Exotic Structures on Open 4-Manifolds

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If \( n \neq 4 \), then \( \mathbb{R}^n \) has one smooth structure (this means if \( R \) is a smooth \( n \)-manifold and \( R \) is homeomorphic to \( \mathbb{R}^n \) then \( R \) is diffeomorphic to \( \mathbb{R}^n \))

in dimension 4 we have

**Thm 1:**

There exist a 2-parameter family

\[
\{ \mathcal{R}_{s,t} \mid s,t \in (0,1) \}
\]

such that \( \mathcal{R}_{s,t} \) is homeo to \( \mathbb{R}^4 \) but there is no embedding of \( \mathcal{R}_{s,t} \) to \( \mathcal{R}_{s,t'} \)

if \( s > s' \) or \( t > t' \)

(\text{i.e. if } \mathcal{R}_{s,t} \text{ diffeo to } \mathcal{R}_{s',t'} \text{ then } s = s' \text{ and } t = t' \)

but if \( s \leq s' \) and \( t \leq t' \) then \( \mathcal{R}_{s,t} \rightarrow \mathcal{R}_{s',t'} \)

Moreover each \( \mathcal{R}_{s,t} \) contains a compact set that does not embed in \( \mathbb{R}^4 \) so they are not diffeo to \( \mathbb{R}^4 \)

**Remarks:**

1) the existence of one such exotic \( \mathbb{R}^4 \) follows from an argument of Casson
using work of Donaldson and Freedman

2) 3 such examples were found by Gompf in
   "Three exotic \( \mathbb{R}^4 \)'s, and other anomalies"
   and a countable family was found by Gompf in
   "An infinite set of exotic \( \mathbb{R}^4 \)'s"

4) An uncountable family \( \{ \mathcal{R}_t : t \in (0,1) \} \) was found by Taubes in
   "Gauge theory on asymptotically periodic manifolds"

   in the second paper of Gompf above he gave
   the family in Thm 1 using Taubes work

Thm 1 can be refined, we say \( \mathcal{R} \leq \mathcal{R}' \) if any

compact, smooth, codimension zero
submanifold of \( \mathcal{R} \) embeds in \( \mathcal{R}' \)

we say \( \mathcal{R} \) and \( \mathcal{R}' \) are compactly equivalent if

\( \mathcal{R} \leq \mathcal{R}' \) and \( \mathcal{R}' \leq \mathcal{R} \), denote \( \mathcal{R} \sim \mathcal{R}' \)

it is easy to see \( \leq \) is a partial order on equivalence classes of \( n \)-manifolds

**Exercise:** assume \( \mathcal{R} \) and \( \mathcal{R}' \) are connected

1) if \( \mathcal{R} \sim \mathcal{R}' \), then they are both closed or both non-closed
2) if \( R, R' \) are closed and \( R \sim R' \), then \( R \) is diffeomorphic to \( R' \)

**Theorem 1':**

There exist a 2-parameter family

\[
\{ R_{s,t} \mid s, t \in (0, 1) \}
\]

such that \( R_{s,t} \) is homeo to \( R^4 \) but

\[
R_{s,t} \leq R_{s', t'} \iff s \leq s' \text{ and } t \leq t'
\]

**Remark:** \( \text{Th } 1 \) and \( 1' \) are proven using

A) Freedman's proof that "Casson handles are topological 2-handles" i.e. classifying simply connected 4-manifolds

B) Donaldson's Diagonalization Theorem

C) for the uncountable family, Taubes generalization of B) to "end periodic" manifolds

we will cover these later.

**Remark:** getting a single exotic \( R^4 \) can be done by

uses A) and

B) or

Khovanov homology

finding a topologically (locally flat) slice knot in \( S^3 \) that is not smoothly slice

This can be done with no analysis as we will see later.
there exist a family
\[ \{ R_t : t \in (0,1) \} \]
such that \( R_t \) is homeomorphic to \( \mathbb{R}^4 \) and
1) all \( R_t \) are subsets of \( \mathbb{R}^4 \)
2) \( R_t \rightarrow R_{t'} \) if \( t \leq t' \)
3) uncountably many of the \( R_t \) are not
diffeomorphic
note all \( R_t \sim \mathbb{R}^4 \)

Remark:
1) The first such \( R \) was constructed by Freedman
in unpublished work based on ideas of Casson
2) Thm 2 was proved by DeMichelis and Freedman
   "Uncountably many exotic \( \mathbb{R}^4 \)'s in standard 4-space"

there is a family \( \{ R_t : t \in (0,1) \} \) such that \( R_t \) is
homeomorphic to \( \mathbb{R}^4 \) and \( R_t \sim R_{t'} \iff t = t' \)
and for each \( t \) \exists an uncountable family
\( \{ R_{t,s} \} \) such that \( R_{t,s} \) is homeomorphic
to \( \mathbb{R}^n \), all \( R_{t,s} \) are compactly equivalent and
Remark: This is due to Gompf is

