rho)\) contains an overtwisted disk

so we can't guarantee \(\bar{\sigma}(\rho)\) is tight, we can

clearly see when it is not

we will prove this later but now prove the contact
sphere theorem

Proof of Contact Sphere Thm (Thm 1, from intro)

Recall we have \((M, \sigma)\) compatible with \(g\) and
\(\exists K > 0\) s.t. \(\frac{4}{9} K < \text{sec}(g) \leq K\)

we want to show \(\sigma\) is tight
we pull \(\sigma\) back to universal cover \(\tilde{M}\)
ordinary sphere thm says \(\tilde{M} \cong S^3\)
easy to see if pulled back \(\sigma\) is tight
so is \(\sigma\)

so we assume \(M = S^3\)
for contradiction assume \(\sigma\) is overtwisted

let \(D\) be an overtwisted disk

rescale \(g\) so that \(K = 1\) (note still compatible)

Bonnet-Meyer's Thm says that
\[
\text{diam}(M) < \frac{\pi}{\sqrt{3/1}} = \frac{3\pi}{2}
\]
a result of Klingenberg says that
\[ \text{inj}(g) \geq \frac{\pi}{\sqrt{3}} = \pi \]
and we mentioned above
\[ \text{conv}(g) \geq \frac{\pi}{2\sqrt{3}} = \frac{\pi}{2} \]

using "standard" Toponogov comparison argument
we see that if \( p, q \in M \) such that
\[ d(p, q) = \text{diam}(M) \]
then there are \( r_p < \pi \) and \( r_q < \frac{\pi}{2} \)
so
\[ M = B_r(p) \cup B_r(q) \]
we can assume \( D \) does not contain \( q \)

Thm 4 above says \( B_{r_p}(p) \) is standard contact ball
for standard contact ball there is a vector field \( v \)
whose flow pushes any point \( \neq q \) into small nbhd
of \( \partial B_{r_q}(q) \) c int \( B_{r_p}(p) \)
so we can assume \( D \subset B_{r_p}(p) \)

but Thm 5 above says \( \partial D_{r_p}(p) \) must
now contain an overtwisted disk \( D' \)
so \( D' \subset \partial D_{r_p}(p) \subset B_{r_q}(q) \)
contradicting tightness of $\exists \beta_q(u)$

:. $\exists$ tight contact str on $S^2$

(Eliashberg says $\exists$ standard)

for the proof of Thm 5 we need some preliminaries

**Lemma 6:**

if $(\eta, \beta)$ is compatible with $g$ and $r < inj_\eta (g)$, then the characteristic foliation $(\partial B_r(p))$ has only 2 singular points (and they are $\partial \gamma \cap \partial B_r(p)$ where $\gamma$ is a Reeb flow line through $p$)

**Proof:**

suppose $x \in \partial B_r(p)$ is a singular point

so we have

let $\gamma$ be a geodesic starting at $p$ st: $\gamma(t) = x$

by the Gauss lemma we know

$T_x(\partial B_r(p)) = T_x$

is orthogonal to $\gamma'(r)$

:. $\gamma'(r) = R$ the Reeb field

and since the Reeb flow is tangent to geodesics we see

$\gamma$ is a Reeb flow line through $p$
We call a surface $\Sigma$ in $(M,3)$, 3-convex if there is a vector field transverse to $\Sigma$ whose flow preserves $\Sigma$.

We say a sphere $S$ is simple if $S_3$ contains only two singular points (we call the positive one the north pole and the other the south pole).

$S_3$ is almost horizontal if, in addition, all closed leaves of $S_3$ are oriented as the boundary of the disk containing the north pole.

**Examples:**

- [Almost horizontal](#)
- [Not almost horizontal](#)

**Lemma 7 (Giroux):**

If $S_3$ is simple, then

$\Sigma_3$ is 3-convex $\iff$ $S_3$ has no degenerate closed orbits

We are now ready for our main technical result.

**Proposition 8:**

Let $B$ be a ball in $(M,3)$.

