# Exotic Structures on Open 4-Manifolds

If $n \neq 4$, then $\mathbb{R}^n$ has one smooth structure (this means if $R$ is a smooth $n$-manifold and $R$ is homeomorphic to $\mathbb{R}^n$ then $R$ is diffeomorphic to $\mathbb{R}^n$)

in dimension 4 we have

<u>Thm 1:</u>

there exist a 2-parameter family

$$\{ R_{s,t} \mid s,t \in (0,1) \}$$

such that $R_{s,t}$ is homeo to $\mathbb{R}^4$ but there is no embedding of $R_{s,t}$ to $R_{s',t'}$ if $s > s'$ or $t > t'$

(i.e. if $R_{s,t}$ diffeo to $R_{s',t'}$ then $s = s'$ and $t = t'$)

but if $s \leq s'$ and $t \leq t'$ then $R_{s,t} \hookrightarrow R_{s',t'}$

moreover each $R_{s,t}$ contains a compact set that does not embed in $\mathbb{R}^4$ so they are not diffeo to $\mathbb{R}^4$

<u>Remarks:</u> 1) the existence of <u>one</u> such exotic $\mathbb{R}^4$ follows from an argument of Casson

using work of Donaldson and Freedman

2) 3 such examples were found by Gompf in

"Three exotic $\mathbb{R}^4$'s, and other anomalies"

and a countable family was found by Gompf in

"An infinite set of exotic $\mathbb{R}^4$'s"

4) An uncountable family $\{R_t : t \in (0,1)\}$ was found by Taubes in

"Gauge theory on asymptotically periodic manifolds"

in the second paper of Gompf above he gave the family in Th$^{\underline{m}}$ 1 using Taubes work

Th$^{\underline{m}}$ 1 can be refined, we say $R \leq R'$ if any compact, smooth, codimension zero submanifold of $R$ embeds in $R'$

we say $R$ and $R'$ are compactly equivalent if $R \leq R'$ and $R' \leq R$, denote $R \sim R'$

it is easy to see $\leq$ is a partial order on equivalence classes of $n$-manifolds

exercise: assume $R$ and $R'$ are connected

1) if $R \sim R'$, then they are both closed or both non-closed

2) if $R, R'$ are closed and $R \sim R'$, then $R$ diffeomorphic to $R'$

**Th$^{\underline{m}}$ 1':**

there exist a 2-parameter family
$$\{ R_{s,t} \mid s,t \in (0,1) \}$$
such that $R_{s,t}$ is homeo to $\mathbb{R}^4$ but
$$R_{s,t} \leq R_{s',t'} \iff s \leq s' \text{ and } t \leq t'$$

**Remark:** Th$^{\underline{m}}$ 1 and 1' are proven using

A) Freedman's proof that "Casson handles are topological 2-handles"
i.e. classifying simply connected 4-manifolds } lots of hard topology

B) Donaldson's Diagonalization Th$^{\underline{m}}$ ) analysis

C) for the uncountable family, Taubes generalization of B) to "end periodic" manifolds } harder analysis

we will cover these later.

**Remark:** getting a single exotic $\mathbb{R}^4$ can be done by

uses A) and B) or Khovanov homology { finding a topologically (locally flat) slice knot in $S^3$ that is not smoothly slice

This can be done with no analysis as we will see later

**Thm 2:**

there exist a family

$$\{R_t : t \in (0,1)\}$$

such that $R_t$ is homeomorphic to $\mathbb{R}^4$ and

1) all $R_t$ are subsets of $\mathbb{R}^4$
2) $R_t \hookrightarrow R_{t'}$ if $t \le t'$
3) uncountably many of the $R_t$ are not diffeomorphic

note all $R_t \sim \mathbb{R}^4$

**Remark:**

1) The first such $R$ was constructed by Freedman in unpublished work based on ideas of Casson
2) Thm 2 was proved by De Michelis and Freedman

"Uncountably many exotic $\mathbb{R}^4$'s in standard 4-space"

**Thm 2':**

there is a family $\{R_t : t \in (0,1)\}$ such that $R_t$ is homeomorphic to $\mathbb{R}^4$ and $R_t \sim R_{t'} \Leftrightarrow t = t'$ and for each $t$ $\exists$ an uncountable family $\{R_{t,s}\}$ such that $R_{t,s}$ is homeomorphic to $\mathbb{R}^4$, all $R_{t,s}$ are compactly equivalent and

$R_{t,s}$ is diffeomorphic to $R_{t,s'}$ iff $s=s'$

**Remark:** This is due to Gomph is

"An Exotic Menagerie"

**Remark:** The proof of Th$^{m}$2 is based on A) above and

D) $\exists$ h-cobordant 4-manifolds that are not diffeomorphic } analysis

we call an exotic $\mathbb{R}^4$ large if it contains compact codimension 0 sets that don't embed in $\mathbb{R}^4$ and we call it small if it is compactly equivalent to $\mathbb{R}^4$

**Open Question (?):**

if $R \sim \mathbb{R}^4$, does $R$ embed in $\mathbb{R}^4$

**Th$^{m}$ 3:**

$\exists$ an exotic $\mathbb{R}^4$, $R_u$, such that any exotic $\mathbb{R}^4$, $R$ embeds in $R_u$

**Remark:** This is due to Freedman and Taylor

"A universal smoothing of four-space"

## Open questions:

1) Does every compact equivalence class of exotic $\mathbb{R}^4$'s have uncountably many representatives?
2) Is $R_u$ the unique representative in its compact equivalence class?
3) Given a compact equivalence class $\mathcal{C}$ of $\mathbb{R}^4$'s does $\exists R_{\mathcal{C}} \in \mathcal{C}$ s.t. any $R \in \mathcal{C}$ embeds in $R_{\mathcal{C}}$?

How can we organize exotic $\mathbb{R}^4$'s?

## Algebraic Structure

let $\mathcal{R}$ be the set of all exotic $\mathbb{R}^4$'s

and $\mathcal{R}_\sim$ be the compact equivalence classes of exotic $\mathbb{R}^4$'s

so far we know these are both uncountable sets

we can define a binary operation called end sum.

given $R_1, R_2 \in \mathcal{R}$ chose proper embeddings

$$\gamma_i : [0,\infty) \to R_i$$

we can take neighborhoods of $\gamma((0,\infty))$

$$N_i : (0,\infty) \times \mathring{D}^3 \longrightarrow R_i$$

now $\partial(R_i - \text{int}\, N_i) = \mathbb{R}^3$

choose an orientation reversing diffeomorphism

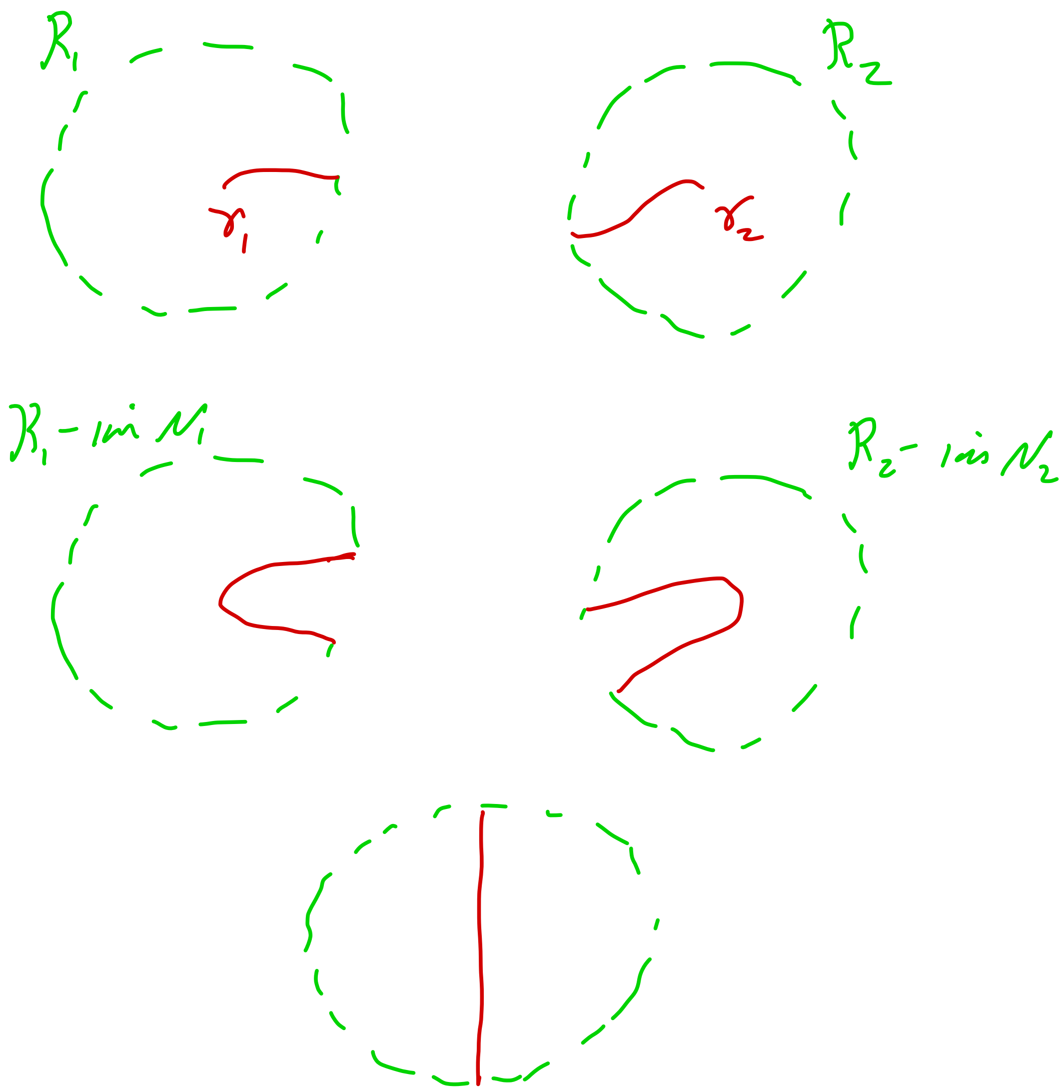
$$\phi : \partial(R_1 - \text{int}\, N_1) \to \partial(R_2 - \text{int}\, N_2)$$

and define the end sum to be

$$R_1 \natural R_2 = (R_1 - \text{int}\, N_1) \cup (R_2 - \text{int}\, N_2) \big/ \sim$$

where $x \in \partial(R_1 - \text{int}\, N_1)$ is glued to

$$\phi(x) \in (R_2 - \text{int}\, N_2)$$



<u>note:</u> any 2 choices fore $\phi$ are isotopic so $\natural$ is well-defined if any choices for $\gamma_i$ are isotopic