"An Exotic Menagerie"

Remark: The proof of Th°2 is based on A) above and
D) \( \exists \) h-cobordant 4-manifolds that are not diffeomorphic

we call an exotic \( R^4 \) large if it contains compact codimension 0 sets that don't embed in \( R^4 \) and we call it small if it is compactly equivalent to \( R^4 \)

Open Question (?): if \( R \sim R^4 \), does \( R \) embed in \( R^4 \)

Th°3: \( \exists \) an exotic \( R^4 \), \( R_u \), such that any exotic \( R^4 \), \( R \) embeds in \( R_u \)

Remark: This is due to Freedman and Taylor

"A universal smoothing of four-space"
Open questions:

1) Does every compact equivalence class of exotic $\mathbb{R}^4$'s have uncountably many representatives?
2) Is $R_u$ the unique representative in its compact equivalence class?
3) Given a compact equivalence class $C$ of $\mathbb{R}^4$'s does $\exists R_e \in C$ s.t. any $R \in C$ embeds in $R_e$?

How can we organize exotic $\mathbb{R}^4$'s?

Algebraic Structure

Let $\mathcal{R}$ be the set of all exotic $\mathbb{R}^4$'s and $\mathcal{R}_n$ be the compact equivalence classes of exotic $\mathbb{R}^4$'s. So far, we know these are both uncountable sets. We can define a binary operation called end sum.

Given $R_1, R_2 \in \mathcal{R}$, choose proper embeddings $\gamma_i : [0, \infty) \to R_i$.

We can take neighborhoods of $\partial([0, \infty))$ $N_i : (0, \infty) \times D^3 \to R_i$. 
now \( \partial(R_2 - \text{int } N_2) = \mathbb{R}^3 \)

choose an orientation reversing diffeomorphism

\[ \phi: \partial(R_1 - \text{int } N_1) \rightarrow \partial(R_2 - \text{int } N_2) \]

and define the end sum to be

\[ R_1 \uplus R_2 = (R_1 - \text{int } N_1) \cup (R_2 - \text{int } N_2) / \sim \]

where \( x \in \partial(R_1 - \text{int } N_1) \) is glued to \( \phi(x) \in (R_2 - \text{int } N_2) \)
*note:* any 2 choices for $\phi$ are isotopic so $\eta$ is well-defined if any choices for $V_i$ are isotopic

**exercise:** show two proper embeddings of $[0,\infty)$ into an exotic $\mathbb{R}^4$ are isotopic

**Hint:** first isotop so agree at the integers

![Diagram]

now have lots of loops each bounds disk with a finite number of intersection points use "finger moves" to remove double pts ... 

we now have a map

$$\eta: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

and
\( \forall : \mathcal{R} \times \mathcal{R} \to \mathcal{R} \)

**Lemma 4:**

\( \mathcal{R} \) and \( \mathcal{R}_\sim \) are commutative monoids under \( \forall \), i.e. \( \forall \) is associative and has an identity, no non-trivial element has an inverse.

**Proof:**

Clearly \( \mathcal{R}^4 \) is the identity and \( \forall \) is associative and commutative.

Suppose \( \mathcal{R}_1 \forall \mathcal{R}_2 = \mathcal{R}^4 \)

Then \( \mathcal{R}_1 = \mathcal{R}_1 \forall (\mathcal{R}_2 \forall \mathcal{R}_1) \).

\[
\mathcal{R}_1 = \mathcal{R}_1 \forall (\mathcal{R}_2 \forall \mathcal{R}_1) \forall (\mathcal{R}_2 \forall \mathcal{R}_1) \forall \ldots \\
= (\mathcal{R}_1 \forall \mathcal{R}_2) \forall (\mathcal{R}_1 \forall \mathcal{R}_2) \forall \ldots \\
= \bigwedge_{i=1}^{\infty} \mathcal{R}^4 = \mathcal{R}^4
\]

**Thm 3':**

\( \mathcal{R}_u \) from Thm 3 satisfies \( \forall \mathcal{R}_u = \mathcal{R}_u \) for all \( \mathcal{R} \in \mathcal{R} \).

Clearly Thm 3' \( \Rightarrow \) Thm 3.
Open Question:

Is there a group we can associate to \( \mathbb{R} \) or \( \mathbb{R}_+ \) and \( 92 \)? Some other operation?

