Smooth Structures on 4-manifolds and corks

today only consider simply connected 4-manifolds

Thm: if M_1 and M_2 are closed simply connected smooth 4-mfds then the following are equivalent

① $(H_2(M_1), I_{M_1}) \underset{\text{as innerproduct spaces}}{\cong} (H_2(M_2), I_{M_2})$

here I_M is $\cap$-pairing on $H_2(M)$

② M_1 is homotopy equivalent to M_2

③ M_1 is h-cobordant to M_2

④ M_1 is homeomorphic to M_2

Remarks:

I) h-cobordant means $\exists$ a smooth 5-manifold, M such that $\partial M = \underset{\text{lower }\partial}{(-M_1)} \cup \underset{\text{upper }\partial}{M_2}$ and the inclusion maps $i_j : M_j \to M$ are homotopy equivalences

great topics for student seminar

II) clearly ③ ⇒ ② ⇒ ①

III) Pontryagin?/Whitehead showed ① ⇒ ② (Milnor '58?) '49

Novikov?/Wall showed ② ⇒ ③ '64

IV) clearly ④ ⇒ ② so ④ ⇒ ①, ② and ③

very hard! maybe not

V) Freedman ③ ⇒ ④ '82

but what about M_1 and M_2 diffeomorphic?

another great topic!

(Donaldson + Freedman ⇒ ∃ infinitely many distinct smooth manifolds that are homeomorphic!

Big question is to understand how to organize this mess of smooth mfds. So far no idea, but here is a curious result that might help!

Thm 1:

If M_1 and M_2 are closed, smooth, simply connected 4-manifolds that are homeomorphic, then ∃ smooth contractible manifold $C \subset M_1$ and an involusion $\phi : \partial C \to \partial C$ s.t.

$$M_2 = (M_1 - C) \cup_\phi C$$

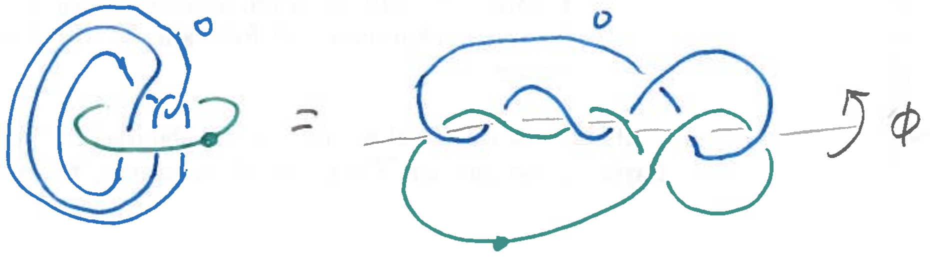
C is called a cork (or Akbulut cork)

he found the first one before Thm known

Remark: I) clearly diffeomorphism can't extend over C (or else M_1 & M_2 diffeo!)

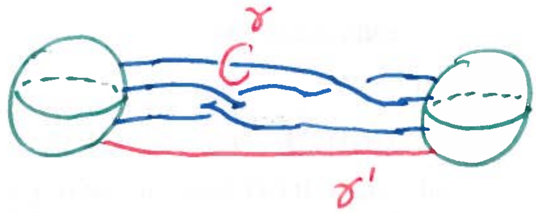
II) Akbulut's Example is the Mazur manifold

C



(fun facts: $D(C) = S^4$
$C \times I = B^5$)

small detour if you know contact/symplectic geromethy

$C =$ 

<u>note</u>:

① ϕ takes γ to γ'

② γ bounds a dish

③ $tb(\gamma') = 0$

④ slice-Bennequin $\Rightarrow$
$tb(\gamma') \leq -\chi\left(\begin{smallmatrix}\text{slice}\\ \text{sfc}\end{smallmatrix}\right.$

so ϕ can't extend to diffeo of C or γ' violate ④

Akbulut's original proof was very involved (21 pages of gauge theory!)

III) lots of recent work by

<u>Tange</u>: $\exists\, C$ and $\phi: \partial C \to \partial C$ order n
s.t. $C \hookrightarrow M$ s.t. $(M - C) \cup_{\phi^k} C$
$\uparrow$ some mfd
different for all $0 \leq k < n$

<u>Auckley-Kim-Melvin-Ruberman</u>:
G any finite subgroup of $SO(4)$
then $\exists\, C$ and action of G on ∂C
s.t. $(M - C) \cup_g C$ all distinct

<u>Gompf, Akbulut</u>: "infinite order corks"

$Th^m 1$ above clearly follows from $Th^m 2$ below

Th^{m}2:-

Let M be a 5-dimensional h-cobordism between simply connected closed 4-manifolds M_0 and M_1

Then $\exists$ a subcobordism $A \subset M$ from $A_0 \subset M_0$ to $A_1 \subset M_1$ st.

