Contact Topology and Riemannian Geometry

I. Introduction

for quite some time it has been clear that there are deep connections between the topology of 3-manifolds and Riemannian metrics (i.e. Thurston's geometrization program)

more recently there have also been deep connections between the topology of 3-manifolds and contact geometry

but there seems to be few results relating properties of contact structures (like tightness) and Riemannian geometry

in these talks we will explore such connections

among other things we will prove

Th^m 1 (E-Komendarczyk-Massot):

> let (M,ξ,g) be a contact metric 3-manifold
> If g is a complete metric and $\exists K>0$ st.
> the sectional curvature of g satisfies
> $$\tfrac{4}{9}K < \sec(g) \leq K$$
> then the universal cover of (M,ξ) is (S^3,ξ_{std}) where ξ_{std} is the unique tight contact structure on S^3

define later

recall curvature

tight later

- Ge-Huang improved $4/9$ to $1/4$

- the classical Sphere theorem said "curvature can controle topology" here we see it can also controle contact topology!

Th$^{\underline{m}}$ 2 (EKM):

let (M, ξ) be a contact 3-manifold weakly compatible with a complete Riemannian metric g

if

$$sec(g) \leq -m_g^2$$

then (M, ξ) is universally tight

here

$v^{\perp}$ is component of v perp. to ξ

$$m_g = \sup_M \| \nabla(\ln \theta')^{\perp} - \nabla \ln \rho \|$$

where ρ is the length of a Reeb vector field

θ' is the instantanious rotation of ξ

define later

- one might hope this might be useful in finding tight contact structures on hyperbolic 3-manifolds

another theorem that might help with this is

Th$^{\underline{m}}$ 3 (EKM):

let (M, ξ) be a closed contact manifold

suppose M admits a complete metric g such that the sectional curvature of g is bounded above by $-K$ for some $K > 0$

and $\exists$ a Reeb vector field R for ξ such that

$N = R/\|R\|$ satisfies

covariant derivative (recall later)

$$\| \nabla_N N \| < \sqrt{K}$$

then the universal cover of (M, ξ) is tight

there are several other results and conjectures we will discuss later but first we give some Riemannian and contact background

II. Riemannian geometry

recall the curvature of a curve:

given $\gamma : \mathbb{R} \to \mathbb{R}^3$ unit speed

$\gamma(0) = p$

then the curvature is

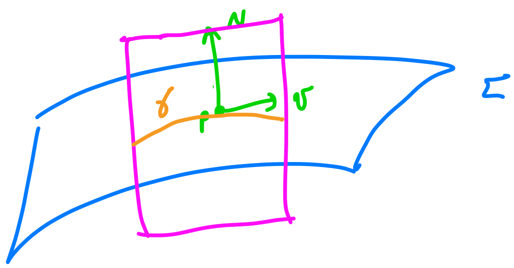
$$\kappa_p = |\gamma''(0)|$$

how fast γ bending from a line

given a surface $\Sigma \subset \mathbb{R}^3$ a point $p \in \Sigma$ and a unit vector $v \in T_p\Sigma$

let γ be the curve $\Sigma \cap \operatorname{span}\{v, N\}$

normal vector to Σ



parameterize γ so it is unit speed

the curvature of Σ in direction v is

$$\kappa_p(v) = \gamma''(0) \cdot N$$

note: $\kappa_p : S^1 \to \mathbb{R}$

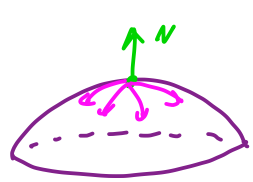
unit circle in $T_p\Sigma$

so κ_p has a max and min: $\kappa_{min}, \kappa_{max}$

the Gauss curvature of Σ at p is $K = \kappa_{min}\kappa_{max}$

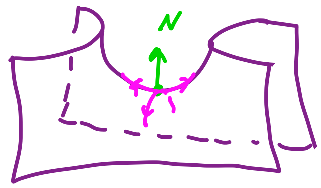
examples:

1) if $K > 0$ then at p, Σ "locally curves to one side of $T_p\Sigma$"



i.e. if you tried to flatten it on table it would rip

2) if $K<0$ then at p, Σ is "locally on both sides of $T_p\Sigma$



i.e. if you try to flatten it would wrinkle

In general, you can define K for any surface with a Riemannian metric (i.e. inner product on tangent vectors)

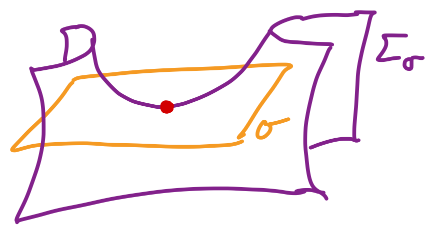
does not have to be in $\mathbb{R}^3$, but this give intuition

the "curving in on itself" can be made rigorus by saying

if Σ a compact oriented surface and $K>0$ on Σ then $\Sigma \cong S^2$

more generally: M an n-manifold, g a metric on M, σ a plane in T_pM $\Big\} \Rightarrow$ Σ_σ a surface in M made of geodesics tangent to σ

recall these are "straight lines" "locally shortest paths"



Σ_σ gets metric from M

define

$$K(\sigma) = \text{Gauss curvature of } \Sigma_\sigma \text{ at } p$$

this is the sectional curvature of (M,g) along σ

a vast generalization of above observation is

Sphere Th$^{\underline{m}}$ (Rauch, Klingenberg, Berger)

if M is a compact, simply connected, Riemannian n-manifold s.t. $\exists$ a constant $C>0$ s.t.

$$\tfrac{1}{4} C < K(\sigma) \leq C$$

for all σ, then M is homeomorphic to S^n

- Brendle-Schoen 2007 $\Rightarrow$ diffeo!
- if $<$ changed to $\leq$ then not true! eg $\mathbb{C}P^n$

Th$^{\underline{m}}$ (Cartan-Hadamard):

a simply connected manifold with a complete non positively curved metric is diffeomorphic to $\mathbb{R}^n$

these are two prototypical examples of the interplay between geometry and topology!

more types of curvature:

Ricci curvature is a "average" of sectional curvature:

given unit vector $v \in T_pM$
let $v_1, \ldots, v_{n-1} \in T_pM$ s.t.
$v, v_1, \ldots v_{n-1}$ is an orthonormal basis

$$Ric_p(v) = \sum_{i=1}^{n-1} K(\mathrm{span}\{v, v_i\})$$

some put $\frac{1}{n-1}$ here

Scalar curvature is an "average" of Ricci curvature

if $v_1, \ldots, v_n$ an orthonormal basis for T_pM
then
$$S_p = \sum_{i \neq j} K(\mathrm{span}\{v_i, v_j\})$$

recall a geodesic in a Riemannian manifold (M, g)

is a path γ that is locally length minimizing

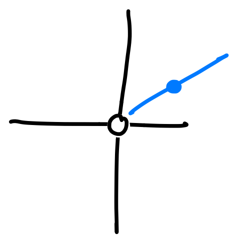
fact: given $v \in T_pM$ there is a unique geodesic

$$\gamma_v : (-\varepsilon, \varepsilon) \to M \quad \text{s.t. } \gamma_v(0) = p, \ \gamma_v'(0) = v$$

we say g is complete if each geodesic can be extended to a geodesic defined on $\mathbb{R}$ ie

$$\gamma_v : \mathbb{R} \to M$$

eg: $\mathbb{R}^2 - \{0\}$ with standard "flat" metric is not complete



we can define a map

$$\exp_p : T_pM \to M$$

by sending $v \in T_pM$ $(v \neq 0)$ to $\gamma_v(1)$

(and 0 to p)