Note: Knots in \( S^3 \) under \# is a monoid without inverses, but knots up to cobordism is a group under \#

So question above is asking for something like this for \( 92 \)

Recall: one way to get a group from a commutative monoid is to form its Grothendieck group where you “add inverses” this is how you get \( \mathbb{Z} \) from \( \mathbb{N} \cup \{0\} \)

Specifically if \((M,+)\) is a commutative monoid, then its Grothendieck group is constructed as follows:

- start with \( M \times M \)
  - think of \((m,n)\) as “\( m-n \)"
- define the equivalence relation
  \((m_1,n_1) \sim (m_2,n_2) \) if \( \exists k \in M \)
\[ \text{s.t. \ } m_1 + n_2 + k = m_2 + n_1 + k \]
\[ \text{set } K = M \times M / \sim \]

**note:** if \((m_1 \neq m_2 \Rightarrow m_1 + n \neq m_2 + n)\) then don't need \(\sim\)

* define \((m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)\)

\((K, +)\) is the **Grothendieck group** of \((M, +)\)

**Exercise:** show the Grothendieck group of \(\mathbb{N} \cup \{0\}\) is \(\mathbb{Z}\)

**Lemma 5:**

The Grothendieck group of \((\mathbb{R}, \times)\) and \((\mathbb{R}_n, \times)\) is trivial

**Proof:**

\[ R = R^+ \cdot R = (R^+ \cdot R) \cdot R = R^+ \cdot (R \cdot R) \]
\[ = R^+ \cdot R = R \]

**Partial Order:**

**Lemma 6:**

if \(R_1 \leq R_2\) and \(R_3 \leq R_4\), then

\[ R_1 \cdot R_3 \leq R_2 \cdot R_4 \]

and \(\leq\) is a partial order on \(R_n\)
**Exercise:** Convince yourself of this. What more can you say about \( \leq \) on \( R \) and \( R_\sim \)?

**Note:** minimal elements:
- \( R^4 \leq R \quad \forall R \in R \)
  so \( R^4 \) is minimal elt in both \( R \) and \( R_\sim \)
- if \( R_m \) is another minimal elt in \( R_\sim \) then \( R_m \leq R^4 \) and \( R^4 \leq R_m \)
  \[ R^4 \sim R_m \]
  i.e. \( [R^4] \) is unique minimal elt in \( R_\sim \)
- if \( R \) such that \( R \in R^4 \) then \( R \) is also a minimal element in \( R \)

**Question:** if \( R \in R \) is a minimal element does \( R \) embed in \( IR^4 \) ?

**Maximal elements:**
- \( R \leq R_u \quad \forall R \in R \)
  so \( R_u \) is maximal element for both \( R \) and \( R_\sim \).
as above \([R_u]\) is the unique maximal element in \(\mathcal{R}\)

**Question:**

if \(R \in \mathcal{R}\) is a maximal element does \(R_u\) embed in \(\mathcal{R}\)?
or is \(R_u = \mathcal{R}\)?

gaps:

**Question:**

- if \(R \leq \mathcal{R}'\) is there always some \(\mathcal{R}''\) such that \(R \leq \mathcal{R}'' \leq \mathcal{R}'\)?
- infinitely many such \(\mathcal{R}''\)?

comparability:

**Question:**

- given \(R\) are there \(\mathcal{R}'\) that are not comparable to \(\mathcal{R}\)?
  - uncountably many?
- given a family \(\{R_a\}\) are there (uncountably many) \(\mathcal{R}'\) that are not comparable to any \(R_x\)?
**Question:**

- What else can you say about $\leq$ on $\mathbb{R}$ or $\mathbb{R}^\infty$?
- Is there another order on $\mathbb{R}$ or $\mathbb{R}^\infty$?
- Is there an order on a compact equivalence class?

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**Topology**

We can put a topology on $\mathbb{R}^\infty$ using $\leq$ for all $R \in \mathbb{R}^\infty$ let

$$K_R = \{ R' \in \mathbb{R}^\infty : R' \leq R \}$$

$$L_R = \{ R' \in \mathbb{R}^\infty : R \leq R' \}$$

These form a closed subbasis.

Let $B = \{ \text{all finite unions of } K_R \text{ and } L_R \text{'s} \}$

and $T_\infty = \{ \text{all infinite intersections of elts of } B \}$

so open sets in $T_\infty$ are complements of elts of $T_\infty$.

Not much is known about this topology, but Gompf in

"A moduli of exotic $R^4$'s" gave a refinement of $T_\infty$ for any compact oriented 4-manifold $X$ let

$$U_X = \{ R \in \mathbb{R}^\infty : X \text{ embeds in } R^3 \}$$

Now let $T$ be the topology with closed...
subbasis all $\mathbb{R}_n - U$ and $L_R$

Gompf proved:

1) $\mathcal{I}$ is regular
2) $\mathcal{I}$ is 2nd countable [follows from 1), 2) by Urysohn Metrization]
3) $\mathcal{I}$ is metrizable
4) every increasing sequence converges

Open Problems:

- is $\mathcal{I} = \mathcal{I}_S$?
- what more can be said about $\mathcal{I}$ or $\mathcal{I}_S$?
- is there a "better" topology on $\mathbb{R}_n$?
- is there a "good" topology on $\mathcal{I}$?