<u>exercise:</u> show two proper embeddings of $[0,\infty)$ into an exotic $\mathbb{R}^4$ are isotopic

<u>Hint:</u> first isotop so agree at the integers


now have lots of loops
each bounds disk with a finite number of intersection points
use "finger moves" to remove double pts ...

we now have a map

$$\natural : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$$

and

$$\natural : \mathcal{R}_\sim \times \mathcal{R}_\sim \to \mathcal{R}_\sim$$

**lemma 4:**

> $\mathcal{R}$ and $\mathcal{R}_\sim$ are commutative monoids under $\natural$
>
> i.e. $\natural$ is associative and has an identity
>
> no non-trivial element has an inverse

**Proof:**

Clearly $\mathbb{R}^4$ is the identity and $\natural$ is associative and commutative

suppose $R_1 \natural R_2 = \mathbb{R}^4$

then
$$
\begin{aligned}
R_1 &= R_1 \natural \left( \natural_{i=1}^{\infty} \mathbb{R}^4 \right) \\
&= R_1 \natural (R_2 \natural R_1) \natural (R_2 \natural R_1) \natural \dots \\
&= (R_1 \natural R_2) \natural (R_1 \natural R_2) \natural \dots \\
&= \natural_{i=1}^{\infty} \mathbb{R}^4 = \mathbb{R}^4
\end{aligned}
$$
∎

**Th$^{\underline{m}}$ 3':**

> $R_u$ from Th$^{\underline{m}}$ 3 satisfies $R \natural R_u \cong R_u$
>
> for all $R \in \mathcal{R}$

clearly Th$^{\underline{m}}$ 3' $\Rightarrow$ Th$^{\underline{m}}$ 3

## Open Question:

Is there a group we can associate to $\mathcal{R}$ or $\mathcal{R}_{\sim}$ and $\natural$? Some other operation?

note: Knots in $S^3$ under # is a monoid without inverses, but knots upto cobordism is a group under #

So question above is asking for something like this for $\natural$

Recall: one way to get a group from a commutative monoid is to form its Grothendieck group where you "add inverses" this is how you get $\mathbb{Z}$ from $\mathbb{N} \cup \{0\}$

Specifically if $(M,+)$ is a commutative monoid, then its Grothendieck group is constructed as follows:

- start with $M \times M$

    think of $(m,n)$ as "$m-n$"

- define the equivalence relation

    $(m_1,n_1) \sim (m_2,n_2)$ if $\exists\, k \in M$

s.t. $m_1+n_2+k = m_2+n_1+k$

set $K = M \times M/\sim$

note: if $(m_1 \neq m_2 \Rightarrow m_1+n \neq m_2+n)$
then don't need $\sim$

- define $(m_1,n_1)+(m_2,n_2) = (m_1+m_2, n_1+n_2)$

$(K,+)$ is the Grothendieck group of $(M,+)$

exercise: show the Grothendieck group of $\mathbb{N} \cup \{0\}$ is $\mathbb{Z}$

lemma 5:

> The Grothendieck group of $(\mathcal{R}, \lhd)$ and $(\mathcal{R}_\sim, \lhd)$ is trivial

Proof:

$$\mathcal{R} = \mathbb{R}^4 \lhd \mathcal{R} = (\mathcal{R}_u^{-1} \lhd \mathcal{R}_u) \lhd \mathcal{R} = \mathcal{R}_u^{-1} \lhd (\mathcal{R}_u \lhd \mathcal{R})$$

$$= \mathcal{R}_u^{-1} \lhd \mathcal{R}_u = \mathcal{R}_u$$

## Partial Order:

lemma 6:

> if $\mathcal{R}_1 \leq \mathcal{R}_2$ and $\mathcal{R}_3 \leq \mathcal{R}_4$, then
> $$\mathcal{R}_1 \lhd \mathcal{R}_3 \leq \mathcal{R}_2 \lhd \mathcal{R}_4$$
> and $\leq$ is a partial order on $\mathcal{R}_\sim$

exercise: Convince yourself of this

What more can you say about $\leq$ on $\mathcal{R}$ and $\mathcal{R}_\sim$?

note: minimal elements:

- $\mathbb{R}^4 \leq R \quad \forall R \in \mathcal{R}$

  so $\mathbb{R}^4$ is minimal elt in both $\mathcal{R}$ and $\mathcal{R}_\sim$

- if $R_m$ is another minimal elt in $\mathcal{R}_\sim$

  then $R_m \leq \mathbb{R}^4$ and $\mathbb{R}^4 \leq R_m$

  $\therefore \mathbb{R}^4 \sim R_m$

  i.e. $[\mathbb{R}^4]$ is unique minimal elt in $\mathcal{R}_\sim$

- if $R$ such that $R \subseteq \mathbb{R}^4$ then $R$ is also a minimal element in $\mathcal{R}$

Question:

$\boxed{\text{if } R \in \mathcal{R} \text{ is a minimal element does } R \text{ embed in } \mathbb{R}^4\text{?}}$

rephrase of earlier question

maximal elements:

thm 3 above

- $R \leq R_u \quad \forall R \in \mathcal{R}$

  so $R_u$ is maximal element for both $\mathcal{R}$ and $\mathcal{R}_\sim$

- as above $[R_u]$ is the unique maximal element in $\mathcal{R}_\sim$

Question:

> if $R \in \mathcal{R}$ is a maximal element does $R_u$ embed in $R$?
> or is $R_u \cong R$?

gaps: Question:

> - if $R \leq R'$ is there always some $R''$ such that $R \leq R'' \leq R'$?
> - infinitely many such $R''$?

comparability:

Question:

> - given $R$ are there $R'$ that are not comparable to $R$?
>   uncountably many?
> - given a family $\{R_\alpha\}$ are there (uncountably many) $R'$ that are not comparable to any $R_\alpha$?

other:

**Question:**

- What else can you say about $\leq$ on $\mathcal{R}$ or $\mathcal{R}_\sim$?
- Is there another order on $\mathcal{R}$ or $\mathcal{R}_\sim$?
- Is there an order on a compact equivalence class?

## Topology

We can put a topology on $\mathcal{R}_\sim$ using $\leq$

for all $R \in \mathcal{R}_\sim$ let

$$\left.\begin{aligned} K_R &= \{R' \in \mathcal{R}_\sim : R' \leq R\} \\ L_R &= \{R' \in \mathcal{R}_\sim : R \leq R'\} \end{aligned}\right\} \text{ these form a closed subbasis}$$

let $\mathcal{B} = \{\text{all finite unions of } K_R \text{ and } L_R\text{'s}\}$ ← closed basis

and $\tau_\leq^c = \{\text{all infinite intersections of elts of } \mathcal{B}\}$ ← closed topology

so open sets in $\tau_\leq$ are complements of elts of $\tau_\leq^c$

not much is known about this topology, but Gompf in "A moduli of exotic $\mathbb{R}^4$'s" gave a refinement of $\tau_\leq$

for any compact oriented 4-manifold $X$ let

$$U_X = \{R \in \mathcal{R}_\sim : X \text{ embeds in } R\}$$

now let $\tau$ be the topology with closed subbasis all $\mathcal{R}_\sim - U_x$ and $L_R$

## Gompf proved:

1) $\tau$ is regular
2) $\tau$ is $2^{nd}$ countable
3) $\tau$ is metrizable ⟸ follows from 1), 2) by Urysohn Metrization
4) every increasing sequence converges

## Open Problems:

- is $\tau = \tau_\leq$ ?
- what more can be said about $\tau$ or $\tau_\leq$ ?
- is there a "better" topology on $\mathcal{R}_\sim$ ?
- is there a "good" topology on $\mathcal{R}$ ?

## Symmetries:

note if $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by $\pi$

then $\mathbb{R}^2/_{x \sim \tau(x)} \cong \mathbb{R}^2$ and $\mathbb{R}^2 \to \mathbb{R}^2/_\tau$

this is the standard 2-fold cover of $\mathbb{R}^2$ branched over a point

so $\tau' = \tau \times id : (\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^4 : (x, y) \mapsto (\tau(x), y)$

satisfies $\mathbb{R}^4/_{x \sim \tau'(x)} \cong \mathbb{R}^4$ and $\mathbb{R}^4 \to \mathbb{R}^4/_{\tau'}$

this is the standard 2-fold cover of $\mathbb{R}^4$ branched over a plane

## Theorem 7:

there exists $\mathbb{R}^4$'s $R_1, R_2$ and a smooth involution $\sigma : R_1 \to R_1$ that is

1) topologically standard
2) $R_1 \to R_1/\sigma \cong R_2$

and we can independently choose $R_1$ and $R_2$ to be large or small exotic $\mathbb{R}^4$'s as long as $R_1$ is not $\mathbb{R}^4$ (we can take $R_2$ to be standard)

## Open Question:

Can you find such an $R_1 \to R_2$ with $R_1 \cong \mathbb{R}^4$ and $R_2$ not?