① [1] A_0 (hence A and A_1) are compact contractible manifolds

② [1] $M - \text{int}\, A_0$ is a product cobordism i.e. $\cong (M_0 - \text{int}\, A_0) \times [0,1]$

③ [1] $M - A$ is simply connected

④ [3] $A \cong B^5$

⑤ [2] $A_0 \times [0,1] \cong A_1 \times [0,1] \cong B^5$

⑥ [2] $A_0 \cong A_1$ (diffeo restricted to ∂ is involution)

[1] Curtis - Freedman - Hsiang - Stong '9

[2] Matveyev '96

[3] Bižaca unpublish

Before going into the proof of the theorem let's recall some facts about cobordisms

Facts about cobordisms/handlebodies:

"recall" an <u>n-dimensional</u> <u>handle of index k</u>

is $h^k = D^k \times D^{n-k}$ with attaching region

$$\partial_- h^k = \partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$$

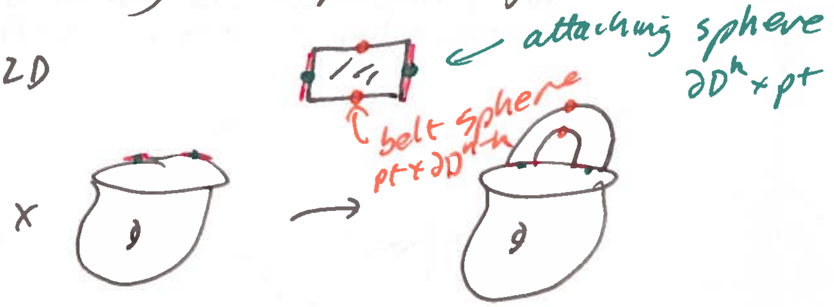
h^n is attached to the ∂ of an n-manifold X by an embedding $\phi : \partial_- h^k \to \partial X$

so attaching h^n to X is

$$X \cup_\phi h^k$$

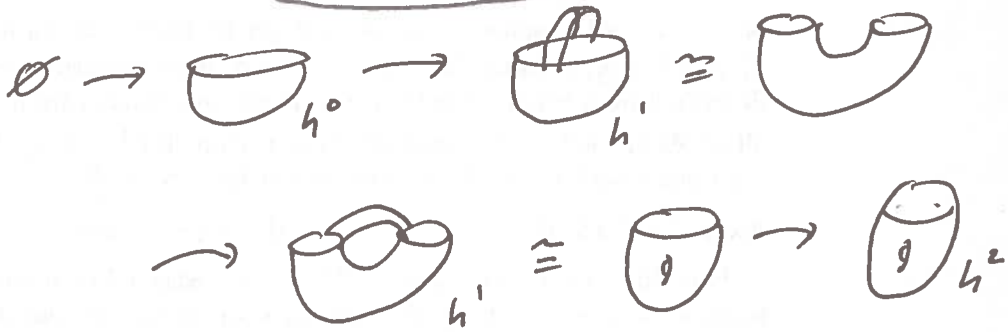
e.g. 0-handle h^0 is just D^n and "attaching" it is just disjoint union

1-handle in 2D



a handle body is a manifold X^n such that X^n is build from $\emptyset$ $\boxed{\text{or } M^{n-1} \times [0,1]}$ by attaching a sequence of handles

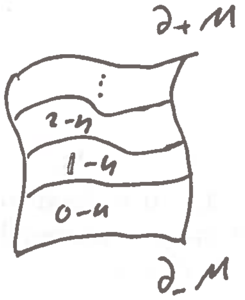
eg



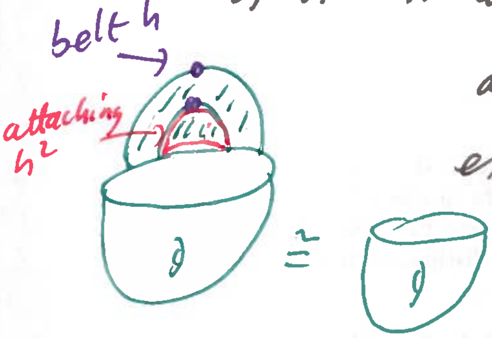
<u>Facts about handle bodies:</u>

1) any compact smooth manifold has structure of a handle body
(for a cobordism can assume handlebody built on (lower ∂) $\times [0,1]$)