Symmetries:

note if $\gamma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is rotation by $\pi$
then $\mathbb{R}^2/_{x \sim 2\pi(x)} \cong \mathbb{R}^2$ and $\mathbb{R}^2 \rightarrow \mathbb{R}^2/\gamma$
this is the standard 2-fold cover of $\mathbb{R}^2$
branched over a point

so $\gamma' = \gamma \times \text{id}: (\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^4$ $(x, y) \mapsto (2\pi(x), y)$
satisfies $\mathbb{R}^4/_{x \sim 2\pi(x)} \cong \mathbb{R}^4$ and $\mathbb{R}^4 \rightarrow \mathbb{R}^4/\gamma'$
this is the standard 2-fold cover of $\mathbb{R}^4$
branched over a plane
Theorem 7:

there exists \( R^4 \)'s \( R_1, R_2 \) and a smooth involution \( \sigma : R_1 \rightarrow R_1 \) that is

1) topologically standard
2) \( R_1 \rightarrow R_1/\sigma \cong R_2 \)

and we can independently choose \( R_1 \) and \( R_2 \) to be large or small exotic \( R^4 \)'s as long as \( R_1 \), is not \( R^4 \) (we can take \( R_2 \) to be standard)

Open Question:

**can you find such an \( R_1 \rightarrow R_2 \) with \( R_1 \cong R^4 \) and \( R_2 \) not?**

Remark:

1) \( R_1 \) small

- is due to Freedman in unpublished work

2) rest of theorem due to Gompf in "An exotic menagerie"
more group actions (from Gompf)

let $G$ be I) a group that acts properly discontinuously on $\mathbb{R}^3$

i.e. $\forall x \in \mathbb{R}^3 \exists$ a nbhd $U$ of $x$ st. $g(U) \cap U = \emptyset$ if $g \neq 1$

clearly $G$ acts on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$

or II) a finite group acting on $\mathbb{R}^4$

let $a = \{(0,0,0)\} \times [0,\infty)$ and

$A = \bigcup_{g \in G} g(a)$

let $R$ be an exotic $\mathbb{R}^4$ (large or small)

set $R' = \mathbb{R}^4 \sqcup_{A} (1124 \{1\} \mathbb{R})$

(i.e. end sum one $R$ into $\mathbb{R}^4$ along each $g(a)$)

clearly $G$ acts on $R'$ and is topologically equivalent to the $G$ action on $\mathbb{R}^4$

if $\mathbb{R}^4 / G = \mathbb{R}^4$, then

$R' / G \cong \mathbb{R}$
Note: \( R', R \) are both either large or small and \( R' \geq R \)

**Open Question:**

Can an exotic \( \mathbb{R}^4 \) cover a compact 4-manifold?

Now, for any \( \mathbb{R}^4, R \) set

\[
D(R) = \{ \text{diffeomorphisms of } \mathbb{R}^3/\text{isotopy} \}
\]

and \( D_+(R) \subset D(R) \) the subgroup of orientation preserving isotopy classes

Note: \( D_+(\mathbb{R}^n) = \{ \text{id} \} \) \( \forall n \)

**Thm 8:**

There are uncountably many \( R \) (both large and small) for which there is an uncountable subgroup of \( D_+(R) \)

**Remark:** This is due to Gompf in "Group actions, corks and exotic smoothings of \( \mathbb{R}^4 \)"

When studying manifold with boundary we are interested in how diffeos of the boundary interact with diffeos of the mfd
for non-compact manifolds we introduce "diffeomorphisms at infinity"

a closed neighborhood of infinity (cni) in a manifold M is a codimension one submanifold $E \subset M$ that is closed and $M - E$ is compact
given $cni$ $E_1 \subset M_1$ and $E_2 \subset M_2$, $i = 1, 2$

and diffeomorphisms $f_i : E_i \to E_i$

we say $f_1 \sim f_2$ if there is a $cni$ $E \subset M$ st. $E_i \subset E$ and $f_i|_E = f_2|_E$

$f_1, f_2$ have same "germ at $\infty$"

a diffeomorphism at infinity from $M$ to $M'$ is an equivalence class of such diffeos

when $M, M'$ have a single end we also say this is a diffeomorphism of the ends

let $D^\infty(M) = \{\text{isotopy classes of diffeos at } \infty\}$

and $D^\infty_+(M)$ the orient pres. subgroup

there is an obvious restriction map

$r : D^\infty(M) \to D^\infty_+(M)$
Lemma 9: for any $R \in \mathbb{R}$, ker $r$ and coker $r$ are countable.