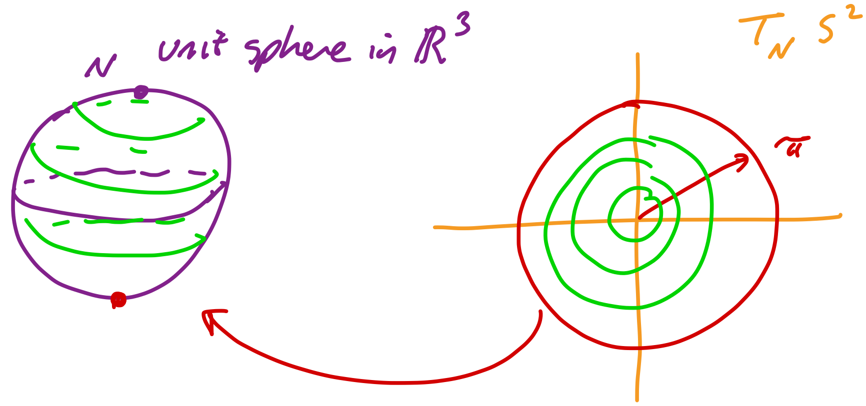
it is known $\exp_p$ is a diffeomorphism from a nbhd of $0 \in T_pM$ to a nbhd of $p \in M$

we define the injectivity radius at p to be

$$\text{inj}_p = \sup\{ r \text{ s.t. } \exp_p|_{B_0(r)} \text{ is a diffeo. onto its image} \}$$

ball of radius r about $0 \in T_pM$

example:



so $\mathrm{inj}_N = \pi$

if $r < \mathrm{inj}_p$ then $B_p(r) = \mathrm{im}\, \exp_p(B_0(r))$

is called the geodesic ball of radius r

and its boundary $S_p(r)$ the geodesic sphere

one last thing we will need (for now) is the Hodge star operator

let V be a vector space with inner product

and $e_1, \dots, e_n$ an oriented orthonormal basis

and $e^1, \dots, e^n$ the dual basis for V^*

$$*: \Lambda^k V^* \to \Lambda^{n-k} V^*$$

is defined by sending the basis element $e^{i_1} \wedge \dots \wedge e^{i_k}$

to $e^{j_1} \wedge \dots \wedge e^{j_{n-k}}$

where $e_{i_1}, \dots, e_{i_k}, e_{j_1}, \dots, e_{j_{n-k}}$ is an oriented basis for V

exercise: 1) $*1 = e^1 \wedge \dots \wedge e^n$

so $*: \Lambda^0 V \to \Lambda^n V : r \mapsto r e^1 \wedge \dots \wedge e^n$

2) $*e^i = (-1)^{i-1} e^1 \wedge \dots \wedge \hat{e^i} \wedge \dots \wedge e^n$

3) $** : \Lambda^p V^* \to \Lambda^p V^*$ is multiplication by $(-1)^{p(n-p)}$

4) $\langle v, w \rangle = *(v \wedge *w) = *(w \wedge *v)$

inner product

now if g is a metric on M then it gives an innerproduct on T_pM for all $p \in M$ so we can apply the Hodge star to each T_pM to get a Hodge star operator

$$*: \Omega^k(M) \to \Omega^{n-k}(M)$$

exercise:

given a metric g we get an isomorphism

$$\phi_g : TM \to T^*(M) : v \mapsto g(v, \cdot)$$

if we let $\mathfrak{X}(M)$ = vector fields on M

and $C^\infty(M)$ = functions on M

then for a 3-manifold we have

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "D_1"] \arrow[d, "id"] & \mathfrak{X}(M) \arrow[r, "D_2"] \arrow[d, "\phi_g"] & \mathfrak{X}(M) \arrow[r, "D_3"] \arrow[d, "* \circ \phi_g"] & C^\infty(M) \arrow[d, "id"] \\
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M)
\end{tikzcd}$$

the vertical arrows are isomorphisms

define D_i using isomorphisms and d

Show for $\mathbb{R}^3$ with standard metric

$$D_1 = \text{gradient}$$

$D_2 = \text{curl}$

$D_3 = \text{divergence}$

III Contact Geometry

a contact structure on a 3-manifold M is a plane field

$$\begin{array}{ccc} \xi^2 & \subset & TM \\ & \searrow \swarrow & \\ & M & \end{array}$$

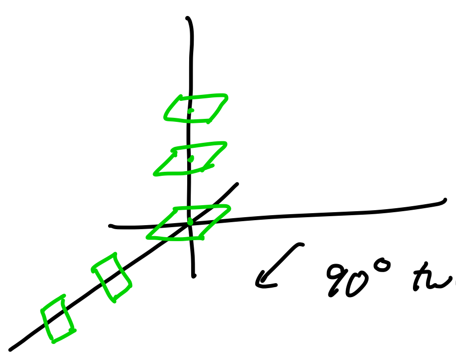
that is non-integrable
(i.e. not tangent to a surface along an open set in the surface)

one can show ξ is contact $\iff \exists$ (locally) a 1-form α s.t.
$\xi = \ker \alpha$
$\alpha \wedge d\alpha > 0$

(we always assume α can be defined globally)

examples:

1) $\mathbb{R}^3$ $\quad \xi_{std} = \ker(dz - r^2 d\theta) = \text{span}\left\{\frac{\partial}{\partial r},\ r^2\frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}\right\}$

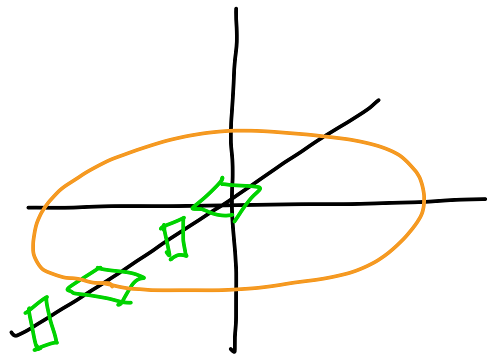


$\swarrow$ $90°$ twist

2) S^3 = unit sphere in $\mathbb{C}^2$

ξ_{std} = complex tangencies to S^3
= $\ker(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)$
= orthogonal planes to Hopf fibration

3) $\mathbb{R}^3$ $\quad \xi_{ot} = \ker(\cos r\, dz + r \sin r\, d\theta)$



note $D = \{(r,\theta,z) \mid z=0, r \leq \pi\}$
has ∂D tangent to ξ_{ot}

such a disk is called an overtwisted disk

If a contact structure has such a disk
it is called overtwisted
otherwise called tight

Facts:

1) (Darboux 1882) every contact structure is locally equivalent to $(\mathbb{R}^3, \xi_{std})$

2) (Lutz, Martinet 1970) every closed oriented 3-manifold admits a contact structure

3) (Bennequin 1982)

(S^3, ξ_{std}) and $(\mathbb{R}^3, \xi_{std})$ are tight

(Birth of contact topology!)