2) can assume handles are all attached with increasing index:



just ⋔ of belt and attaching sphere



3) if h^k and h^{k+1} are attached to ∂M so that attaching sphere of h^{k+1} $\cap$ belt sphere of h^k exactly once, then

$$M \cup h^k \cup h^{k+1} \cong M$$

4) if M^n connected and $\partial_- \neq \emptyset$ then can assume <u>no</u> 0-handles

($\partial_+ \neq \emptyset$ then no n-handles)

just cancellat above.

5) if $C_k(M, \partial_- M)$ is generated by k-handles and $\partial_k : C_k \to C_{k-1}$ sends h^k to $\sum \langle h^k_i, h^{k-1}_j \rangle$

where $\langle h^k_i, h^{k-1}_j \rangle$ = algebraic $\cap$ of attaching sphere of h^k & belt sphe of h^{k-1}_j

then homology of this is ordinary homology $H_k(M, \partial_- M)$

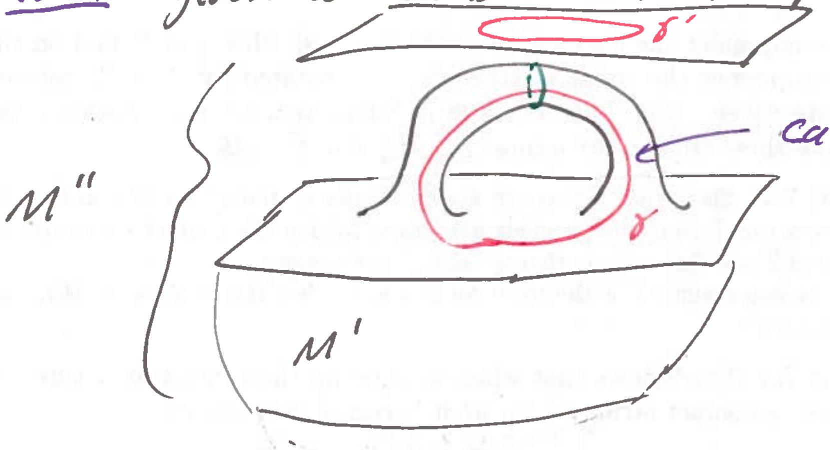
6) by "handle slides" (isotoping attaching maps) can effect elementary column operations on ∂_k

<u>Note</u>: This ⇒ for an h-cobordism (i.e. $H_k(M, \partial_- M) =$ $\forall k$) can assume ∂_k diagonal $\begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix}$

⑦

7) if M^n a cobordism w/ $\partial_- \neq 0$, $\pi_1(M) = 1$, and $n \geq 5$ then we can assume there are no 1 or $(n-1)$ handl

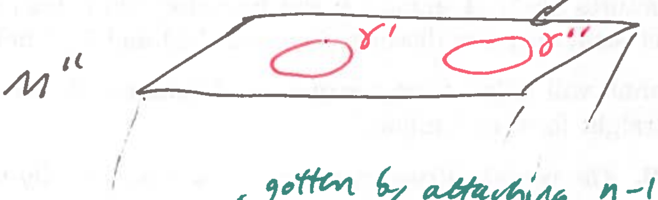
Idea: given a 1-handle consider loop



← level after all 1&2-handles attach

can make disjoint fro

add cancelling 2/3 handle pair to M'' (still ha M'' & M



attaching handle for cancelling 2-handle

gotten by attaching $n-1$ and $n-2$ (≥ 3) handles

$$\pi_1(\partial M'') \cong \pi_1(M'') \cong \pi_1(M) = \{1\}$$

gotten by attaching > 2 handles

so γ' and γ'' homotopic ∴ isotopic since $\dim \partial M'' \geq 4$

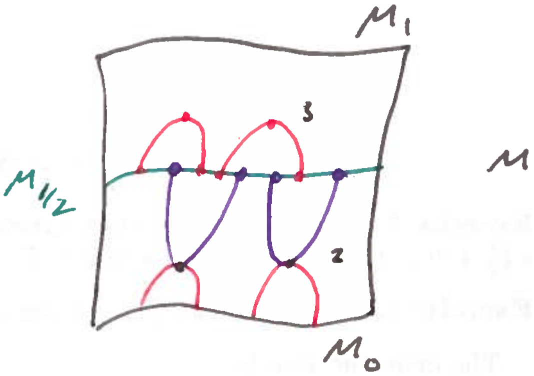
so can assume cancelled 2-handle is attached to γ' and hence γ

∴ 2-handle cancels 1-handle and left w/ 3-h.