Theorem 9: there are uncountably many $R^+$'s $R$ (large and small) such that:
1) $r(D(R)) \subseteq D^\infty(R)$ is uncountable
2) There exist nonfinitely generated groups in coker $r = D^\infty(R)/r(D(R))$
3) for the universal Ru

Coker $r = \{1\}$

Open Question: What can you say about ker $r$? Is it always trivial?

Open Question: Can $S^1$ or $R$ act non-trivially on an exotic $\mathbb{R}^4$?
Geometry:

There is not much known about Riemannian metrics on exotic $R^4$'s. Here are a few: suppose $R \in \mathbb{R}$ but $\neq R^*$

1) Can't have a constant curvature metric (since this $\Rightarrow R^4$ or $S^4$)

2) Can't have curvature $\leq 0$ (since $\Rightarrow R^4$)

3) Can have a complete metric with negative Ricci curvature and one with sectional curvature any constant negative number (Lohkamp)

Questions:

Can you construct invariants of exotic $R^4$'s using Riemannian metrics?

Eg. for $g$ on $R$ let

$$G_g = \max \text{ curvature} - \min \text{ curvature}$$

$$G_R = \inf_g G_g$$

Can this distinguish exotic $R^4$'s? Can this be 0?
**Theorem 10:** There exist $\mathbb{R}^4$'s that admit metrics whose isometry group contains an uncountable subgroup.

**Remark:** Due to Gompf in last mentioned paper an exotic $\mathbb{R}^4$, $\mathbb{R}$, is called **full** if there exists a compact subset that cannot be embedded in its complement or into any homology $S^4$ (not hard to construct these).

**Theorem 11:**

1) any metric on a full $\mathbb{R}^4$ has finite isometry group

2) there exist $\mathbb{R}$, full exotic $\mathbb{R}^4$'s, with $\text{D}^1(\mathbb{R})$ and $\text{D}^0(\mathbb{R})$ uncountable

**Open Question:** Does every isometry group of an exotic $\mathbb{R}^4$ inject into its diffeotopy group?

**Note:** not true for $\mathbb{R}^4$!
Remark: 1) in Thms due (mostly) to Taylor in "Smooth Euclidian 4-spaces with few isometries"
2) is due to Gompf in above paper

Let's move to symplectic geometry.

Recall a symplectic structure on a 4-manifold $M$ is a 2-form $\omega$ st.

(closed) $d\omega = 0$ and

(non-degenerate) $\omega \wedge \omega$ is never zero (i.e. volume form)

A complex manifold $X$ is **Stein** if

\[ \exists \text{ an exhausting pluri-subharmonic function } \phi : X \to \mathbb{R} \]

i.e. $\phi^{-1}(-\infty, c]$ is compact $\forall c$ and

\[ [d(d\phi \circ J)](v, Jv) > 0 \ \forall v \neq 0 \text{ in } TX \]

This $\Rightarrow d(d\phi \circ J)$ is a symplectic form.

Here $J : TX \to TX$ is the action of multiplication by $i$ on $TX$. 
There are uncountably many small exotic \( IR^4 \)'s that admit Stein structures.

There are uncountably many large \( IR^4 \)'s that embed into Stein surfaces.

There are exotic \( IR^4 \)'s that do not embed into Stein surfaces.

**Remark:** 1\(^{st} \) result is due to Gompf in “Handlebody constructions of Stein surfaces”
other results due to Bennett in “Exotic smoothings via large \( IR^4 \)'s in Stein surfaces”

**Questions:**

- Do all small \( IR^4 \)'s admit Stein structures?
  (Symplectic with convex “boundary”)

- Does any large \( IR^4 \) admit a Stein structure?

**Invariants:**

Given \( R \) homeomorphic to \( IR^4 \), for any compact subset \( C \subseteq R \) there is a compact 3-manifold \( M \) separating \( C \) from \( \infty \).
(just take any smooth proper $f: \mathbb{R} \to [0, \infty)$
then there is some regular value $c$
so $C \subset f^{-1}([0, C])$ so $M = f^{-1}(c)$ works)

let $b_c = \min \{ b_1(M) \}$ over all such $M$

*note: if $C \subset C' \subset \mathbb{R}$ then $b_c \leq b_{c'}$

Bičakča and Gompf defined the engulfing index of $\mathbb{R}$
to be
\[ e(\mathbb{R}) = \sup \{ b_c \mid C \subset \mathbb{R} \} \]
easy to see $e(4R_i) \leq \Sigma e(R_i)$
in many examples it is $\infty$

**Thm 13:**

\[ \exists \text{ exotic } \mathbb{R}^4 \text{'s with } e \text{ finite} \]

**Question:**

\[ \text{Which values of } e \text{ can be realize?} \]

(1 think $\infty$, many can)