4) (Eliashberg 1992)

classified overtwisted contact structures

$\left\{ \begin{array}{c} \text{ot structures upto} \\ \text{isotopy} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{plane fields upto} \\ \text{homotopy} \end{array} \right\}$

can understand via algebraic topology

5) (Eliashberg 1992)

there is a unique tight contact structure on S^3

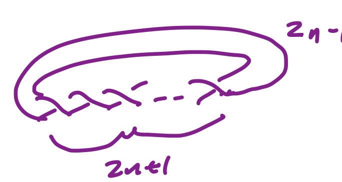
6) (Etnyre-Honda 2001)

not all closed orientable 3-manifolds have tight contact structures

doesn't

(Poincaré homology sphere with opposite orientation)

later Lisca-Stipsicz: all Seifert fibered spaces have tight contact structures except



7) tight contact structures are important
in CR-geometry
as boundaries of symplectic manifolds
in fluid mechanics
in knot theory
in 3-manifold topology

and they have a rich and subtle structure

Major open question

$\boxed{\text{Do hyperbolic manifolds admit tight contact structures}}$

IV Metrics and Contact Structures

let ξ be a plane field on a 3-manifold M

exercise: the Frobenius theorem says ξ is integrable iff the flow of a non-zero vector field tangent to ξ preserves ξ

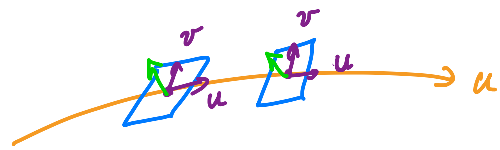
so if ξ contact then it must twist as you flow along a vector field tangent to ξ

let's see how to measure this with a Riemannian metric

let g be a metric on M and
ξ be a plane field

fix an orthonormal basis u, v for ξ and
let n = oriented unit normal to ξ

we want to measure how much v twists as we flow along v



let ϕ_t be the flow of u

$$g((\phi_{-t})_* v, n)$$

says how much v twists but to nomalize we scale and define

$$\theta(t) = \cos^{-1}\left(-\frac{g((\phi_{-t})_* v, n)}{\|(\phi_{-t})_* v\|}\right) \qquad (1)$$

<u>note</u> sign incorrect in EKM papen

for ξ to be a (positive) contact structure $\theta(t)$ must be increasing

we call $\theta' = \theta'(0)$ the <u>instantanious rotation</u> of ξ

$\boxed{\theta' \text{ is a function on } M \text{ and } \xi \text{ is contact} \iff \theta' > 0}$

<u>note</u>: θ' might depend on v, u

to see this is not the case we can rewrite (1) as

$$\cos\theta(t) = -\frac{g((\phi_{-t})_* v, n)}{\|(\phi_{-t})_* v\|}$$

differentiate with respect to t to get

$$\theta'(t)\sin(\theta(t)) = \frac{d}{dt}\,\frac{g((\phi_{-t})_* v, n)}{\|(\phi_{-t})_* v\|}$$

$$= \frac{d}{dt}\left(\frac{1}{\|(\phi_{-t})_* v\|}\right) g((\phi_{-t})_* v, n) + \frac{1}{\|(\phi_{-t})_* v\|}\frac{d}{dt} g((\phi_{-t})_* v, n)$$

at $t=0$

$$\theta' = (\quad)\, g(\cancel{u,n}) + \frac{1}{1} g(\mathcal{L}_u v, n)$$

only consider one point so no derivative of g

$$\theta' = g(n, [u,v])$$

if α is any contact form for ξ then $m\alpha(\cdot) = g(n,\cdot)$ for some function m

so $\theta' = m\,\alpha([u,v])$

$= m\big(u\cdot\alpha(v) - v\cdot\alpha(u) + \alpha([u,v])\big)$ 0 since $v \in \ker\alpha$

$= m\,d\alpha(u,v)$

now if u', v' two other oriented orthonormal vector fields spanning ξ then

$u' = au + bv$ for functions a,b,c,d

$v' = cu + dv$ s.t. $ac - bd = 1$

now $m\,d\alpha(u',v') = m\,d\alpha(au+bv, cu+dv)$

$= m\big((ad-bc)\,d\alpha(u,v)\big)$

$= m\,d\alpha(u,v)$

so $\boxed{\theta' \text{ only depends on } \xi \text{ and } g}$

definition:

we say a metric g on M is weakly compatible with a contact structure ξ if there is a Reeb vector field R for ξ such that $R \perp_g \xi$

(recall R is a Reeb vector field for ξ if R is transvers to ξ and flow of R preserves ξ

given a contact form α for ξ, $\exists$ a unique Reeb field R_α satisfying $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = 0$)

Proposition 1:

let α be a contact form on M

g a Riemannian metric

R_α the Reeb field of α

Then the following are equivalent

1) $R_\alpha \perp_g \xi$ (i.e. g weakly compat. with ξ)

2) $*d\alpha = \theta'\alpha$ Hodge star

3) $g(u,v) = \frac{\rho}{\theta'} d\alpha(u, \phi(v)) + \rho^2 \alpha(u)\alpha(v)$

where $\rho = \|R_\alpha\|$

J is complex structure on ξ given by rotation by $\pi/2$

$\phi : TM \to \xi$ is projection to ξ followed by J

note: given any contact form α with Reeb field R_α

any positive functions $\rho, \theta' : M \to \mathbb{R}$

and any complex structur $J : \xi \to \xi$

such that $d\alpha(v, Jv) > 0$ for $v \neq 0$

and $d\alpha(Jv, Jv) = d\alpha(v,v)$

J is said to be compatible with $d\alpha$ (lots of these)

define
$$\begin{tikzcd} TM = \xi + \mathrm{span}\{R_\alpha\} \arrow[r, "proj"] \arrow[rr, bend right, "\phi"'] & \xi \arrow[r, "J"] & \xi \end{tikzcd}$$

then $g(u,v) = \frac{\rho}{\theta'} d\alpha(u, \phi(v)) + \rho^2 \alpha(u)\alpha(v)$ is weakly compatible with ξ

so every ξ has lots of weakly compatible metrics!

Proof: 1) $\Rightarrow$ 2)

set $\rho = \|R_\alpha\|$

unit orthogonal to ξ is $n = R_\alpha / \rho$

so $\rho\alpha(v) = g(n, v)$

from above computation we have

$$\theta' = \rho\, d\alpha(u, v)$$

for any oriented orthonormal basis u, v for ξ (might only exist locally)

$\{e_1, e_2, e_3\} = \{u, v, n\}$ is an orthonormal basis for TM

let $\{e^1, e^2, e^3\}$ be the dual basis for T^*M

so $e^3 = \rho\alpha$ and $\alpha = \frac{1}{\rho} e^3$

write $d\alpha = a\, e^1 \wedge e^2 + b\, e^1 \wedge e^3 + c\, e^2 \wedge e^3$

note: $\iota_{e_3} d\alpha = \iota_{R_\alpha/\rho} d\alpha = \frac{1}{\rho} \iota_{R_\alpha} d\alpha = 0$

$\therefore b = c = 0$

$a = d\alpha(e_1, e_2) = d\alpha(u, v) = \theta'/\rho$

$$d\alpha = \theta'/\rho \; e^1 \wedge e^2$$

exercise: $* e^1 \wedge e^2 = e^3$

so $* d\alpha = \theta'/\rho \; e^3 = \theta' \alpha$ ✓

2) ⇒ 1) let v, u be an oriented orthonormal basis for ξ

n the unit normal to ξ

u, v, n an orthonormal basis for TM

denote it e_1, e_2, e_3

let e^1, e^2, e^3 be the dual basis

$* d\alpha = \theta' \alpha = \theta' m\, e^3$ some m

$\therefore d\alpha = ** d\alpha = * \theta' \alpha = \theta' m\, e^1 \wedge e^2$

so $\iota_n d\alpha = 0$ $\therefore$ n parallel to X_α

i.e. X_α orthogonal to ξ ✓

3) $\Rightarrow$ 1) is obvious so we are left to show

1) $\Rightarrow$ 3) let $\Lambda^1_\xi = \{\beta \in \Lambda^1 : \beta(R_\alpha) = 0\}$

exercise: $\phi_{d\alpha} : \xi \to \Lambda^1_\xi : v \mapsto d\alpha(v, \cdot)$

$\phi_g : \xi \to \Lambda^1_\xi : v \mapsto g(v, \cdot)$

are both isomorphisms

set $A = \phi_g^{-1} \circ \phi_{d\alpha}$

Claim: $A^2 = -\frac{1}{m^2} id$ for some positive $m : M \to \mathbb{R}$

indeed at a point pick a symplectic basis for $d\alpha$

i.e. e_1, e_2 st. $d\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

let f^1, f^2 be the algebraic dual basis

$$\phi_{d\alpha} : \begin{array}{l} e_1 \mapsto f^2 \\ e_2 \mapsto -f^1 \end{array}$$

g is represented by some positive definite $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $ac - b^2 > 0$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

so $\phi_g : \begin{array}{l} e_1 \longmapsto a f^1 + b f^2 \\ e_2 \longmapsto b f^1 + c f^2 \end{array}$