Proof of Th^{m}2:

given cobordism M

can assume no
0,1,4, or 5 handles



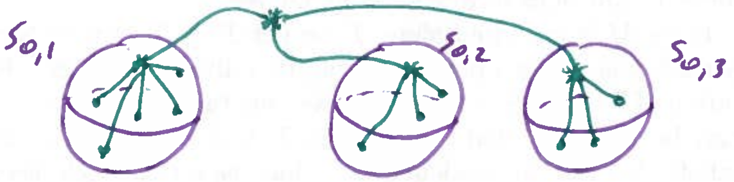
on $M_{1/2}$ let $S_{0,i}$ $i=1 \dots n$ be belt spheres of 2-handles
$S_{1,i}$ $i=1 \dots n$ be attaching " " 3- "

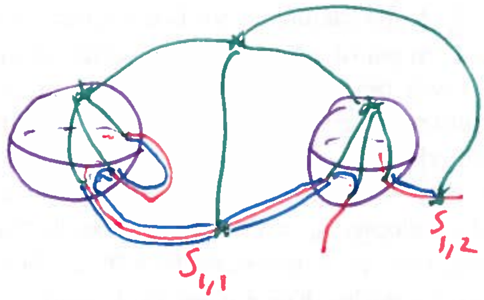
since M h-cobordism can assume $S_{0,i} \cap S_{1,j} = \delta_{ij}$ ← alg ∩.

Choose a base point $* \in M_{1/2}$ disj from spheres
and base points $*_{i,j}$ on $S_{i,j}$

let $\gamma_{i,j}$ be disjoint arcs in $M_{1/2}$ from $*$ to $*_{i,j}$
(only ∩ spheres in endpts)

let η_α be arcs in $S_{i,j}$ connecting $*_{i,j}$ to intersection
points w/ other $S_{i';k}$





let B = small nbhd of $\cup(\gamma_{0,i}) \cup (\eta_\alpha \text{ in } S_{0,i}\text{'s}) = B^4$ a <u>0-h</u>

<u>note</u>: nbhd of $\gamma_{1,i}$ and η_α in $S_{1,i}$'s attached to
B like 1-handles $h_1', h_2' \dots h_k'$

now nbhd of rest of $S_{1,j}$ are 2-handles

$$\underbrace{h_1^2, \dots h_n^2}_{\text{from } S_{0,i}}, \underbrace{\dots, h_{2n}^2}_{\text{from } S_{1,i}}$$

so $\overbrace{\text{nbhd } S_{ij} \text{ \& arcs}}^{N}$ is 0-h $\cup$ (1-h)s $\cup$ (2-h)s

note: 1) $h_1^2 \dots h_n^2$ do not go over 1-handles!

2) h_j^2, $j > n$ go over 1-handles like

note: $\pi_1(N) = \langle x_1 \dots x_n \mid r_1 \dots r_{2n} \rangle$

1 for each 2-handle

1 for each 1-handle

$r_1 \dots r_n$ trivial

and $r_{n+1} \dots r_{2n}$ products of $x_1 \bar{x}_1$ and $\bar{x}_1 x_1$

so trivial too!

extend handlebody structure to $M_{1/2}$

add 1-handles $\tilde{h}_1^1 \dots \tilde{h}_{n'}^1$

2-h $\tilde{h}_1^2 - \tilde{h}_{n'}^2$

3-h & 1 4-handle

$\pi_1(M_{1/2})$ has generators given by $h_1^1 \dots h_n^1$ & $\tilde{h}_1^1 \dots \tilde{h}_{n'}^1$

and rel^n given by h_j^2 and $\tilde{h}_i^2$

trivial rel^{n}s so ignore

since $\pi_1(M_{1/2})$ there is a sequence of "Tietze moves" taking relations from $\tilde{h}_1^2$ to trivial presentation

by handle slides and adding 2/3 cancelling pairs

so (after possibly adding 2/3 cancelling pairs) can assume $\tilde{h}_1^2 \cdots \tilde{h}_\ell^2$ kill the gens h_ℓ^1 & h_i and $\tilde{h}_{\ell+1}^2 \cdots \tilde{h}_{n'}^2$ trivial rel$^{\underline{n}}$s