Now given $\mathcal{R}$ consider the set
\[ \text{Sp}(\mathcal{R}) = \{ \text{closed spin } \mathcal{R} \text{-manifolds with} \]
\[ \text{intersection form } \Theta(\mathcal{R}) \]
\[ \text{into which } \mathcal{R} \text{ embeds} \}

if $\text{Sp}(\mathcal{R}) = \emptyset$, set $b_\infty = \infty$
otherwise \( b_e = \frac{1}{2} \min_{N \in \mathcal{S}(\mathcal{R})} \{ b_2(N) \} \)

for any smooth 4-manifold \( M \) let

\[ E(M) = \{ \text{topological embeddings } e : D^4 \to M \]
\[ \text{st } e(D^4) \text{ is bicollared and } \exists p \in D^4 \text{ s.t. } e(\partial D^4) \text{ is smooth} \}

for \( e \in E(M) \) let \( R_e = \text{smooth structure induced on } e(\text{interior of } D^4) \)

finally set

\[ \gamma(R) = \max \{ b_{R_e} \} \quad e \in E(M) \]

this is called the Taylor invariant

roughly \( \gamma(R) \) measures the minimal number \( n \)

\( \text{st } R \text{ embeds in } \#_n S^2 \times S^2 \)

all the complications in the def.\(^2\) are to prove things about \( \gamma \)

we can extend \( \gamma \) to any 4-manifold as follows

if \( M \) is spin, then

\[ \gamma(M) = \max \{ \gamma(E) \} \]

where \( E \in M \text{ open set homeo to } R^4 \)

if \( M \) is orientable but not spin and
1) The 2nd Steifel–Whitney class $w_2(M)$ has no compact dual then

$$\gamma(M) = -\infty$$

2) If there are compact duals then

$$\gamma(M) = \max \{ \gamma(M-F) - \dim H_i(F; \mathbb{Z}_2) \}$$

where $F$ runs over all compact duals to $w_2(M)$

If $M$ is nonorientable, then let $\tilde{M}$ be its orientation double cover, and set

$$\gamma(M) = \gamma(\tilde{M})$$

Properties of $\gamma$:

1) If $w_0 \leq \cdots \leq w_{n/2-1} \leq 0 \leq w_{n/2} \leq \cdots \leq w_n \leq M$ then $\gamma(M) \leq \max \{ \gamma(w_i) \}$ get = if $M$ spin

2) If $M_2$ spin and $M_1 \subseteq M_2$ then

$$\gamma(M_1) \leq \gamma(M_2)$$

3) $\gamma(S^2) = 0$ but $\exists$ exotic $\mathbb{R} \subseteq S^2$ st.

$$\gamma(R) \text{ arbitrarily large}$$

4) $\exists M$ with $\gamma(M) = -\infty$ and

for each $n \geq 0$, $\exists$ uncountably many exotic $\mathbb{R}^n$ with $\gamma = n$ (n could be $\infty$)

5) If $M$ is a Stein manifold then $\gamma(M) \leq b_2(M)$

so for a Stein exotic $\mathbb{R}^4$, $\gamma = 0$

($\therefore$ get lots of exotic $\mathbb{R}^4$ with no Stein str)
6) There are uncountably many exotic $\mathbb{R}^4$ that embed in Stein manifold but have arbitrarily large $\gamma$

7) If $R$ is an exotic $\mathbb{R}^4$ and $R$ nontrivially covers some other manifold then
   \[ \gamma(4^4E) = \gamma(E) \]
   \[ \exists \text{ examples of } R \text{ s.t. } \gamma(4^4E) = \frac{1}{3} \gamma(E) \]
   So these can't cover!

8) \( \gamma(R) \leq \varepsilon(R) \)

**Remark:** All results due to Taylor

"An Invariant of Smooth 4-Manifolds"

except 6) due to Bennett in above paper
and 8) due to Khuzam in

"A comparative study of two fundamental invariants of exotic $\mathbb{R}^4$'s"

**Question:**
Can $\gamma$ take on finite negative values?

**Question:**
- can one find a non discrete invariant of exotic $\mathbb{R}^4$?
- can one define an invariant of exotic $\mathbb{R}^4$'s?
Other Manifolds

**Theo 14:** let $X$ be one of the following

1) $W$-pt for any topological manifold
2) total space of an oriented $\mathbb{R}^2$ bundle over an oriented surface
3) $Y \times \mathbb{R}$ for a 3-manifold $Y$ that topologically locally flatly embeds in $\# n \overline{CP}^2$

then $X$ admits uncountably many smooth structures

**Remark:** 1) is due to Gompf in

"An Exotic Menagerie"

but Furuta and Ohta proved many cases in

"A remark on uncountably many exotic differential structures on one-point punctured topological 4-manifolds"

2) is due to Ding in
"Smooth structures on some open 4-manifolds"

3) is due to Fang in

"Embedding 3-manifolds and smooth structures of 4-manifolds"

also note any rational homology sphere or Seifert fibered space has such embeddings

Theorem 15:

If $Y$ is any compact 3-manifold then $Y \times \mathbb{R}$ admits infinitely many smooth structures.