$\therefore$ $\phi_g^{-1} : \begin{array}{l} f^1 \longmapsto \frac{1}{ac-b^2}(c e_1 - b e_2) \\ f^2 \longmapsto \frac{1}{ac-b^2}(-b e_1 + a e_2) \end{array}$

and $\phi_g^{-1} \circ \phi_{d\alpha} : \begin{array}{l} e_1 \longmapsto \frac{1}{ac-b^2}(-b e_1 + a e_2) \\ e_2 \longmapsto \frac{1}{ac-b^2}(-c e_1 + b e_2) \end{array}$

i.e. its matrix is $A = \frac{1}{ac-b^2}\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}$

$$\therefore A^2 = \frac{1}{(ac+b^2)^2}\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}$$

$$= \frac{1}{(ac+b^2)^2}\begin{pmatrix} b^2-ac & bc-bc \\ -ba+ba & b^2-ac \end{pmatrix} = -\frac{1}{(ac+b)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so if $m = \sqrt{ac-b^2}$, $A^2 = -\frac{1}{m^2}\, id$ ✓

set $J = mA$

note $J^2 = m^2 A^2 = -id_?$

so J is a complex structure on ?

for $v, w \in$?

(1) $g(Av, w) = \phi_g(Av)(w) = \phi_{d\alpha}(v)(w)$
$= d\alpha(v, w) = -d\alpha(w, v)$
$= -g(Aw, v) = -g(v, Aw)$

$$\therefore g(Jv, w) = -g(v, Jw)$$

(2) $g(Jv, Jw) = -g(v, J^2 w) = g(v, w)$

(3) $g(v, w) = -g(v, J^2 w) = -m^2 g(v, A^2 w) \overset{(1)}{=} -m^2 d\alpha(Aw, v)$
$= -m\, d\alpha(Jw, v) = m\, d\alpha(v, Jw)$

let u, v be orthonormal basis for ?, n unit normal to ?

a general vector is

$$w = au + bv + cn$$

we know $\iota_n d\alpha = \iota_{R_\alpha/\|R_\alpha\|} d\alpha = 0$

any vector

so $d\alpha(au+bv+cn, V)$

$$= d\alpha(au+bv, V)$$
$$= d\alpha(w^{\xi}, V)$$

projection of w to ξ

normal component

now $g(U,V) = g(U^{\xi}, V^{\xi}) + g(U^{n}, V^{n})$

$$\overset{(3)}{=} m\, d\alpha(U^{\xi}, JV^{\xi}) + g(g(U,n)n, g(V,n)n)$$
$$= m\, d\alpha(U, \phi(V)) + g(U,n)g(V,n)$$

(4) $$= m\, d\alpha(U, \phi(V)) + \rho^2 \alpha(U)\alpha(V)$$

(since $\underbrace{\|R_\alpha\|}_{\rho}\alpha = \iota_n g$)

we need to determine m

for this note for $v \in \xi$

$$g(v, Jv) = m\, g(v, Av) \overset{(1)}{=} m\, d\alpha(v,v) = 0$$

so Jv orthogonal to v (so J rotation by $\pm\pi/2$)

(5) if $\|v\| = 1$ then $d\alpha(v, Jv) = m\, d\alpha(v, Av)$
$$= m\, g(Av, Av)$$
$$= \frac{1}{m} g(v,v) > 0$$

so v, Jv oriented orthonormal basis for ξ (J rotation by $\pi/2$)

let $\{e_1, e_2\} = \{v, Jv\}$ and $e_3 = n = R_\alpha/\rho$

and e^1, e^2, e^3 be dual basis

as in proof of 1) ⇒ 2) we see

$$d\alpha = \frac{\theta'}{\rho} e^1 \wedge e^2$$

$$\therefore \frac{\theta'}{\rho} = d\alpha(e_1, e_2) = d\alpha(v, Jv)$$

$$\underset{(5)}{=} \frac{1}{m} g(v,v) = \frac{1}{m}$$

so $m = \frac{\rho}{\theta'}$

plug into (4) to get

$$g(U,V) = \frac{\rho}{\theta'} d\alpha(U, \phi(V)) + \rho^2 \alpha(U)\alpha(V)$$

definition:

a contact structure ξ and a metric g are compatible if there is a contact form α for ξ such that

$$\|\alpha\| = 1 \quad \text{and}$$

$$\star d\alpha = \theta' \alpha$$

for some constant θ'

(this is equivalent to saying the unit orthogonal to ξ is a Reeb field and the instantanious rotation is constant)

Remark:

This is the same as Chern and Hamilton's definition from 1984 if $\theta' = 2$

This form of compatibility has been extensively studied from a Riemannian geometry perspective (see book of David Blare and below) we note that given a contact structure ξ there is a projection

$$\{\text{all metrics}\} \xrightarrow{\pi_\xi} \{\text{compatible metrics w/ } \theta' = 1\}$$

to define π_ξ note that if g_ξ is an inner product on ξ then we can get a compatable metric as follows:

- g_ξ gives area form Ω on ξ
- take any contact form α_0 for ξ

  note $d\alpha_0$ an area form on ξ

  and for any $f > 0$, $d(f\alpha_0)|_\xi = f\, d\alpha_0|_\xi$

  so $\exists!$ function f_0 s.t. $d(f_0\alpha)|_\xi = \Omega$

  set $\alpha = f_0\alpha_0$
- define g to be g_ξ on ξ and R_α to be orthogonal to ξ and of unit length

now define $\pi_\xi(g) =$ extension of $g|_\xi$ to a compatible metric

exercise: if g is compatible with ξ and $\theta' = 1$ then $\pi_\xi(g) = g$

Questions:

1) how do geometric quantities, like various curvatures, of g and $\pi_\xi(g)$ compair? what if $\pi_\xi(g)$ is "close" to g?
2) if ξ_t is a family of contact structures how do $\pi_{\xi_t}(g)$ compair?
3) What can you say about image(π_ξ)
4) if you fix g st $g \in \operatorname{im} \pi_\xi \cap \operatorname{im} \pi_{\xi'}$ what can you say about ξ and ξ'?