## Remark:

1) $\begin{array}{c} R_1 \\ \downarrow \\ \mathbb{R}^4 \end{array}$ small is due to Freedman in unpublished work

2) rest of theorem due to Gompf in "An exotic menagerie"

more group actions (from Gompf)

let $G$ be I) a group that acts properly discontinuously on $\mathbb{R}^3$

i.e. $\forall x \in \mathbb{R}^3$ $\exists$ a nbhd $U$ of $x$ s.t. $g(U) \cap U = \emptyset$ if $g \neq 1$

clearly $G$ acts on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$

or II) a finite group acting on $\mathbb{R}^4$

let $\alpha = \{(0,0,0)\} \times [0,\infty)$ and

$$A = \bigcup_{g \in G} g(\alpha)$$

let $R$ be an exotic $\mathbb{R}^4$ (large or small)

set $R' = \mathbb{R}^4 \natural_A \left( \coprod_G R \right)$

(i.e. end sum one $R$ into $\mathbb{R}^4$ along each $g(\alpha)$)

clearly $G$ acts on $R'$ and is topologically equivalent to the $G$ action on $\mathbb{R}^4$

if $\mathbb{R}^4/G = \mathbb{R}^4$, then

$$R'/G \cong R$$

note: $\mathcal{R}'$, $\mathcal{R}$ are both either large or small and $\mathcal{R}' \geq \mathcal{R}$

Open Question:

$\boxed{\text{Can an exotic } \mathbb{R}^4 \text{ cover a compact 4-manifold?}}$

now for any $\mathbb{R}^4$, $\mathcal{R}$ set

$$D(\mathcal{R}) = \{\text{diffeomorphisms of } \mathcal{R}\}/_{\text{isotopy}}$$

and $D_+(\mathcal{R}) \subset D(\mathcal{R})$ the subgroup of orientation preserving isotopy classes

note: $D_+(\mathbb{R}^n) = \{id\}$ $\forall n$

Th$^m$8: There are uncountably many $\mathcal{R}$ (both large and small) for which there is an uncountable subgroup of $D_+(\mathcal{R})$

Remark: This is due to Gompf in "Group actions, corks and exotic smoothings of $\mathbb{R}^4$"

When studying manifold with boundary we are interested in how diffeos of the boundary interact with diffeos of the mfd for non-compact manifolds we introduce "diffeomorphisms at infinity"

a closed neighborhood of infinity (cni) in a manifold $M$ is a codimension one submanifold $E \subset M$ that is closed and $\overline{M-E}$ is compact

given cni $E_i \subset M_i$ and $E_i' \subset M_i'$ $i=1,2$

and diffeomorphisms $f_i : E_i \to E_i'$

we say $f_1 \sim f_2$ if $\exists$ a cni $E \subset M$

s.t. $E_i \subset E_i$ and $f_1|_E = f_2|_E$

$f_1, f_2$ have same "germ at $\infty$"

a diffeomorphism at infinity from $M$ to $M'$ is an equivalence class of such diffeos

when $M, M'$ have a single end we also say this is a diffeomorphism of the ends

let $\mathcal{D}^\infty(M) = \{$isotopy classes of diffeos at $\infty\}$

and $\mathcal{D}^\infty_+(M)$ the orient pres. subgroup

there is an obvious restriction map

$$r : \mathcal{D}_{(+)}(M) \to \mathcal{D}^\infty_{(+)}(M)$$

## Lemma 9:

for any $R \in \mathcal{R}$, $\ker r$ and $\operatorname{coker} r$ are countable

## Theorem 9:

there are uncountably many $\mathbb{R}^4$'s $R$ (large and small) s.t.

1) $r(\mathcal{D}(R)) \subset \mathcal{D}^\infty(R)$ is uncountable
2) $\exists$ nonfinitely generated groups in $\operatorname{coker} r = \mathcal{D}^\infty(R) / \underline{r(\mathcal{D}(R))}$

   only a set

3) for the universal $R_u$

$$\operatorname{coker} r = \{1\}$$

## Open Question:

What can you say about $\ker r$? is it always trivial?

## Open Question:

Can $S^1$ or $\mathbb{R}$ act non-trivially on an exotic $\mathbb{R}^4$?

## Geometry:

There is not much known about Riemannian metrics on exotic $\mathbb{R}^4$'s

here are a few: suppose $R \in \mathcal{R}$ but $\neq \mathbb{R}^4$

1) Can't have a constant curvature metric (since this $\Rightarrow \mathbb{R}^4$ or $S^4$)
2) Can't have curvature $\leq 0$ (since $\Rightarrow \mathbb{R}^4$)
3) Can have a complete metric with negative Ricci curvature and one with sectional curvature any constant negative number (Lohkamp)

## Questions:

Can you construct invariants of exotic $\mathbb{R}^4$s using Riemannian metrics?

e.g. for $g$ on $R$ let

$$G_g = \text{max curvature} - \text{min curvature}$$

$$G_R = \inf_g G_g$$

can this distinguish exotic $\mathbb{R}^4$'s?
can this be 0?

**Th$^{m}$ 10:**

$\exists$ exotic $\mathbb{R}^4$'s that admit metrics whose isometry group contains an uncountable subgroup

**Remark:** Due to Gompf in last mentioned paper an exotic $\mathbb{R}^4$, $\mathcal{R}$, is called <u>full</u> if there exists a compact subset that cannot be embedded in its complement or into any homology $S^4$ (not hard to construct these)

**Th$^{m}$ 11:**

1) any metric on a full $\mathbb{R}^4$ has finite isometry group
2) there exist $\mathcal{R}$, full exotic $\mathbb{R}^4$'s, with $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}^{\infty}(\mathcal{R})$ uncountable

**Open Question:**

Does every isometry group of an exotic $\mathbb{R}^4$ inject into its diffeotopy group?

**note:** not true for $\mathbb{R}^4$!

Remark: 1) in Th$^{m}$ due (mostly) to Taylor in "Smooth Euclidian 4-spaces with few isometries"
2) is due to Gompf in above paper

let's move to symplectic geometry

recall a symplectic structure on a 4-manifold $M$ is a 2-form $\omega$ st.

(closed) $d\omega = 0$ and

(non-degenerate) $\omega \wedge \omega$ is never zero (i.e. volume form)

a complex manifold $X$ is Stein if

$\exists$ an exhausting pluri-subharmonic function $\phi : X \to \mathbb{R}$

i.e. $\phi^{-1}((-\infty, c])$ is compact $\forall c$ and

$[d(d\phi \circ J)](v, Jv) > 0 \ \forall v \neq 0$ in $TX$

this $\Rightarrow d(d\phi \circ J)$ is a symplectic form

here $J : TX \to TX$ is the action of multiplication by $i$ on $TX$

**Thm 12:**

There are uncountably many small exotic $\mathbb{R}^4$'s that admit Stein structures

There are uncountably many large $\mathbb{R}^4$'s that embed into Stein surfaces

There are exotic $\mathbb{R}^4$'s that do not embed into Stein surfaces

**Remark:** 1st result is due to Gompf in "Handlebody constructions of Stein surfaces"

other results due to Bennett in "Exotic smoothings via large $\mathbb{R}^4$'s in Stein surfaces"

**Questions:**

do all small $\mathbb{R}^4$'s admit Stein structures?

(Symplectic with convex "boundary")

does any large $\mathbb{R}^4$ admit a Stein structure?

**Invariants:**

given $R$ homeomorphic to $\mathbb{R}^4$

for any compact subset $C \subset R$ there is a compact 3-manifold $M$ separating $C$ from $\infty$

(just take any smooth proper $f: \mathcal{R} \to [0,\infty)$
then there is some regular value $c$
s.t. $C \subset f^{-1}([0,c])$ so $M = f^{-1}(c)$ works)

let $b_C = \min\{b_1(M)\}$ over all such $M$

$\nwarrow$ 1st Betti number

<u>note</u>: if $C \subset C' \subset \mathcal{R}$ then $b_C \le b_{C'}$

Bižaka and Gompf defined the <u>engulfing index of $\mathcal{R}$</u> to be

$$e(\mathcal{R}) = \sup\{b_C \mid C \subset \mathcal{R}\}$$

easy to see $e(\natural \mathcal{R}_i) \le \sum e(\mathcal{R}_i)$

in many examples it is $\infty$

<u>Thm 13</u>: $\boxed{\exists \text{ exotic } \mathbb{R}^4\text{'s with } e \text{ finite}}$

<u>Question</u>: $\boxed{\text{Which values of } e \text{ can be realize?}}$ (I think $\infty$'ly many can?)

Now given $\mathcal{R}$ consider the set

$$Sp(\mathcal{R}) = \{\text{closed spin 4-manifolds with intersection form } \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ into which } \mathcal{R} \text{ embeds}\}$$

if $Sp(\mathcal{R}) = \emptyset$, set $b_E = \infty$

otherwise $b_E = \frac{1}{2} \min_{N \in Sp(R)} \{b_2(N)\}$ 2nd Betti number

for any smooth 4-manifold $M$ let

$$\mathcal{E}(M) = \{ \text{topological embeddings } e: D^4 \to M \text{ s.t. } e(\partial D^4) \text{ is bicollared and } \exists p \in \partial D^4 \text{ s.t. } e|_{nbhd(p)} \text{ is smooth} \}$$

for $e \in \mathcal{E}(M)$ let $R_e$ = smooth structure induced on $e$(interior of $D^4$)

finally set $$\gamma(R) = \max_{e \in \mathcal{E}(M)} \{ b_{R_e} \}$$

this is called the <u>Taylor invariant</u>

roughly $\gamma(R)$ measures the minimal number $n$ st $R$ embeds in $\#_n S^2 \times S^2$

all the complications in the def$^{n}$ are to prove things about $\gamma$

we can extend $\gamma$ to any 4-manifold as follows

if $M$ is spin, then

$$\gamma(M) = \max \{ \gamma(E) \}$$

where $E \subset M$ open set homeo to $\mathbb{R}^4$

if $M$ is orientable but not spin and

1) the $2^{nd}$ Steifel-Whitney class $w_2(M)$ has no comact dual then

$$\gamma(M) = -\infty$$

2) if there are compact duals then

$$\gamma(M) = \max\{\gamma(M-F) - \dim H_1(F;\mathbb{Z}_2)\}$$

where $F$ runs over all compact duals to $w_2(M)$

if $M$ is nonorientable, then let $\tilde{M}$ be its orientation double cover and set

$$\gamma(M) = \gamma(\tilde{M})$$

## Properties of $\gamma$:

1) if $W_0 \subset \dots \subset W_{i-1} \subset \mathring{W}_i \subset W_i \subset \dots \subset M$ then $\gamma(M) \leq \max\{\gamma(W_i)\}$ get $=$ if $M$ spin