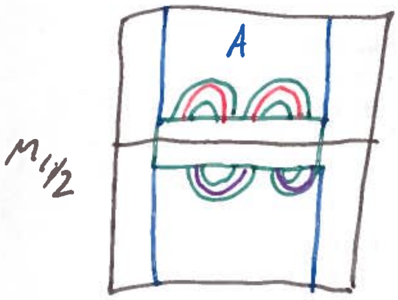
let B_1 = all 0 and 1-handles of $M_{1/2} \cup (\tilde{h}_1^2 \cdots \tilde{h}_\ell^2)$

note: $\pi_1(B_1) = 1$ and $\tilde{H}_*(B_1) = 0$ so B_1 contractible

$B_2 = B_1 \cup (h_1^2 \cup \cdots h_{2n}^2)$

now let $B_3 = B_2 \times [1/2 - \varepsilon, 1/2 + \varepsilon] \cup$ 3-handles attached to $S_{0,i}$ & $S_{1,i}$

upside down 2-handle



let $A = B_3 \cup$ (all M above and below B_3)

note ① $M - A$ has no handles so is a product

② A is contractible since 3-handles cancel h_i^2's

also $B_1 \times [0,1]$ diffeo to A since added 2 & 3 handles cancel

since $B_1 \times [0,1]$ is 5-mfd w/ "same" handle str as B_1 and there attaching spheres of 2-h are S^1 that represent generators in π_1 ∂(1-handle body) $= \#S$

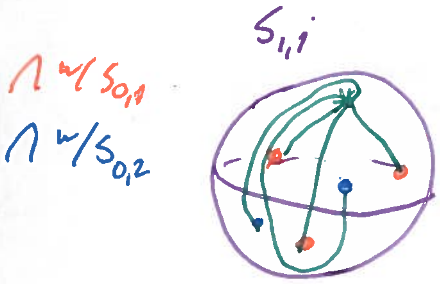
we can isotop them to these genera

$\therefore$ 1 & 2 handles cancel so ④ $B_1 \times [0,1]$ a ball! $\therefore$ so is to see ⑤, that $A_0 \times [0,1] \cong B^5$ need a handle decomposition of A_0

<u>note:</u> get from B_2 to A_0 by attaching 3-handles to $S_{0,i}$ $\Leftrightarrow$ surgering $S_{0,i}$ from 2-handle to 1-handle

so new 1-handles give new gens to $\pi_1(A_0)$ $y_1 \ldots y_n$

and 2-handles $h^2_{n+1} \cdots h^2_{2n}$ give relations that kill $y_1 - y_n$



if when choosing arcs on $S_{1,i}$'s you first connect $*_{1,i}$ to all $\cap$ w/ $S_{0,1}$ then $S_{0,2}, \ldots$

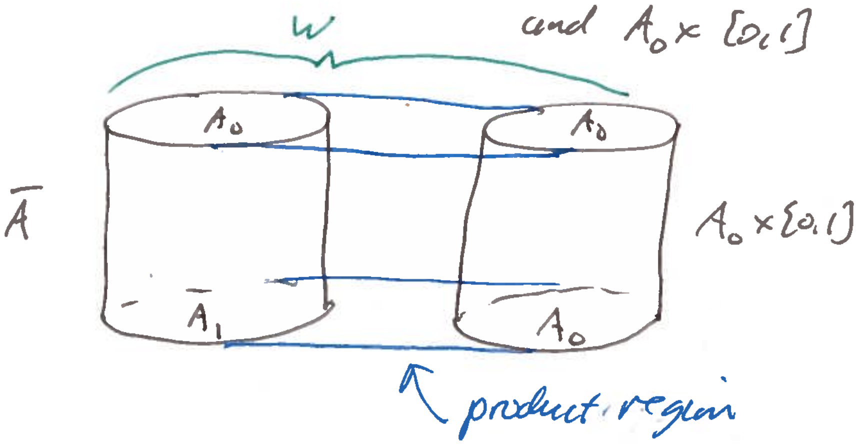
then easy to see rel^{n}s from $S_{1,i}$ is $w_1 w_2 \cdots w_n$ where w_j is word in y_j & $\bar{y}_j$ with exponent sum $= \begin{cases} 0 & \text{for } i \neq j \\ 1 & i = j \end{cases}$

so in $A_0 \times [0,1]$ where $h^2_{n+1} \cdots h^2_{2n}$ attached to circles in 4-mfd (homotopy $\Rightarrow$ isotopy. the h^2_i cancel new 1-han