recall if $E \to M$ is a bundle then a connection on E is a way to differentating sections of E

more specifically if $\Gamma(E)$ = sections of E

and $\mathfrak{X}(M) = \Gamma(TM)$ = vector fields on M

then a connection is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$
$$(v, \sigma) \longmapsto \nabla_v \sigma$$

satisfying

1) $\nabla_v \sigma$ is linear in $\mathfrak{X}(M)$ as a $C^\infty(M)$ module

   ie. $f, g \in C^\infty(M)$, $v, w \in \mathfrak{X}(M)$

$$\nabla_{fv+gw} \sigma = f \nabla_v \sigma + g \nabla_w \sigma$$

2) $\nabla_v \sigma$ is linear in $\Gamma(E)$ as an $\mathbb{R}$-vector space

   ie. $\nabla_v a\sigma + b\eta = a\nabla_v \sigma + b \nabla_v \eta$

   for $a, b \in \mathbb{R}$

3) $\nabla_v \sigma$ satisfies a product rule

$$\nabla_v (f\sigma) = f \nabla_v \sigma + (v.f)\sigma$$

   for $f \in C^\infty(M)$

a linear connection is a connection on $\mathfrak{X}(M) = \Gamma(TM)$

Facts: 1) connections exist for any E

2) given a linear connetion there exist unique connections on $\underbrace{TM \otimes \dots \otimes TM}_{k} \otimes \underbrace{T^*M \otimes \dots \otimes T^*M}_{\ell}$

such that

1) on $C^\infty(M)$ (= 0^{th} tensor power of TM)

$$\nabla_v f = v.f = df(v)$$

2) $\nabla_v(\sigma \otimes \eta) = (\nabla_v \sigma) \otimes \eta + \sigma \otimes \nabla_v \eta$

3) $\nabla_v(\mathrm{tr}\,\sigma) = \mathrm{tr}(\nabla_v \sigma)$

there trace means plug one of the setions of TM into one of the sections of T^*M

3) given a metric g there is a unique linear connection satisfying

1) $\nabla_v g(u,w) = g(\nabla_v u, w) + g(u, \nabla_v w)$ (compatible)

2) $\nabla_v w - \nabla_w v = [v,w]$ (symmetric)

this is called the Levi-Civita connection of g

we will always use this connection

lemma 2:

If ξ and g are weakly compatible and n is the unit vector field normal to ξ then

$$\nabla_n n = -(\nabla \ln \rho)^\xi$$

gradient

recall def^n ∇f is unique v.f. s.t. $df(v) = g(\nabla f, v)\,\forall v$

where v^ξ is the component of v in ξ

and $\rho = \|R\|$ where R is the Reeb vector field showing weak compatibility

Proof: for any vector field u

$$0 = \nabla_u g(n,n) = 2g(\nabla_u n, n)$$

so $\nabla_n n$ is tangent to $\xi = n^\perp$

now for $v \in \xi$

$$g(\nabla_n n, v) = -g(n, \nabla_n v) \qquad (g(n,v)=0 \text{ so } g(\nabla_n v, n) + g(v, \nabla_n n) = 0)$$

$$= -g(n, \nabla_n v - \nabla_v n) \qquad (\text{from above } g(n, \nabla_v n) = 0)$$

$$= -g(n, [n,v])$$

$$= -\rho\,\alpha([n,v]) \qquad (\text{recall } g(n,\cdot) = \rho\,\alpha(\cdot))$$

$$= \rho\,\big(\overset{n=\frac{1}{\rho}R_\alpha}{\cancel{d\alpha(n,v)}} - \overset{\alpha(v)=0}{\cancel{n\cdot\alpha(v)}} + v\cdot\alpha(n)\big)$$

$$(d\alpha(u,v) = u\cdot\alpha(v) - v\cdot\alpha(u) - \alpha([u,v]))$$

$$= \rho\, v\cdot\left(\tfrac{1}{\rho}g(n,n)\right) \qquad (\text{as above})$$

$$= \rho\, v\cdot\left(\tfrac{1}{\rho}\right) = -\rho\left(\tfrac{1}{\rho^2}d\rho(v)\right)$$

$$= -d(\ln\rho)(v)$$

$$= -g(\nabla \ln\rho, v) \qquad (\text{def}^{\text{n}} \text{ gradient})$$

so $\nabla_n n$ is in ξ and pairs with all vectors in ξ the same as $-\nabla \ln\rho$, $\therefore$

$$\nabla_n n = -(\nabla \ln\rho)^\xi \quad \blacksquare$$

Remark:

1) note that is ξ and g are compatible then the flow of the associated Reeb vector field is tangent to geodesics

2) Zeghib showed that no closed hyperbolic manifold can have a non-singular vector field whose flow traces out geodesics

this is one of the major motivations for introducing weakly compatible metrics!

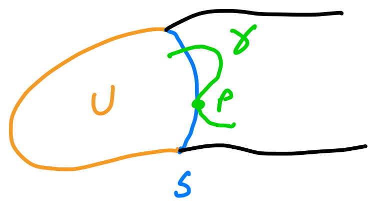
V Convexity

let S be a hypersurface in a Riemannian manifold (M, g) that bounds a domain U

U is geodesically convex if for any geodesic γ tangent to S at a point p we have

$\gamma \cap U = \{p\}$

locally ie nbhd of p in γ only intersects U at p



lemma 1:

if $f: M \to \mathbb{R}$ s.t. $c \in \mathbb{R}$ is a regular value and

$$S = f^{-1}(c)$$

$$U = f^{-1}((-\infty, c])$$

then U is geodesically convex

$$\Leftrightarrow$$

$$\nabla^2 f(v, v) > 0 \quad \forall v \in TS$$

here $\nabla^2 f(v, v)$ is the Hessian of f and is defined by

$$\nabla^2 f(u, v) = (\nabla_u df)(v)$$

$$\left(= \nabla_u \nabla_v f - \nabla_{\nabla_X Y} f = \frac{1}{2} \mathcal{L}_{\nabla f}\, g(u, v) \right)$$

idea: compose f with a geodesic to get map $(-\varepsilon, \varepsilon) \to \mathbb{R}$

its first derivative is 0 since tangent to S

its second derivative is positive ...

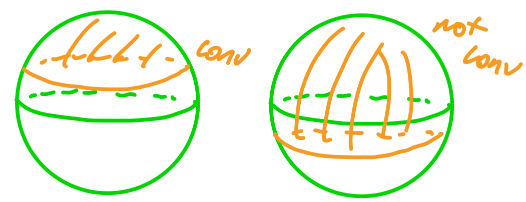
let

$$\mathrm{conv}(g) = \sup_r \left\{ \begin{array}{l} B_r(p) \text{ is geodesically convex} \\ \text{for all } p \in M \end{array} \right\}$$

convexity radius

<u>example:</u> unit $S^2 \subset \mathbb{R}^3$ with induced metric has

$$\mathrm{conv} = \pi/2$$



<u>Thm 2:</u>

if $K > 0$ and $\sec(g) \leq K$ then

$$\mathrm{conv}(g) \geq \min\left\{ \mathrm{inj}(g), \frac{\pi}{2\sqrt{K}} \right\}$$

where $\mathrm{inj}(g)$ is the injectivity radius

if $\sec(g) \leq 0$, then $\mathrm{conv}(g) = \mathrm{inj}(g)$

now for symplectic convexity

let (W, J) be an almost complex manifold

Ω a domain in W bounded by Σ

let $\mathcal{C} \subset T\Sigma$ be the complex tangencies to Σ

i.e. $\mathcal{C} = T\Sigma \cap J(T\Sigma)$

we say Σ (or Ω) is <u>(strongly) pseudoconvex</u> if $\mathcal{C}$ is

a positive contact structure (and Σ oriented as $\partial\Omega$)

if $f: W \to \mathbb{R}$ is a function and c a regular value s.t.

$$\Sigma = f^{-1}(c)$$

$$\Omega = f^{-1}((-\infty, c])$$

then

$$\boxed{\mathcal{C} = \ker(-df \circ J)}$$

so ξ a contact structure

$\Longleftrightarrow$

$L(v,v) > 0$ for $v \in \xi$

where

$L(u,v) = -d(df \circ J)(u, Jv)$

is the Levi form

Why do we care about pseudoconvex hypersurfaces?