2) if $M_2$ spin and $M_1 \subseteq M_2$ then

$$\gamma(M_1) \leq \gamma(M_2)$$

3) $\gamma(\mathbb{C}P^2) = 0$ but $\exists$ exotic $R \subset \mathbb{C}P^2$ st. $\gamma(R)$ arbitrarily large

4) $\exists$ $M$ with $\gamma(M) = -\infty$ and for each $n \geq 0$, $\exists$ uncountably many exotic $\mathbb{R}^4$ with $\gamma = n$ ($n$ could be $\infty$)

5) if $M$ is a Stein manifold then $\gamma(M) \leq b_2(M)$ so for a Stein exotic $\mathbb{R}^4$, $\gamma = 0$ ($\therefore$ get lots of exotic $\mathbb{R}^4$ with no Stein str)

6) $\exists$ uncountably many exotic $\mathbb{R}^4$ that embed in Stein manifold but have arbitrarily large $\gamma$

7) If $R$ is an exotic $\mathbb{R}^4$ and $R$ nontrivially covers some other manifold then

$$\gamma(\mathcal{G}^r E) = \gamma(E)$$

$\exists$ examples of $R$ s.t. $\gamma(\mathcal{G}^r E) = \frac{2}{3} r\, \gamma(E)$

so these can't cover!

8) $\gamma(R) \leq e(R)$

Remark: All results due to Taylor "An Invariant of Smooth 4-Manifolds"

except 6) due to Bennett in above paper
and 8) due to Khuzam in
"A comparative study of two fundamental invariants of exotic $\mathbb{R}^4$'s"

Question:

Can $\gamma$ take on finite negative values?

Question:

- can one find a non discrete invariant of exotic $\mathbb{R}^4$?
- can one define an invariant of exotic $\mathbb{R}^4$'s

in a compact equivalence class?
(note $\gamma(R) = \gamma(R')$ if $R \sim R'$
is this true of $e$?)

## Other Manifolds

**Th^m 14:**

let $X$ be one of the following
1) $W$-pt for *any* topological manifold
2) total space of an oriented $\mathbb{R}^2$ bundle over an oriented surface
3) $Y \times \mathbb{R}$ for a 3-manifold $Y$ that topologically locally flatly embeds in $\#^n \overline{\mathbb{C}P}^2$

then $X$ admits uncountably many smooth structures

**Remark:** 1) is due to Gompf in
"An Exotic Menagerie"

but Furuta and Ohta proved many cases in
"A remark on uncountably many exotic differential structures on one-point punctured topological 4-manifolds"

2) is due to Ding in

"Smooth structures on some open 4-manifolds"

3) is due to Fang in

"Embedding 3-manifolds and smooth structures of 4-manifolds"

also note any rational homology sphere or Seifert fibered space has such embeddings

Th$^{\underline{m}}$ 15:

If $Y$ is any compact 3-manifold then $Y \times \mathbb{R}$ admits infintely many smooth structures

if $X$ is any open 4-manifold with at least one end that is topologically $Y \times \mathbb{R}$ and $X$ has only finitely many ends homeomorphic to $Y \times \mathbb{R}$ then $X$ admits infinitely many smooth structures

Remark: This is due to Bižaca and Etnyre in

"Smooth structures on collarable ends of 4-manifolds"

Questions:

1) can th$^{\underline{m}}$ be upgraded to get uncountably many smooth structures?

2) Can th$^{\underline{m}}$ be upgraded so $Y$ is <u>any</u> 3-manifold?

3) Can th$^{\underline{m}}$ be upgraded so $X$ is <u>any</u> open 4-manifold?

## 3 constructions of exotic $\mathbb{R}^4$'s

### I) Restrictions of the intersection form of 4-manifolds (failure of smooth surgery)

recall given an oriented closed 4-manifold $X$

$H_4(X) = \mathbb{Z}$ and a generator $[X]$ is called a fundamental class

Poincaré Duality says

$$H^2(X)/\text{tor} \times H^2(X)/\text{tor} \xrightarrow{I_X} \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \alpha \cup \beta([X])$$

is a (symmetric) non-degenarate pairing called the <u>intersection form</u>

using Poincaré duality we can reinterperate this in $H_2(X)$

recall if $\Sigma \subset X^4$ is an embedded oriented surface then

$[\Sigma] \in H_2(X^4)$ (actually $i : \Sigma \to X$ inclusion $i_*([\Sigma]) \in H_2(X^4)$)

<u>Fact</u>: for any $h \in H_2(X^4)$ $\exists$ some surface $\Sigma^2 \subset X$ s.t. $h = [\Sigma]$

now given $h, h' \in H_2(X^4)$ let $h = [\Sigma]$ and $h' = [\Sigma']$

we can isotop $\Sigma'$ so that $\Sigma$ is transverse to $\Sigma'$

so $\Sigma \cap \Sigma' = \{p_1 \dots p_k\}$

let $\varepsilon(p_i)$ = sign of intersection

{i.e. does or$^{n}$ on $T_{p_i}\Sigma$ followed by or$^{n}$ on $T_{p_i}\Sigma'$ agree or not with or$^{n}$ on $T_{p_i}X$}

define $\Sigma \cdot \Sigma' = \sum_{i=1}^{k} \varepsilon(p_i)$

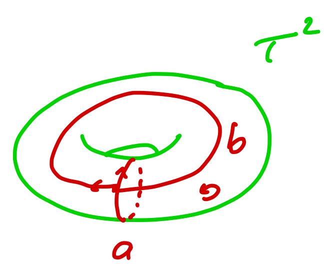
now $I_X : H_2(X) \times H_2(X) \to \mathbb{Z}$

$([\Sigma], [\Sigma']) \mapsto \Sigma \cdot \Sigma'$

is Poincaré dual to pairing above

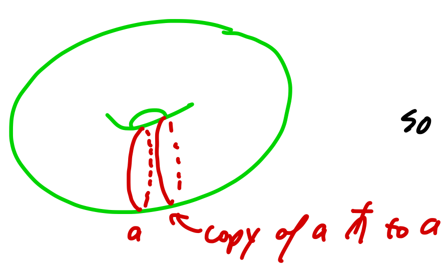
example:

in dimension 2:




$H_2(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ (a, b)

$a \cdot a = ?$ :




so $a \cdot a = 0$

$a \cdot b = 1$

so $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

similarly for $S^2 \times S^2$ we have $H_2(S^2 \times S^2) = \mathbb{Z} \oplus \mathbb{Z}$

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can also check $H_2(\mathbb{CP}^2) = \mathbb{Z}$ and $I = [1]$

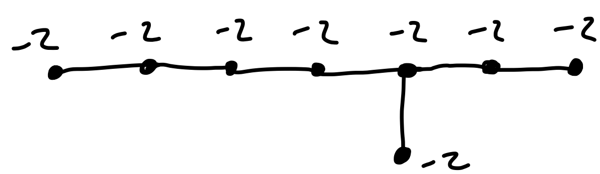
**Theorem [Donaldson]:**

$$\boxed{\begin{array}{l}\text{if } X \text{ is an oriented, simply connected, smooth} \\ \text{4-manifold with } I_X \text{ negative definite} \\ (\text{i.e. } I_X(e,e) < 0 \ \forall e) \text{ then } I_X = \oplus [-1]\end{array}}$$

i.e. $I_X$ is diagonalizable

**Remark:** there are lots of non-diagonalizable symmetric pairings, e.g.

$$\begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & 0 & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & 0 & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & 1 \\ & & & & & 1 & -2 & 0 \\ & & & & & 1 & 0 & -2 \end{pmatrix}$$




is not diagonalizable over $\mathbb{Z}$

so it cannot be $I_X$ for $X$ as in theorem

**Idea of Proof:**

Given $X$ as in th$^{\underline{m}}$ except $I_X$ is pos. definite (just revers or$^{n}$)

there is a $SU(2)$-bundle $E$ over $X$ with Chern class $c_2(P) = 1$

let $\mathcal{B} = \{$ connections $D$ on $E$ with self-dual curvature $R_D = *R_D \}$

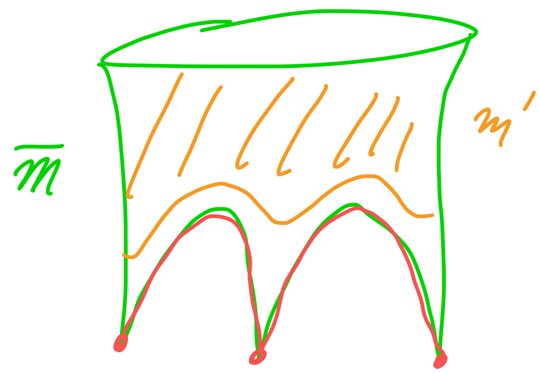
Yang-Mills equations

$\mathcal{M} = \mathcal{B}/\mathcal{G}$ ⟵ gauge group (symmetries of sol$^{n}$s)

Using work of many people, in particular Taubes and Uhlenbeck one can show:

- $\overline{\mathcal{M}}$ is a compact oriented 5-manifold with singular points
  number $= \frac{1}{2}\#$ solutions to $I_X(h,h) = 1$
  call this number $m$
- Singular points have nbhd that are cones on $\mathbb{C}P^2$
- $\partial \overline{\mathcal{M}} = X$

so $\mathcal{M}' = \overline{\mathcal{M}}$ - nbhd of sing pts and arcs connecting them




has $\partial \mathcal{M}' = X \cup -(\#_m \mathbb{C}P^2)$

$I_M$ positive definite so $\operatorname{rank}(I_M) = \sigma(I_M)$

signature

but signature is cobordism invariant so

$$\operatorname{rank}(I_M) = \sigma(I_M) = \sigma(\#_m \mathbb{C}P^2) = m$$

note: for any symmetric pos. definite unimodular form $I$, if $m = \frac{1}{2}\#$ solutions $I(h,h) = 1$

then $m \leq \operatorname{rank}(I)$ and equality $\Leftrightarrow$ $I$ diagonal

indeed given sol$^{n}$ $h$ consider

$$I = (\operatorname{span} h) \oplus (\operatorname{span} h)^{\perp}$$

if $k \neq \pm h$ is another sol$^{n}$ then

$$I(h \pm k, h \pm k) = 1 + 1 \pm 2I(h,k)$$

since $I$ pos. definite $I(h,k) = 0$

$\therefore$ all other sol$^{n}$ in $(\operatorname{span} h)^{\perp}$

result follows by induction ✓

Donaldson's th$^{\underline{m}}$ now follows! $\blacksquare$

## Existence of 1 large exotic $\mathbb{R}^4$:

let $K$ be "the" K3 surface

i.e. $K = \{ z_0^4 + z_1^4 + z_2^4 + z_3^4 \text{ in } \mathbb{C}P^3 \}$

one can compute $\pi_1(K) = 1$, $H_2(K) \cong \oplus_{22} \mathbb{Z}$, and

$$I_K = E_8 \oplus E_8 \oplus_3 H \qquad \text{where } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

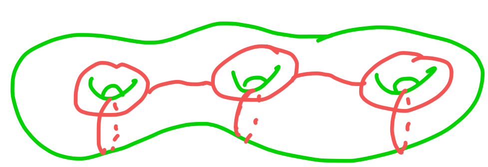
since $\pi_1 = 1$ we know $\pi_2(K) \cong H_2(K)$

so $\exists$ immersions $e_i : S^2 \to K$ generating $\oplus_3 H$

after work of Casson we have

**Th$^{\underline{m}}$ (Freedman):**

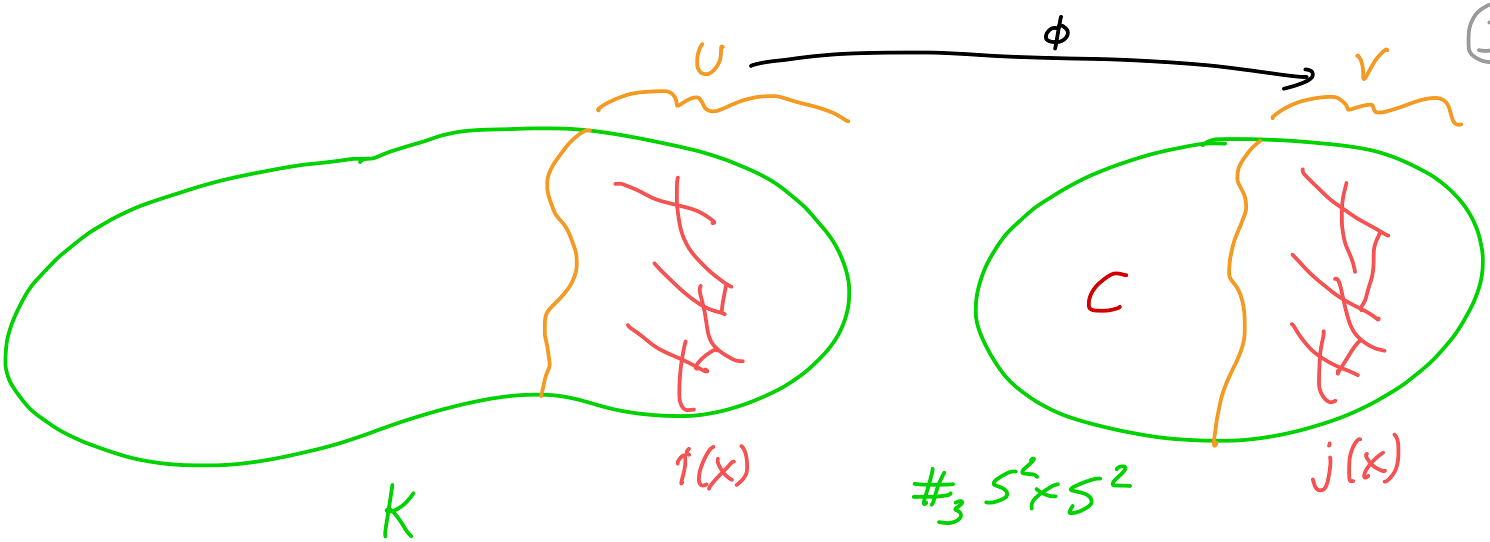
let $X$ be 3 copies of $S^2 \times \{p\} \cup \{p\} \times S^2$ in $\#_3 S^2 \times S^2$ connected by arcs




$\exists$ a topological embedding $i : X \to K$ realizing $\oplus_3 H$,

a top embedding $j : X \to \#_3 S^2 \times S^2$ top isotopic to $X$, a nbhd $U$ of $i(X)$, a nbhd $V$ of $j(X)$, and a diffeomorphism $\phi: U \to V$






let $\mathcal{R} = (\#_3 S^2 \times S^2) - j(X)$

Claim: $\mathcal{R}$ is an exotic $\mathbb{R}^4$

1st: $\mathcal{R}$ homeo. to $\mathbb{R}^4$

for this we use

Th$^{\underline{m}}$ (Freedman):

any open manifold with $\pi_1 = 0$ and $H_2 = 0$ and one end topologically equivalent to $S^3 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$

we can easily check these properties for $\mathcal{R}$ ✓

2nd: $\mathcal{R}$ not diffeo to $\mathbb{R}^4$

assume $\mathcal{R}$ diffeo to $\mathbb{R}^4$

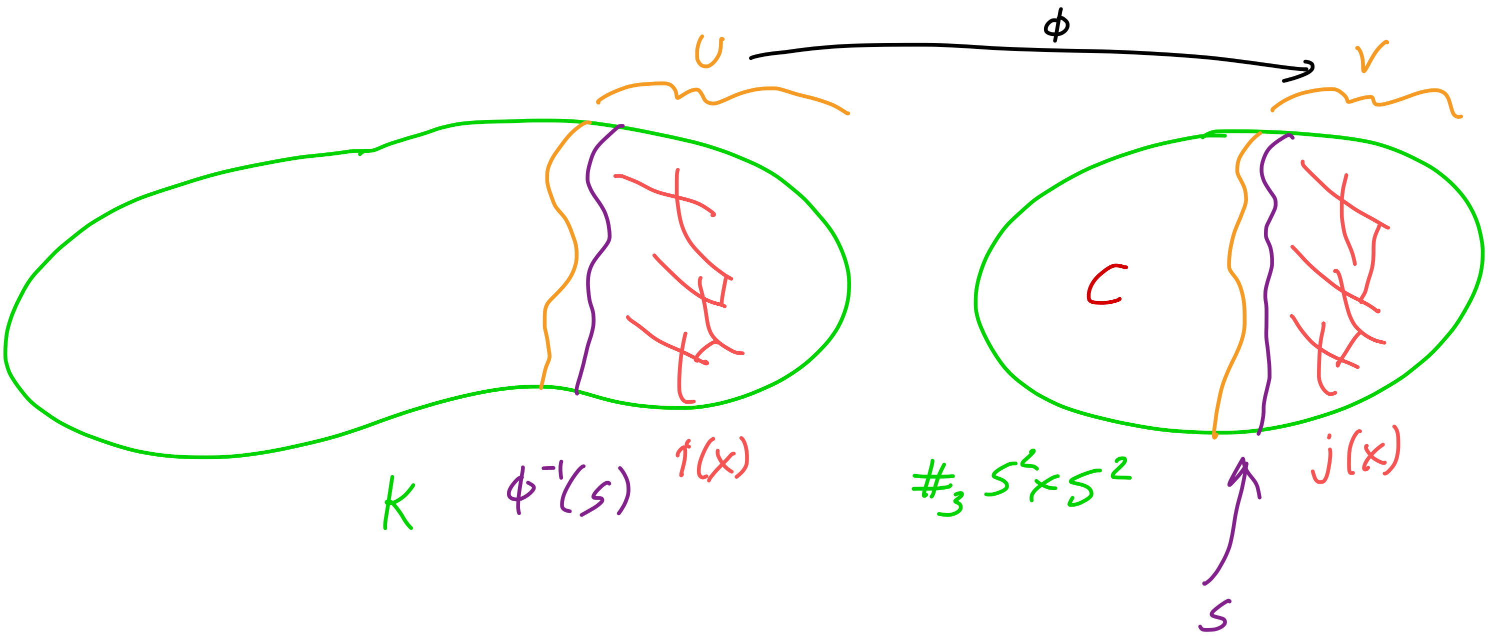
let $C = \#_3 S^2 \times S^2 - V$

this is a compact set

in $\mathbb{R}^4$ we know given any compact set $C$

$\exists$ a 3-sphere $S$ that separates $C$

form $\infty$




now $\phi^{-1}(C)$ breaks $K$ into $K_1 \cup K_2$ with

$i(X) \subset K_2$

set $K' = K_1 \cup B^4$ glued along $S^3$

$I_{K'} \cong \oplus_2 E_8$

contradicts Donaldson's Th$^{\underline{m}}$

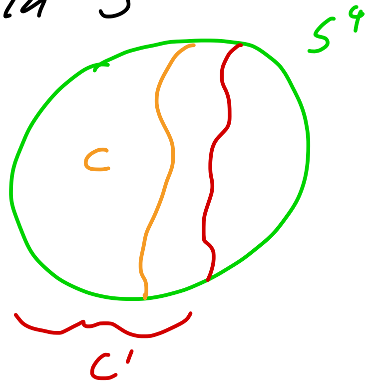
so $\mathcal{R}$ not diffeo to $\mathbb{R}^4$

now let's check $\mathcal{R}$ is large

let $C'$ be a compact set in $\mathcal{R}$ containing $C$

and homeomorphic to $\mathbb{R}^4$

if $C'$ smoothly embedded in $\mathbb{R}^4$ then it embeds in $S^4$




now we can glue $S^4 - C$ to $K - \phi^{-1}(C'-C)$

by $\phi$ on $(C'-C)$ to $\phi^{-1}(C'-C)$

giving a closed smooth mfd with $I = \oplus_2 E_8$ #

There are many ways to get an infinite family, one uses

If $X$ is a smooth closed spin 4-manifold

trivialization over 2-skeleton

if simply connected $\Leftrightarrow I_X$ even

then by Donaldson $I_X$ is $\oplus_k E_8 \oplus_\ell H$

(Rochlin's th$^{\underline{m}}$ says $\sigma \equiv 0 \bmod 16$ so $k$ is even)