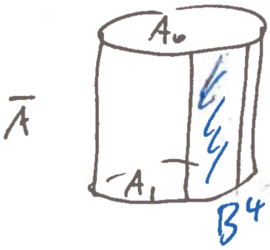
so $A_0 \times [0,1] \cong B^5$

to see ⑥, from ④ and ⑤ consider $\bar{A}$ = a upsidedown and $A_0\times[0,1]$

so



note: top is $A_0 \cup_{B^3} A_0 = (\underbrace{\partial(A_0\times[0,1])}_{A_0\cup_{S^4}A_0} - B^4) \cong (\partial B^5) - B^4$
$\cong B^4$

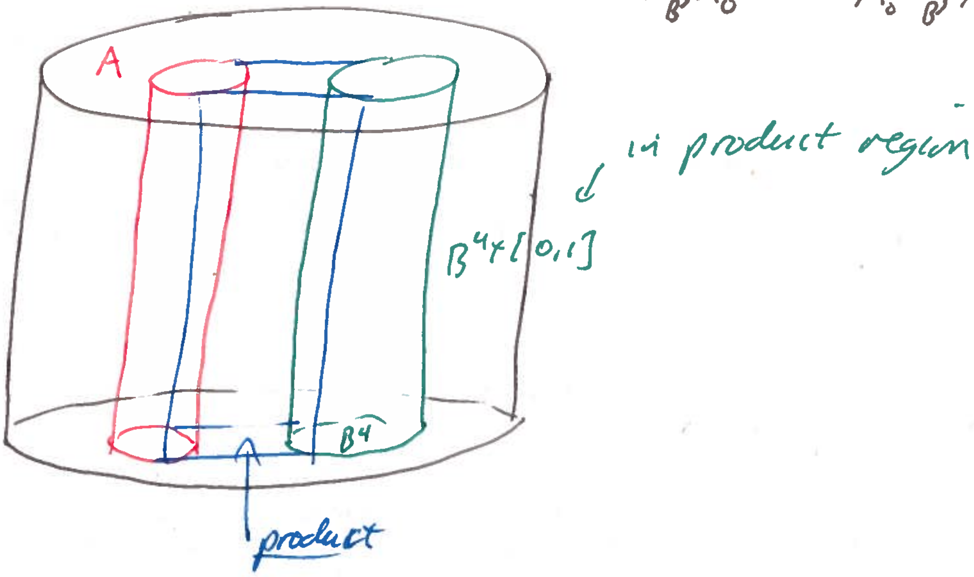


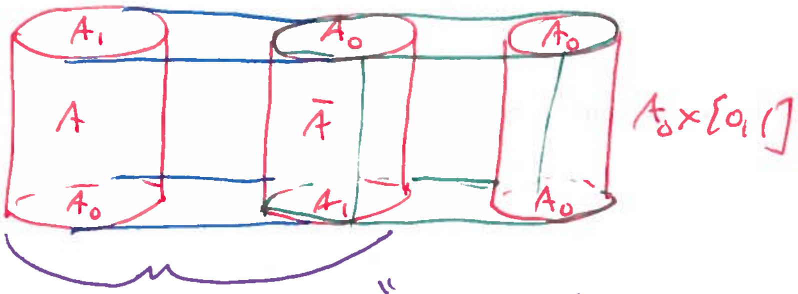
and bottom $A_1 \cup_{B^3} A_0 = (\partial\bar{A} - B^4) \cong B^4$

$$W = \bar{A} \cup_{B^4} (A_0\times[0,1]) \cong B^5$$

so W a cobordism $\cong (B^4\times[0,1], \underset{\substack{\downarrow \\ A_1\cup_{B^3}A_0}}{B^4_0}, \underset{\substack{\downarrow \\ A_0\cup_{B^3}A_0}}{B^4_1})$

now in M





call this "new A"

note clear Involution of bottom to get top!