If $X$ is any open 4-manifold with at least one end that is topologically $Y \times \mathbb{R}$ and $X$ has only finitely many ends homeomorphic to $Y \times \mathbb{R}$, then $X$ admits infinitely many smooth structures.

Remark: This is due to Bičaka and Etnyre in

"Smooth structures on collarable ends of 4-manifolds"

Questions:

1) can this be upgraded to get uncountably many smooth structures?
2) Can the be upgraded so Y is any 3-manifold?

3) Can the be upgraded so X is any open 4-manifold?

3 constructions of exotic $\mathbb{R}^4$'s

I) Restrictions of the intersection form of 4-manifolds (failure of smooth surgery)

recall given an oriented closed 4-manifold $X$

$$H_4(X) = \mathbb{Z}$$

and a generator $[X]$ is called a fundamental class

Poincaré Duality says

$$H^2(X) \times H^2(X) \xrightarrow{I_X} \mathbb{Z}$$

$$(\alpha, \beta) \mapsto \alpha \cup \beta ([X])$$

is a (symmetric) non-degenerate pairing called the intersection form

using Poincaré duality we can reinterpret this in $H_2(X)$

recall if $\Sigma \subset X^4$ is an embedded oriented surface then

$$[\Sigma] \in H_2(X^4)$$

(actually $i^*: H_2(X^4) \to H_2(X)$ is an inclusion

$$i_*([\Sigma]) \in H_2(X^4)$$

Fact: for any $h \in H_2(X^4)$ $\exists$ some surface $\Sigma^2 \subset X$ s.t. $h = [\Sigma]$
now given \( h, h' \in H_2(X) \) let \( h = [\Sigma] \) and \( h' = [\Sigma'] \)

we can isotop \( \Sigma' \) so that \( \Sigma \) is transverse to \( \Sigma' \)
so \( \Sigma \cup \Sigma' = \{ p_1, \ldots, p_k \} \)

let \( \epsilon(p_i) = \) sign of intersection
\[ \{ \text{i.e. does or \( \circ \) on } T_{p_i} \Sigma \text{ followed by} \]
\[ \text{or \( \circ \) on } T_{p_i} \Sigma' \text{ agree or not} \]
\[ \text{with or \( \circ \) on } T_{p_i} X \} \]

define \( \Sigma \cdot \Sigma' = \sum_{i=1}^{k} \epsilon(p_i) \)

now \( I_X : H_2(X) \times H_2(X) \to \mathbb{Z} \)
\[ ([\Sigma], [\Sigma']) \mapsto \Sigma \cdot \Sigma' \]

is Poincaré dual to pairing above

**example:**

in dimension 2:

\[ H_2(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \]

\[ a \quad b \]

\[ a \cdot a = ? : \]
so \( a \cdot a = 0 \)

\[ a \cdot b = 1 \]

so \( I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)
similarly for $S^2 \times S^2$ we have $H_2(S^2 \times S^2) = \mathbb{Z} \oplus \mathbb{Z}$

$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

can also check $H_2(CP^2) = \mathbb{Z}$ and $I = [1]$

**Theorem [Donaldson]:**

If $X$ is an oriented, simply connected, smooth 4-manifold with $I_X$ negative definite (i.e. $I_X(e,e) < 0 \forall e$) then $I_X = \oplus [-1]$

**Remark:** there are lots of non-diagonalizable symmetric pairings, e.g.

$$
\begin{pmatrix}
-2 & 1 \\
1 & -2 \\
1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
0 & 1 & -2 \\
\end{pmatrix}
$$

is not diagonalizable over $\mathbb{Z}$

so it cannot be $I_X$ for $X$ as in theorem

**Idea of Proof:**

Given $X$ as in thm except $I_X$ is pos. definite (just reverses or $\mathbb{C}$)

there is a $SU(2)$-bundle $E$ over $X$ with Chern class $c_2(P) = 1$
let \( B = \{ \text{connections } D \text{ on } E \text{ with self-dual curvature } R_D = * R_D \} \)

\[ M = B/\mathbb{Q} \leq \text{gauge group (symmetries of solv's)} \]

Using work of many people, in particular Taubes and Uhlenbeck one can show:

- \( \overline{M} \) is a compact oriented 5-manifold with singular points.
  
  number = \( \frac{1}{2} \# \) solutions to \( I_X(h,h) = 1 \)
  
  Call this number \( m \)

- Singular points have nbhd that are cones on \( \mathbb{CP}^2 \)

- \( \partial \overline{M} = \overline{X} \)

so \( M' = \overline{M} - \text{nbhd of sing pts and arcs connecting them} \)

has \( \partial M' = X \cup - (\# \mathbb{CP}^2) \)

If positive definite so \( \text{rank}(I_M) = 0 \text{ (Im)} \)

but signature is cobordism invariant so

\[ \text{rank}(I_M) = 0 \text{ (Im)} = 0 \text{ (}\# \mathbb{CP}^2\text{)} = m \]

**note:** for any symmetric positive definite unimodular form \( I \), if \( m = \frac{1}{2} \# \) solutions \( I(h,h) = 1 \)

then \( m \leq \text{rank}(I) \) and equality \( \iff \) \( I \) diagonal.