Th$^{\underline{m}}$ (Furuta):

$\boxed{\text{must have } \ell \geq k+1}$

uses the Seiberg-Witten equations and restrictions on equivariant maps between spheres

## Countably infinite family of large $\mathbb{R}^4$'s

let $\mathcal{R}_1 = \mathcal{R}$ from above

$\mathcal{R}_n = \mathcal{R}_{n-1} \natural \mathcal{R}$

and $\mathcal{R}_\infty = \natural_\infty \mathcal{R}$

note: $\mathcal{R}_n$ contains $n$ copies of $C$ and $C'$

let $C_n$ = the boundary sum of all these
$C_n'$ = " "

we could alternately construct the $\mathcal{R}_n$ and $C_n$, $C_n'$ by doing the above construction to

$\#_n K$ and $\#_{3n} S^2 \times S^2$

lemma:

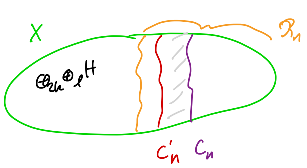
> If $X$ is any closed smooth spin 4-manifold, then $\exists$ an $m > 0$ st. for any $n > m$, $\mathcal{R}_n$ and $C_n$ cannot be smoothly embedded in $X$

Proof:

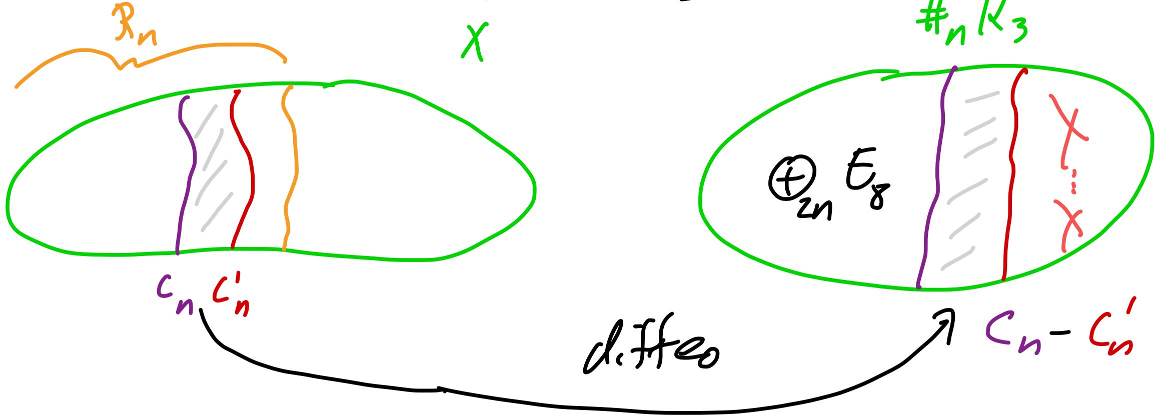
as noted above $I_X = \oplus_{2k} E_8 \oplus_\ell H$

let $m$ be any integer with $2m > \ell - 2k$

if $n \geq m$ then $\mathcal{R}_n$ cannot imbed in $X$




to see this assume it does



So we can glue $X - C_n$ to a piece of $\#_n K$ cut along $\partial C_n'$ to get $Z$

with $I_Z = \oplus_{2n+2h} E_8 \oplus_\ell H$

but $\ell < 2k+2n$ $\cancel{\times}$ Furuta!

now all $\mathcal{R}_n$ are distinct since by construction

$\mathcal{R}_n$ embeds in $\#_{3n} S^2 \times S^2$

but by lemma not in $\#_{2n} S^2 \times S^2$

from this it is clear that infinitely many of the $\mathcal{R}_n$ are different

but now assume $\mathcal{R}_n = \mathcal{R}_m$ for $n < m$

this implies for any $k > m$, $\mathcal{R}_k = \mathcal{R}_\ell$ for $n \leq \ell \leq m$

$\therefore \mathcal{R}_n \neq \mathcal{R}_m$ for $n \neq m$

exercise: $\mathcal{R}_n$ can't embed in any neg. definite mfd

$\mathcal{R}_\infty$ " " " "

or any spin mfd.

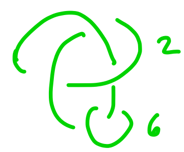
## Infinitely many smooth structures on $M^3 \times \mathbb{R}$

let $M$ be a compact closed smooth

orientable 3-manifold

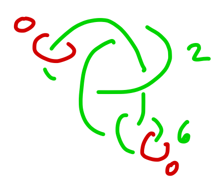
Fact: $\exists\, n$ s.t. $M$ smoothly embeds in $\#_n S^2 \times S^2$ ($\therefore$ $M \times \mathbb{R}$ does too)

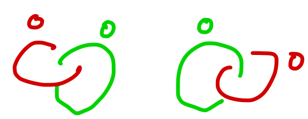
idea: $M$ is obtained from $S^3$ by Dehn surgery on a link with even integer coeff

e.g 

so $M = \partial X$ where $X$ is 4-manifold obtained from $B^4$ by attaching 2-handles

$D(X) = \partial(X \times [0,1])$ has handle diagram

 $\cup$ 4-handle

handle slides give  $\cup$ 4-handle

this is $\#_n S^2 \times S^2$ ✓

now $(M \times \mathbb{R}) \natural \mathcal{R}_{n+1}$ is not diffeo to $M \times \mathbb{R}$ since it contains a set that can't be embedded in $\#_n S^2 \times S^2$

similarly infinitely many of $(M \times \mathbb{R}) \natural \mathcal{R}_k$ must be different ✓

if $M$ has boundary, but is orientable

then $D(M) = \partial(M \times I)$ is closed and as above

embeds in $\#_n S^2 \times S^2$ so $M$ does too

now same argument $\Rightarrow M \times \mathbb{R}$ has infinitely

many smooth str's

if $M$ is non oriented then let $\tilde{M}$ be its orientation

double cover

note the double cover of $(M \times \mathbb{R}) \natural \mathcal{R}_n$ is

$(\tilde{M} \times \mathbb{R}) \natural \mathcal{R}_{2n}$ and a diffeo of $(M \times \mathbb{R}) \natural \mathcal{R}_n$

with $(M \times \mathbb{R}) \natural \mathcal{R}_m$ will lift to a diffeo of

$(\tilde{M} \times \mathbb{R}) \natural \mathcal{R}_{2n}$ to $(\tilde{M} \times \mathbb{R}) \natural \mathcal{R}_{2m}$

$\therefore$ infinitely many of $(M \times \mathbb{R}) \natural \mathcal{R}_k$ must be

different.

## Uncountably many large $\mathbb{R}^4$'s

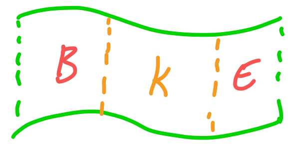
For this we need: an end $E$ of an open manifold $X$

is called periodic if $\exists$ a shift map $\phi : E \to E$

s.t. $\phi : E \to \phi(E)$ is a diffeomorphism

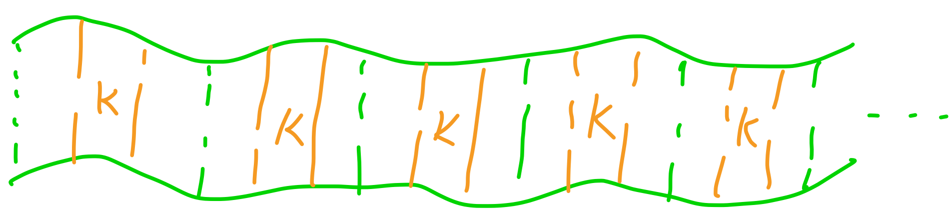
and $\phi^n(E)$ exits any compact set for some $n$

**example:** let $X$ = open 4-manifold with a compact set $K$ s.t. $X - K$ has 2 components $B$ and $E$ as shown




and $\phi : B \to E$ a diffeomorphism s.t. "$\infty$" in $B$ maps to "$\partial K$" in $E$

now let $X_\infty = \coprod X_i /_\sim$ where $X_1 = X$ and $B$ in $X_i$ glued to $E$ in $X_{i-1}$




clearly end periodic

**Thm (Taubes):**

$\boxed{\text{let } X \text{ be a smooth open simply connected 4-manifold with one end. If } X \text{ is end periodic and } I_X \text{ is definite then } I_X \text{ is } \oplus_n(1) \text{ or } \oplus_n(-1)}$

now let $R$ be the first example constructed above

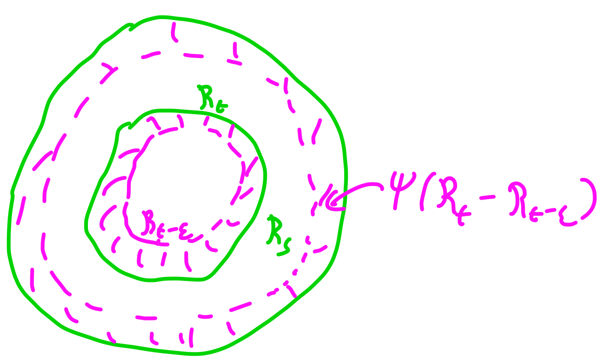
let $f: R \to [0,\infty)$ be a topological radial function