<u>answer:</u> control holomorphic curves

complex str

given a Riemannian surface (F, j)

and an almost complex manifold (X, J)

a map $u: F \to X$ is called holomorphic if

$du \circ j = J \circ du$ (du ~~preserves~~ respects almost complex strs)

if Σ is pseudoconvex surface bounding Ω

and $u(F) \subset \Omega$ then $u(F)$ can't be

tangent to Σ (if $\Sigma = f^{-1}(c)$ as above $f \circ u$ satisfies a "maximum principal")

where do we use holomorphic curves?

<u>Th$^{\underline{m}}$ (Hofer):</u>

$$\boxed{\text{if } M \text{ is closed and } \xi \text{ is an overtwisted contact str on } M, \text{ then any Reeb vector field for } \xi \text{ has a close orbit}}$$

<u>Sketch of proof:</u>

consider $W = (-\infty, 0] \times M$

if α is a contact form for ξ, then $\omega = d(e^t\alpha)$ a symplectic structure on W and $\exists$ an almost complex str. J on W that sends R_α to $\frac{\partial}{\partial t}$ and preserves ξ

Reeb field — coord on $[-\infty,0]$

you can easily check $\{t\}\times M$ pseudoconvex $\forall t$

let D be an overtwisted disk in (M,ξ)

its characteristic foliation is

singular foliation tangent to $\xi \cap TD$

- Bishop proved there are holomorphic disks

$$u_t:(D^2,\partial D^2 \to (W, \{0\}\times D) \qquad t\in[0,\epsilon)$$

such that

u_0 constantly $p\in D$

$\bigcup_t u_t(\partial D^2)$ fill a nbhd of p in D

Fact: $u(\partial D^2)$ must be transverse to leaves of D_ξ (if not constant)

(another max principal)

- "Standard" functional analysis says

  if you extend the family of u_t above they always fill out an open subset of D

  (this is because the holomorphic curve equations are elliptic)

- What happens if we have a Cauchy sequence of holomorphic disks $u_n:(D^2,\partial D^2)\to(W,\{0\}\times D)$

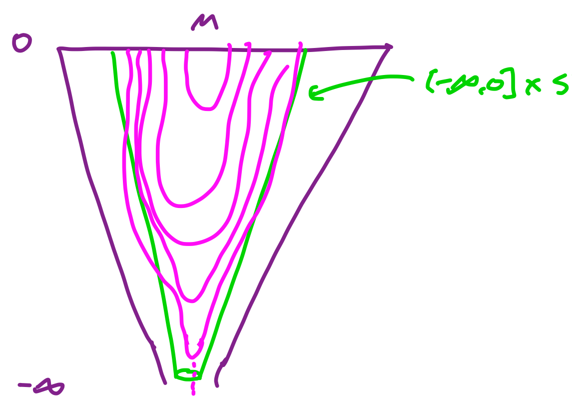
  if im u_n stay in $[a,0]\times W$ for some a then Arzelà-Ascoli says they will converge to another holomorphic disk unless the ∇u_n blows up

but in this situation Gromov says that can't happen ("no bubbling" since all $\{t\} \times M$ convex)

so if u_n don't converge to a holomorphic disk, then image of u_n must "go to $-\infty$"

Hofer says if this is the case then there must be a periodic orbit in R_α

specifically, limit of u_n will be assymptotic to $(-\infty, 0] \times \gamma$ for some periodic orbit



so if no periodic orbits any u_n converge to another holomorphic disk

$\therefore$ subset of D filled by boundaries of holomorphic disks is closed!

- in this case subset of D filled by such boundaries is open and closed $\therefore$ all of D!

but first boundary of holomorphic disk to touch ∂D

will be tangent to ∂D = leaf of $\mathcal{D}_\xi$

this contradicts Fact above

$\therefore$ must have closed orbit in flow of R_α !

note: if M not closed, but $(-\infty, 0] \times \partial M$ pseudo-convex then same argument says must be periodic orbit!

Putting 2 convexities together

Th$^{\underline{m}}$ 3 (E-Komendarczyk-Massot):

let g be weakly compatible with (M, ξ)

S a surface in M cut out by f and

U the sublevel set

$$\Sigma = \mathbb{R} \times S \subseteq \mathbb{R} \times M$$

$$\Omega = \mathbb{R} \times U \subseteq \mathbb{R} \times M$$

let R is the Reeb field for ξ showing weak compatibility with g

J be an almost complex structure on $\mathbb{R} \times M$ that preserves ξ and sends R to $\frac{\partial}{\partial t}$

for any $v \in \mathcal{C}$ ← complex tangencies to Σ

we have

$$L(v,v) = \nabla^2 f(v,v) + \nabla^2 f(Jv, Jv) - \|v\|^2 g\big(\nabla \ln \rho - (\nabla \ln \theta')^\perp, \nabla f\big)$$

$\rho = \|R\|$

θ': instantaneous rotation

the proof is a long computation

we can use this to prove a Darboux theorem with estimates

given (M, ξ) a contact manifold and

g a metric on M

we define

$$\tau(g) = \sup_r \left\{ \xi \text{ restricted to } B_r(p) \text{ is tight for all } p \in M \right\}$$

tightness radius or Darboux radius

Thm 4 (EKM):

$\boxed{\text{if } g \text{ is a metric compatible with } (M, \xi) \text{ then } \tau(g) \geq conv(g)}$

note: if M compact it is easy to use Darboux + Lebesgue number to prove $\tau(g)$ bounded by positive number

but not possible on non-compact manifold and computing a lower bound in compact case would be hard

Proof:

fix a point $p \in M$ for all $r < conv(g)$ we know $B_r(p)$ is geodesically convex

let $\mathcal{C}_r$ = complex tangencies to $\partial(\mathbb{R} \times B_r(p))$

then $\nabla^2 f(v,v), \nabla^2 f(Jv, Jv) \geq 0$ and one must be positive

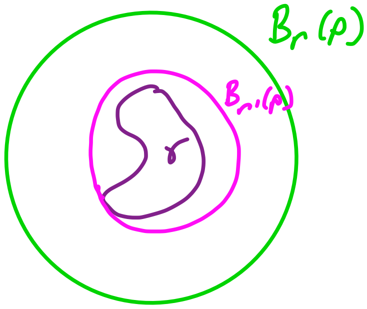
$$\therefore L(v,v) > 0$$

and $\mathcal{C}_r$ pseudoconvex for $r < conv(g)$

we can adapt a theorem of Hofer (see note above) to see that if $\xi|_{B_r(p)}$ is overtwisted then there is a close Reeb orbit γ in $B_r(p)$

recall γ is also a geodesic

but now



let r' be the largest radius s.t.
$\partial B_{r'}(p) \cap \gamma$
must have $\partial B_{r'}(p)$ tangent to γ
and $\gamma \subset B_{r'}(p)$ ⊠ convexity

so $\xi|_{B_r(p)}$ tight!