$B$ is a union of a point $p$ and spheres $S_\epsilon$ for $\epsilon \in (0,1)$. 
1) \((S_t)_1\) is simple } \Rightarrow \text{ (}S_t\text{), almost horizontal } 3 \text{ } \| \text{ } 6 \text{ } \text{ tight}

2) all \((S_t)_3\), almost horizontal \forall t \Rightarrow 3 \| 6 \text{ tight}

3) if \((S_t)_3\), all simple and \(3 \| 6\) is overtwisted, then \(\exists t_0\) such that
   \(\text{(}S_t\text{)}_1\) has a closed leaf for \(t \geq t_0\)
   \(3 \| 6\) tight for \(t < t_0\)

**Proof of Thm 6**:

\[B_r(p) = \bigcup_{t \in (0, 1]} S_t\]

Lemma 7 says \((S_t)_3\) simple since \(r < \text{inj}(g)\)

Part 3) of Prop 8 \Rightarrow if \(B_r(p)\) is overtwisted then we see an overtwisted disk on \(3 \| 6(p)\)

**Proof of Proposition 8**:

1) obvious, if \(\exists\) closed leaf then contact str is overtwisted

2) for small \(t\), \(B_r\) will be tight by Darboux's Thm
   so \((S_t)_1\) has no closed leaves for \(t\) small
   if there are no closed orbits in \((S_t)_3\), for all \(t\), then
   by lemma 7 all of the \(S_t\) are \(\mathfrak{g}\)-convex
   from this it is easy to argue that \(3 \| 6\) is tight
   (can show \(\overline{B_r - B_e} = S^3 \times \{0, 1\}\) has an \(\mathfrak{g}\)-invariant contact str, so \(B_1\) is the result of adding a "collar nbhd" to \(B_e\))
   so \(3 \| 6\) is tight unless some \((S_t)_3\) has a closed leaf
let $t_0$ be smallest $t$ such that $(S_t)_{t_0}$ has a closed orbit

the closed orbit $C$ of $(S_{t_0})_{t_0}$ must be degenerate

(we assume only one orbit, but you can consider other cases)

we can find an nbhd $A$ of $C$ on $S_{t_0}$ st:

we can map $A \times [t_0-\epsilon, t_0+\epsilon]$ into $B$

so that

a) $A_t = A \times \{t\}$ maps to $S_t$

b) leaves of $(S_t)_{t_0}$ enter top of $A$
   and exit bottom of $A$

c) $\{p\} \times [t_0-\epsilon, t_0+\epsilon]$ maps to Legendrian arcs

**note:** $A_t$ has no closed leaves for $t < t_0$

recall the contact planes along $\{p\} \times [t_0-\epsilon, t_0+\epsilon]$
rotate in left-handed way

since $(S_{t_0})_{t_0}$ is almost horizontal we see

just before $t_0$ we see
so by Poincaré-Bendixson there must be a closed leaf in $A_t$

contradicts fact that to smallest such $t$

3) **note:** the above argument say any time a new periodic orbit is born and is closest to north pole it must go east to west

![Diagram](https://via.placeholder.com/150)

same argument says if a northern most periodic orbit dies it must go west to east

now let $t_0$ be the first time a closed leaf appears in $(S_t)_t$

from above it must go east to west

as $t$ increases there can be finitely many birth/deaths of periodic orbits

we inductively see that northern most orbit is always east to west and so can’t die (i.e. all $(S_t)_t$ for $t \geq t_0$ have closed orbit)

let $t_1, \ldots, t_n$ be other birth/death times

from above $t \in [t_0, t_k]$ one orbit east to west

suppose hypothesis true for $t < t_k$

if $t_{k+1}$ is a birth of an orbit closer to north pole than other orbits then done by above observation
(must go east to west)

if not northern most then done since northern most still east to west

if tkt a death it can't involve northern most orbit since deaths of northern most orbit only occur for west to east orbits

i. done

if tc to then \( |b_t| \) tight by 2)