$\exists$ some $A$ s.t. $C' \subset f^{-1}([0,A))$

let $R_t = f^{-1}([0,t))$ for $t > A$

Claim: $R_t \neq R_s$ for $t \neq s$

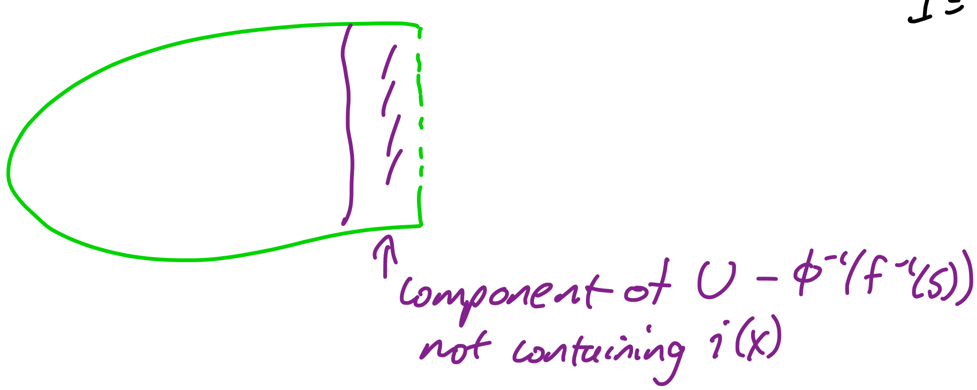
if not let $\psi$ be a diffeo $R_t \to R_s$ $t < s$




$\exists$ $\varepsilon > 0$ s.t. $\psi(R_t - R_{t-\varepsilon}) \subset R_s - R_t$

now consider the component of $K - \phi^{-1}(f^{-1}(s))$

$I = \Theta_2 E_8$




we can now glue $\infty$ copies of $(R_s - R_{t-\varepsilon})$ to this using $\phi^{-1} \circ \psi$ to get $\hat{X}$ an open mfd with periodic end and $I_{\hat{X}} = \oplus_2 E_8$

$\not\exists$ Taubs

## II) Topologically slice not smoothly slice knots
(more large exotic $\mathbb{R}^4$'s)

given a knot $K \subset S^3$

let $X(K) = B^4 \cup$ 2-handle attached to $K$ with framing $0$

i.e. glue $D^2 \times D^2$ to $B^4$ along $S^1 \times D^2$ by an embedding $\phi: S^1 \times D^2 \to S^3$ sending $\phi(S^1 \times \{0\})$ to $K$ and $\phi(S^1 \times \{p \neq 0\})$ to a copy of $K$ linking $0$ times w/ $K$

$X(K)$ is called the zero trace of $K$

**lemma (Trace Embedding Lemma):**

$K$ is smoothly (resp. topologically) slice in $B^4$

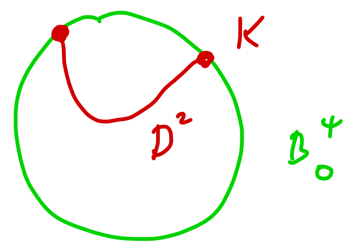
$\Leftrightarrow$

$X(K)$ smoothly (resp. topologically) embeds in $S^4$, or $\mathbb{R}^4$

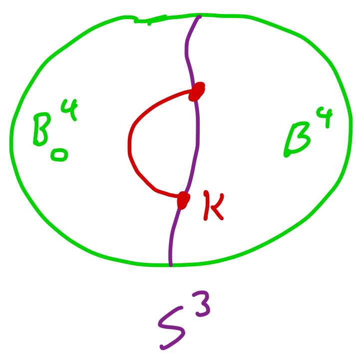
recall $K$ is slice in $B^4$ if $\exists$ an embedded disk $D^2 \subset B^4$ s.t. $\partial D^2 = K$

it is smooth/top slice if the embedding
is smooth/topological (locally flat).

Proof: $(\Rightarrow)$ $K$ slice means we have




glue $B^4$ to this $B^4$ to get



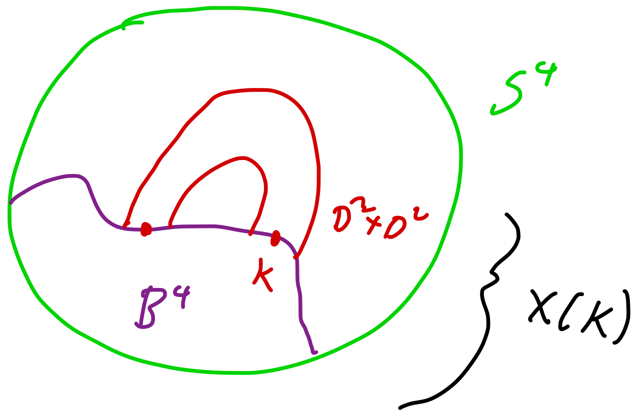

a nbhd of $D^2$ in $B^4_0$ is $D^2 \times D^2$ attached
to the other $B^4$ along $S^1 \times D^2$
i.e. $B^4 \cup D^2 \times D^2$ is result of
a 2-handle attachment to $K$

if framing not zero then
$D^2 \cup$ Seifert surface for $K$
would give a non-trivial

homology class in $H_2(S^4) = 0$
(since self-intersection $\neq 0$)

$\therefore$ we have $X(K)$ embedded in $S^4$ ✓

($\Leftarrow$) if $X(K)$ embeds we see




let $B^4_0 = \overline{S^4 - B^4}$ in $X(K)$

so $B^4_0$ is a 4-ball

and $D^2 \times \{0\}$ in 2-handle

gives slice disk for $K$ in $B^4_0$ ▦

Fact: There are topologically slice knots that are not smoothly slice

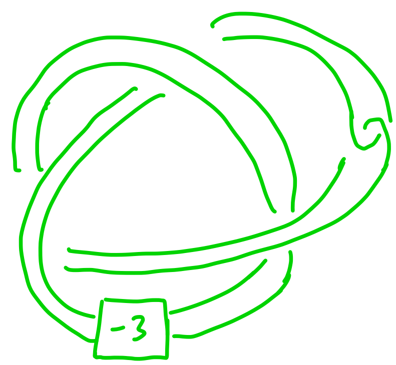
to see this need

Freedman: If $K \subset S^3$ has Alexander polynomial 1, then $K$ is topologically slice

Gompf using Donaldson: $\exists$ Alex poly 1 knots that are not smoothly slice

more resently one can use Khovanov homology to construct examples (this is great! since it does not use any "hard analysis" like all previous methods used)

e.g.




Given a top slice, but not smoothly slice $K \subset S^3$ we can construct a large exotic $\mathbb{R}^4$

Since $K$ is top slice, lemma above says $\exists$ a topological embedding
$$\phi : X(K) \to \mathbb{R}^4$$

let $C = \mathbb{R}^4 - \phi(\mathrm{int}\, X(K))$

Quinn proved that any open 4-manifold has a smooth structure so we can put a smooth str on $C$

$\partial C = -\partial X(K)$ and these are smooth 3-manifolds so they are diffeomorphic

$$\psi : \partial C \to -\partial X(K)$$

let $\mathcal{R} = X(K) \cup_\psi C$

by Freedman's work discussed above we know $\mathcal{R}$ is homeomorphic to $\mathbb{R}^4$

but $X(K)$ smoothly embeds in $\mathcal{R}$ so $\mathcal{R}$ can't be $\mathbb{R}^4$ or $K$ would be smoothly slice by Trace Embedding Lemma

Question:

Can you construct more than one exotic $\mathbb{R}^4$ using such knots?

almost certainly yes, but how do you distinguish them?

# III) Constructing small exotic $\mathbb{R}^4$ using the failure of the smooth 5D h-cobordism theorem

an h-cobordism from $M_0^n$ to $M_1^n$ is a compact $(n+1)$-manifold $W$ such that

lower $\partial$ upper $\partial$

$$\partial W = -M_0 \cup M_1$$

and the inclusions $i_j : M_j \to W$ are homotopy equiv.