Th$^{\underline{m}}$ 5 (EKM):

this is Th$^{\underline{m}}$ 2 from introduction section

$\boxed{\begin{array}{l}\text{let } (M,\xi) \text{ be a contact 3-manifold weakly compatible} \\ \quad \text{with a complete Riemannian metric } g \\ \text{if} \\ \qquad \sec(g) \leq -m_g^2 \\ \text{then } (M,\xi) \text{ is universally tight} \\ \text{here} \\ \qquad m_g = \sup_M \| \underbrace{\nabla \ln \rho - \nabla(\ln \theta')^{\perp}}_{\text{call this } D_g} \| \\ \text{where} \quad \rho \text{ is the length of a Reeb vector field} \\ \qquad\quad \theta' \text{ is the instantanious rotation of } \xi \end{array}}$

$v^{\perp}$ is component of v perp. to ξ

$-\sec \geq m_g^2$

$m_g \leq \sqrt{-\sec}$

$\leq \sqrt{K}$

Proof:

pull everything back to the universal cover $\tilde{M} \cong \mathbb{R}^3$

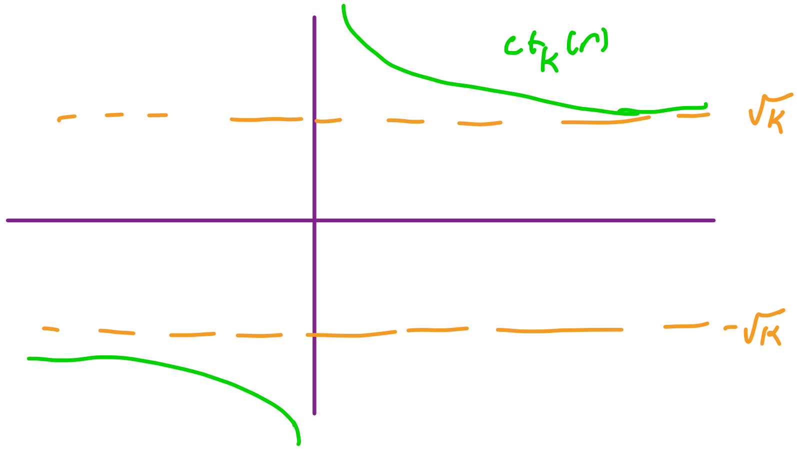
let $B_p(r)$ be ball of radus r about p

if $\sec(g) \leq -K$ for some $K > 0$, then

below we see for $ct_K(r) > m_g$

$$\partial(\mathbb{R} \times B_r(p)) \text{ is pseudo-convex}$$

where $ct_K(r) = \sqrt{K} \coth(\sqrt{K} r)$



but we are assuming $\sqrt{K} \geq \sqrt{-\sec(g)} \geq m_g$

so $\partial(\mathbb{R} \times B_r(p))$ is pseudo-convex for all r

now arguing as in last proof if $(\mathbb{R}^3, \xi)$ is overtwisted

$\exists$ a closed Reeb orbit in $B_r(p)$ for some r

note: $\mathbb{R} \times \gamma$ is holomorphic in $\mathbb{R} \times \tilde{M}$ (with J used above)

Start shrinking $B_r(p)$ to first r_0 where $\partial B_{r_0}(p) \cap \gamma \neq \emptyset$

there $\mathbb{R} \times \gamma$ will be tangent to $\mathbb{R} \times \partial B_{r_0}(p)$

but this contradicts pseudoconvexity

$\therefore M = \mathbb{R}^3$ is tight

now for the above claim about pseudoconvexity

fix p and let

$$r_p : M \to \mathbb{R} : x \longmapsto d(p, x)$$

if $K > 0$ and $\sec(g) \leq -K$ then it is known that

$$\nabla^2 r_g \geq ct_K(r)\, g$$

now for $v \in \mathcal{C}_r$ we can write it as

$$v = v^\xi + aR + b\frac{\partial}{\partial t}$$

recall J is an isometry on ξ so

$$g(v,v) = g(v^3, v^3) + a^2 + b^2$$
$$= g(Jv^3, Jv^3) + a^2 + b^2 = g(Jv, Jv)$$

so
$$L(v,v) = \nabla^2 r_p(v,v) + \nabla^2 r_p(Jv, Jv) - g(D_g, \nabla r_p)\|v\|^2$$

Th$^{\underline{m}}$ 3

unit normal

$$\geq 2\,ct_K(r)\|v\|^2 - \|\nabla r_p\|\, g(D_g, n_p)\|v\|^2$$
$$\geq 2\,ct_K(r)\|v\|^2 - \|D_g\|\,\|v\|^2$$

larger than $g(D_g, n_p)$ and $\|\nabla r_p\| = 1$

$$\geq 2\,ct_K(r)\|v\|^2 - m_g\,\|v\|^2$$
$$\geq (ct_K(r) - m_g)\|v\|^2$$

so $\partial(\mathbb{R} \times B_p(r))$ is convex if $ct_K(r) \geq m_g$

VI Seeing Overtwisted Disks and the Contact Sphere Th^{m}

Th^{m} 6 (EKM):

> let (M,ξ) be a contact manifold compatible with g
> if $r < inj_p(g)$ and ξ is overtwisted on $B_r(p)$
> then $\partial B_r(p) = S_r(p)$ contains an overtwisted disk

so we can't guarantee $\xi|_{B_r(p)}$ is tight, we can clearly see when it is not

we will prove this later but now prove the contact sphere theorem

Proof of Contact Sphere Th^{m} (Th^{m} 1, from intro)

Recall we have (M,ξ) compatible with g and

$\exists K > 0$ st. $\frac{4}{9}K < sec(g) \le K$

we want to show ξ is tight

we pull ξ back to universal cover $\tilde{M}$

ordinary sphere th^{m} says $\tilde{M} \cong S^3$

easy to see if pulled back ξ is tight so is ξ

so we assume $M = S^3$

for contradiction assume ξ is overtwisted

let D be an overtwisted disk

rescale g so that $K = 1$ (note still compatible)

Bonnet-Meyer's Th^{m} says that

$$\operatorname{diam}(M) < \pi/\sqrt{4/9} = \frac{3\pi}{2}$$

a result of Klingenberg says that

$$\mathrm{inj}(g) \geq \pi/\sqrt{1} = \pi$$

and we mentioned above

$$\mathrm{conv}(g) \geq \frac{\pi}{2\sqrt{1}} = \frac{\pi}{2}$$

using "standard" Toponogov comparison argument

we see that if $p,q \in M$ such that

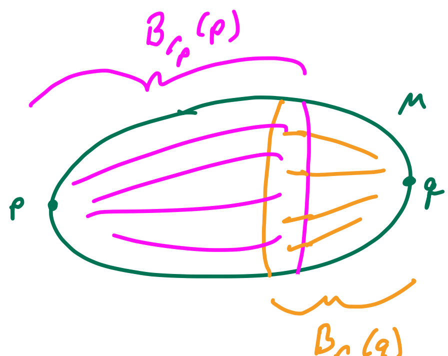
$$d(p,q) = \mathrm{diam}(M)$$

then there are $r_p < \pi$ and $r_q < \frac{\pi}{2}$

s.t.

$$M = B_{r_p}(p) \cup B_{r_q}(q)$$

so



we can assume D does not contain q

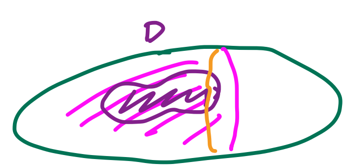
Th$^{\underline{m}}$ 4 above says $B_{r_q}(q)$ is standard contact ball

for standard contact ball there is a vector field v

whose flow pushes any point $\neq q$ into small nbhd

of $\partial B_{r_q}(q) \subset \mathrm{int}\, B_{r_p}(p)$

so we can assume $D \subset B_{r_p}(p)$



but Th$^{\underline{m}}$ 5 above says $\partial D_{r_p}(p)$ must

now contain an overtwisted disk D'!

so $D' \subset \partial B_{r_p}(p) \subset B_{r_q}(q)$

contradicting tightness of $\xi|_{B_{r_q}(q)}$

$\therefore$ ξ tight contact str on S^3

(Eliashberg says ξ standard)

for the proof of Th$^{\underline{m}}$ 5 we need some preliminaries

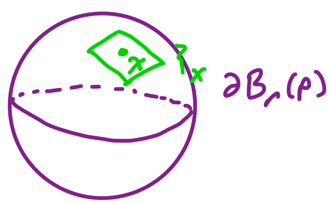
lemma 6:

> if (M,ξ) is compatible with g and $r < \mathrm{inj}_p(g)$, then the characteristic foliation $(\partial B_r(p))_\xi$ has only 2 singular points (and they are $\gamma \cap \partial B_r(p)$ where γ is a Reeb flow line through p)

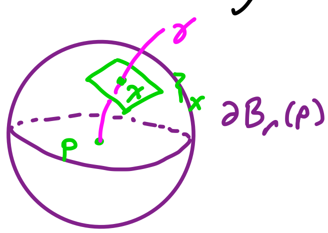
Proof:

suppose $x \in \partial B_r(p)$ is a singular point

so we have



let γ be a geodesic starting at p s.t. $\gamma(r) = x$



by the Gauss lemma we know

$$T_x(\partial B_r(p)) = \xi_x$$

is orthogonal to $\gamma'(r)$

$\therefore$ $\gamma'(r) = R$ the Reeb field

and since the Reeb flow is tangent to geodesics we see γ is a Reeb flow line through p

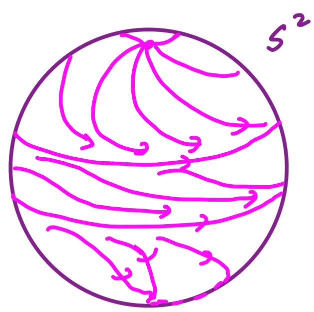
$\therefore$ can only be 2 singularities in $(\partial B_r(p))_\xi$

We call a surface Σ in (M,ξ), <u>ξ-convex</u> if there is a vector field transverse to Σ whose flow preserves ξ

We say a sphere S is <u>simple</u> if S_ξ contains only two singular points (we call the positive one the <u>north pole</u> and the other the <u>south pole</u>)

S_ξ is <u>almost horizontal</u> if, in addition, all closed leaves of S_ξ are oriented as the boundary of the disk containing the north pole

<u>examples:</u>



almost horizontal

<u>not</u> almost horizontal

<u>lemma 7 (Giroux):</u>

> If S_ξ is simple, then
>
> S_ξ is ξ-convex $\Leftrightarrow$ S_ξ has no degenerate closed orbits

we are now ready for our main technical result

<u>Proposition 8:</u>

> let B be a ball in (M,ξ)
>
> B is a union of a point p and spheres S_t for $t \in (0,1]$

1) $(S_t)_\xi$ is simple and $\xi|_B$ tight $\Rightarrow$ $(S_t)_\xi$ almost horizontal

2) all $(S_t)_\xi$ almost horizontal $\forall t$ $\Rightarrow$ $\xi|_B$ tight

3) if $(S_t)_\xi$ all simple and $\xi|_B$ is overtwisted, then $\exists\, t_0$ such that

$(S_t)_\xi$ has a closed leaf for $t \geq t_0$

$\xi|_{B_t}$ tight for $t < t_0$

Proof of Th$^{\underline{m}}$ 6:

$$B_r(p) = p \cup \bigcup_{t \in (0,r]} S_t$$

lemma 7 says $(S_t)_\xi$ simple since $r < \mathrm{inj}(g)$

part 3) of Prop 8 $\Rightarrow$ if $B_r(p)$ is overtwisted then we see an overtwisted disk on $\partial B_r(p)$

Proof of Proposition 8:

1) obvious, if $\exists$ closed leaf then contact str is overtwisted

2) for small t, B_t will be tight by Darboux's Th$^{\underline{m}}$

so $(S_t)_\xi$ has no closed leaves for t small

if there are no closed orbits in $(S_t)_\xi$ for all t, then

by lemma 7 all of the S_t are ξ-convex

from this it is easy to argue that $\xi|_B$ is tight

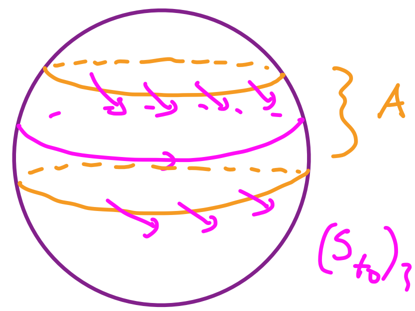
(can show $\overline{B_1 - B_\varepsilon} = S^2 \times [\varepsilon, 1]$ has an $[\varepsilon,1]$-invariant contact str, so B_1 is the result of adding a "collar nbhd" to B_ε)

so $\xi|_B$ is tight unless some $(S_t)_\xi$ has a closed leaf let t_0 be smallest t such that $(S_t)_\xi$ has a closed orbit

the closed orbit C of $(S_{t_0})_\xi$ must be degenerate

(we assume only one orbit, but you can consider other cases)

we can find an nbhd A of C on S_{t_0} s.t.



we can map $A \times [t_0-\varepsilon, t_0+\varepsilon]$ into B

so that a) $A_t = A \times \{t\}$ maps to S_t

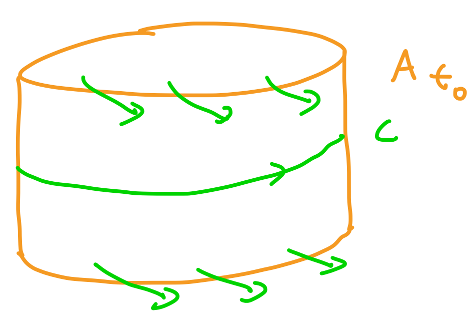
b) leaves of $(S_t)_\xi$ enter top of A and exit bottom of A

c) $\{p\} \times [t_0-\varepsilon, t_0+\varepsilon]$ maps to Legendrian arcs

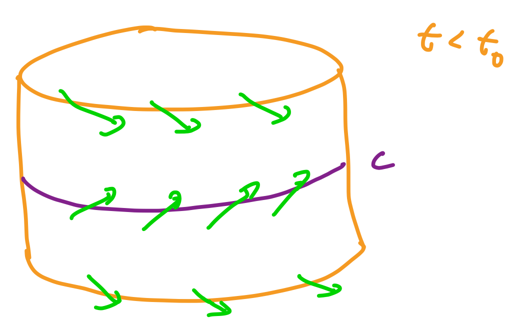
<u>note</u>: A_t has no closed leaves for $t < t_0$

recall the contact planes along $\{p\} \times [t_0-\varepsilon, t+\varepsilon]$ rotate in left-handed way

since $(S_{t_0})_\xi$ is almost horizontal we see



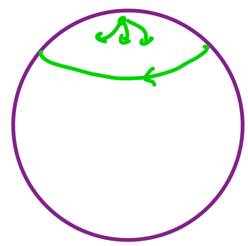
just before t_0 we see



so by Poincaré-Bendixson there must be a closed leaf in A_t

contradicts fact that t_0 smallest such t ✓

3) note: the above argument say any time a new periodic orbit is born and is closest to north pole it must go east to west



same argument says if a northern most periodic orbit dies it must go west to east

now let t_0 be the first time a closed leaf appears in $(S_t)_?$

from above it must go east to west

as t increases there can be finitely many birth deaths of periodic orbits

we inductively see that northern most orbit is always east to west and so can't die (i.e. all $(S_t)_?$ for $t \geq t_0$ have closed orbit)

let $t_1, \ldots, t_n$ be other birth/death times

from above $t \in [t_0, t_1)$ one orbit east to west

suppose hypothesis true for $t < t_k$

if t_{k+1} is a birth of an orbit closer to north pole than other orbits then done by above observation

(must go east to west)

if not northern most then done since northern most still east to west

if t_{k+1} a death it can't involve northen most orbit since deaths of northern most orbit only occur for west to east orbits

$\therefore$ done

if $t < t_0$ then $\exists l_{B_t}$ tight by 2)