**Fact:** if $M_0$ and $M_1$ are homotopy equivalent then they are h-cobordant (Novikov\Wall)

**Facts about h-cobordisms & handlebodies:**

"recall" an n-dimensional k-handle is

$$h^k = D^k \times D^{n-k}$$
$$\partial_- h^k = (\partial D^k) \times D^{n-k} = S^{k-1} \times D^{n-k}$$

$h^k$ is attached to the $\partial$ of an n-manifold $X$ by an embedding $\phi : \partial_- h^k \to \partial X$

so attaching $h^k$ to $X$ is

$$X \cup_\phi h^k = X \amalg h^k / x \in \partial_- h^k \sim \phi(x) \in \partial X$$

**example:**

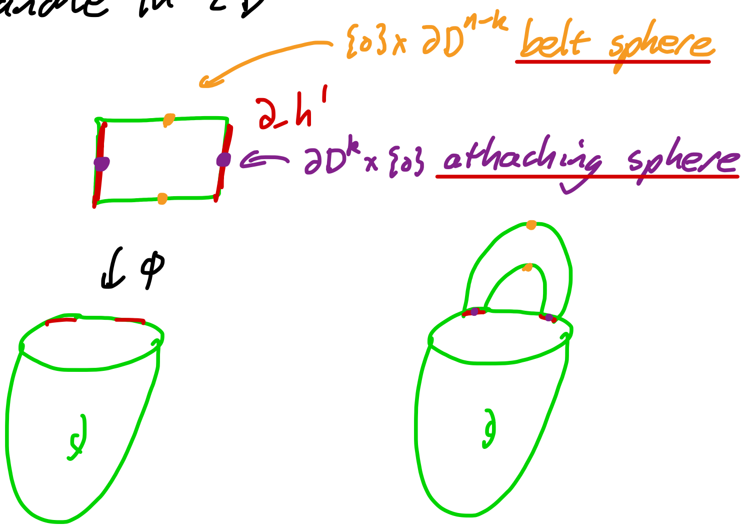
a 0-handle is just $D^n$ attached along $\varnothing$

so attaching 0-handle is just

disjoint union with $D^n$ 

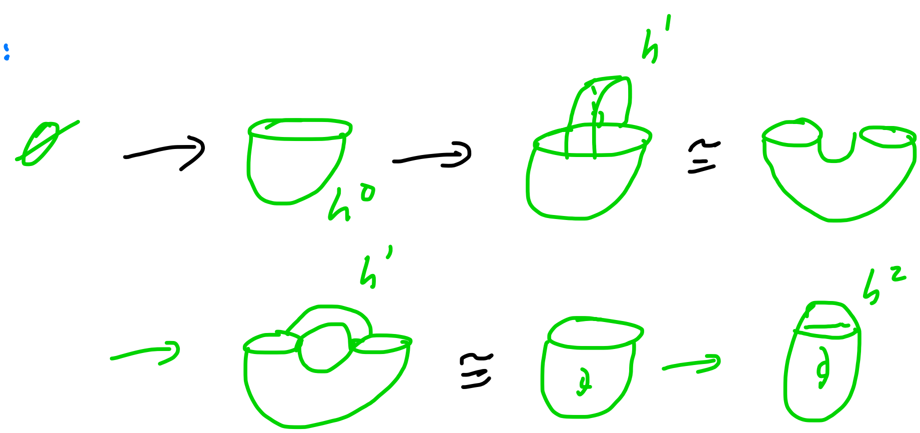
a 1-handle in 2D

$h^1 = D^1 \times D^1$




a handlebody is a manifold $X^n$ built from $\emptyset$ or $M^{n-1} \times [0,1]$ by a sequence of handle attachments

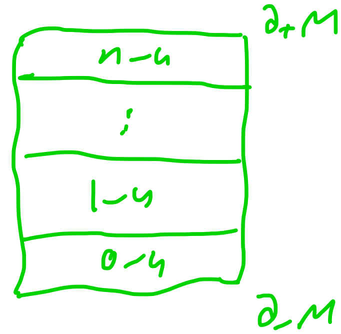
example:




Facts about handlebodies:

1) any compact smooth manifold, or cobordism, has structure of a handlebody

2) handles can be attached with increasing index




this is just $\pi$ for belt and attaching spheres

3) if $h^k$ and $h^{k+1}$ attached to $\partial M$ so that attaching sphere $h^{k+1} \cap$ belt sphere $h^k$ exactly once (and transversely) then

$$M \cup h^k \cup h^{k+1} \cong M$$


4) if $M^n$ connected and $\partial_- \neq \emptyset$ then can assume no 0-handles

($\partial_+ \neq 0$ then no $n$-handles)

(just cancel as above)

5) if $X$ is a cobordism, $\pi_1(X) = 1$, and $n \geq 5$, then can assume there are no 1 or $(n-1)$-handles (and no 0 or $n$-handles, by 4))

Now suppose $M$ and $M'$ are homeomorphic non-diffeo. 4-manifolds (such examples exist due to

Freedman and Donaldson)

From above $\exists$ a cobordism $X$ with
$$\partial X = -M \cup M'$$

and we can assume there are no 0, 1, 4, and 5 handles

so $X = M \times [0,1] \cup$ 2-h's $\cup$ 3-h's

the CW-chain complex $C_k(X, M)$ = generated by k-handles

and $\partial_k h^k = \sum_i \langle h^k, h_i^{k-1} \rangle h_i^{k-1}$

where $\langle h^k, h_i^{k-1} \rangle$ = algebraic intersection of attaching sphere of $h^k$ and belt sphere of $h_i^{k-1}$

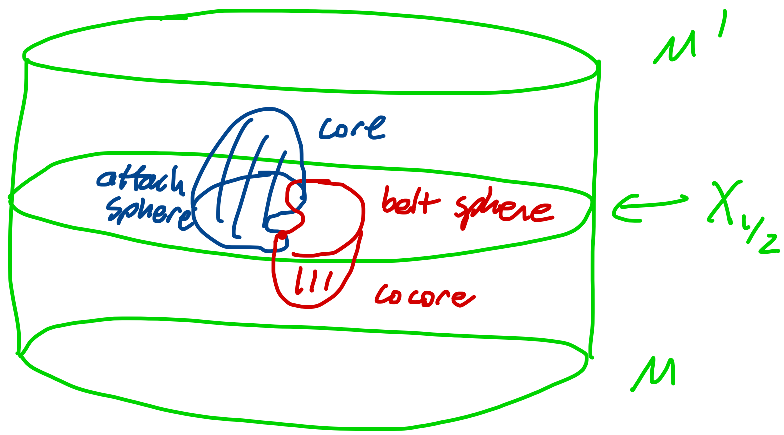
since $H_2(X, M) = H_3(X, M) = 0$ (since $M \hookrightarrow X$ a homotopy equiv)

we know $\partial_3$ is an isomorphism (in particular # 2-h = # 3-h)

after "sliding handles" we can assume
$$(\text{attaching sphere } h_i^3) \cdot (\text{belt sphere } h_j^2) = \delta_{ij}$$

if the geometric $\cap = \delta_{ij}$ then we could cancel all handles and $X = M \times [0,1]$
so $M' = M \times \{1\}$ $\not\!\!\!\!\!\bigcirc$ choice of $M, M'$

From now on assume only <u>one</u> 2 and 3-handles
(argument same if more, and $\exists$ examples like this)

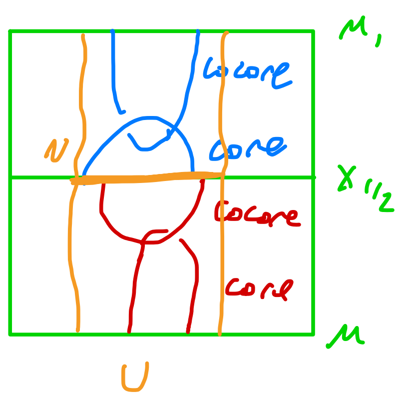



we can find Casson handles in $X_{1/2}$ to cancel extra intersections between $A$ = attaching sphere of $h^3$ and $B$ = belt sphere of $h^2$

and arrange that $N$ = open nbhd of $A \cup B \cup$ Casson handles

is homeomorphic (by Freedman) to $S^2 \times S^2$ − Ball

let $U$ = everything above and below $N$



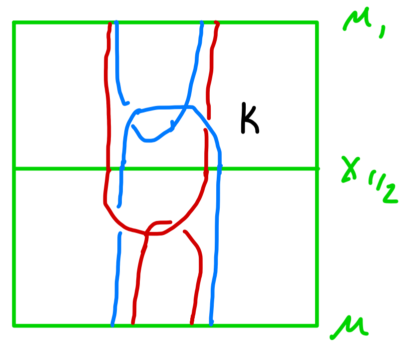

set $R_- = M \cap U$ and $R_+ = M' \cap U$

note: $R_-$ is obtained from $N$ by "surgering $B$"

so is topologically $\mathbb{R}^4$

similarly for $R_+$

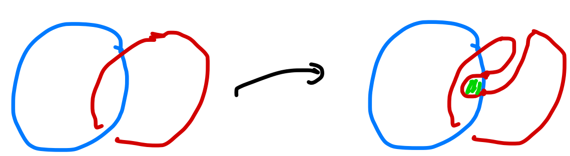
let $K$ = union of all cores and cocores of handles together with points above and below $A \cup B$




$U - K$ is a trivial cobordism from $R_- - \overbrace{(K \cap R_-)}^{K_-}$ to $R_+ - \underbrace{(K \cap R_+)}_{K_+}$

<u>Claim:</u> $R_\pm \subset \mathbb{R}^4$

indeed, we can build $S^4 \times [0,1]$ from $S^4 \times [0,\varepsilon]$ by attaching a cancelling pair of 2 and 3-handles now add double points to attaching & belt spheres



now since we added the double pts $\exists$ embedded disks to cancel them and a nbhd of these disks are 2-handles

recall any Casson handle embeds in a real

2-handle

so we can find $N$ in $(S^4\times[0,1])_{1/2}$ and $U$ in $S^4\times[0,1]$

now $U\cap(S^4\times\{0\})$ is $R_-$ an is clearly contained in $\mathbb{R}^4 = S^4 - pt$

same for $R_+$ ✓

Claim: $R_\pm$ not diffeomorphic to $\mathbb{R}^4$

If $R_-$ is diffeomorphic to $\mathbb{R}^4$ then $\exists$ a 4-ball $D_-$ in $R_-$ s.t. $K_-\subset D_-$

since $U-K$ is a product, above $\partial D_-$ is an $S^3$ in $R_+$ bounding some comact set $D_+$ in $R_+$ (and $K_+\subset D_+$)

so we see

diffeomorphic

$$M-\mathrm{int}(D_-)\cong M'-\mathrm{int}(D_+)$$

and

$$S^4-\mathrm{int}(D_-)\cong S^4-\mathrm{int}(D_+)$$

now $D^4 = S^4-\mathrm{int}(D_-)\cong S^4-\mathrm{int}(D_+)$

recall $\cong D^4$

so $D_+$ also a 4-ball

$\therefore$ $M=(M-\mathrm{int}(D_-))\cup$ 4-handle

$M'=(M'-\mathrm{int}(D_+))\cup$ 4-handle

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So $M - \text{int}(D) \cong M' - \text{int}(D_t) \Rightarrow M \cong M'$ $\emptyset$