Indeed, given solv \( h \) consider

\[ I = (\text{span } h) \oplus (\text{span } h)^\perp \]
if $k \neq h$ is another solution then

$$I(hk, h^2k) = 1 + 1 = 2I(h, k)$$

since $I$ pos. definite $I(h, k) = 0$

all other solns in $(\text{span } h)^\perp$

result follows by induction

Donaldson's $(4+1)$ now follows!

**Existence of 1 large exotic $\mathbb{R}^4$:**

let $K$ be "the" $K3$ surface

re. $K = \{ z_0^2 + z_1^2 + z_2^2 + z_3^2 \in \mathbb{C}P^3 \}$

one can compute $\pi_1(K) = 1$, $H_2(K) = \mathbb{Z}$, and

$$I_K = E_8 \oplus E_8 \oplus_3 H \quad \text{where } H = (0, 1)$$

since $\pi_1 = 1$ we know $\pi_2(K) = H_2(K)$

so $3$ immersions $e_i : S^2 \rightarrow K$ generating $\oplus_3 H$

after work of Casson we have

**Thm (Freedman):**

let $X$ be $3$ copies of $S^2 \times \{p\} \cup \{p\} \times S^2$ in $\#_3 S^2 \times S^2$

connected by arcs

$\exists$ a topological embedding $i : X \rightarrow K$ realizing $\oplus_3 H$

a top embedding $j : X \rightarrow \#_3 S^2 \times S^2$ top isotopic to $X$. 
a nbhd $U$ of $i(X)$, a nbhd $V$ of $j(X)$, and a diffeomorphism $\phi: U \to V$

\[
\begin{array}{c}
\text{let } R = (\#_3 S^2 \times S^2) - j(X) \\
\text{Claim: } R \text{ is an exotic } \mathbb{R}^4 \\
\text{1st: } R \text{ homeo. to } \mathbb{R}^4 \\
\text{for this we use} \\
\text{Th}^m(\text{Freedman}): \\
\text{any open manifold with } \tau_1 = 0 \text{ and } H_2 = 0 \\
\text{and one end topologically equivalent to } S^3 \times \mathbb{R} \text{ is homeomorphic to } \mathbb{R}^4 \\
\text{we can easily check these properties for } R \\
\text{2nd: } R \text{ not diffeo to } \mathbb{R}^4 \\
\text{assume } R \text{ diffeo to } \mathbb{R}^4
\end{array}
\]
let $C = \mathbb{S}^5 \times \mathbb{S}^2 - V$

this is a compact set

in $\mathbb{R}^4$ we know given any compact set $C$

exists a 3-sphere $S$ that separates $C$

now $\phi$ breaks $K$ into $K_1 \cup K_2$ with

$i'(x) \subset K_2$

set $K' = K_1 \cup B^4$ glued along $S^3$

$I_{K'} \cong \oplus_2 E_8$

contradicts Donaldson's Thm

so $X$ not diffeo to $\mathbb{R}^4$

now let's check $X$ is large

let $C'$ be a compact set in $X$ containing $C$
and homeomorphic to $\mathbb{R}^4$

If $C'$ smoothly embedded in $\mathbb{R}^4$ then it embeds in $S^4$

now we can glue $S^4 - C$ to $K - \phi^{-1}(C' - C)$

by $\phi$ on $(C' - C)$ to $\phi^{-1}(C' - C)$

giving a closed smooth mfd with $I = \oplus E_0$

There are many ways to get an infinite family, one uses

If $X$ is a smooth closed spin $4$-manifold

then by Donaldson $I_X$ is $
\oplus_k E_8 \oplus_0 H$

(Rochlin's theorem says $\sigma = 0 \mod 16$
so $k$ is even)

**Theorem (Furuta):**

must have $l \geq k + 1$

uses the Seiberg-Witten equations and restrictions on
equivariant maps between spheres

Countably infinite family of large $\mathbb{R}^4$'s

let $R_i = \mathbb{R}$ from above
\[ R_n = R_{n-1} \frac{47}{2} R \]

and \[ R_\infty = \frac{47}{2} R \]

**note:** \( R_n \) contains \( n \) copies of \( C \) and \( C' \)

let \( C_n = \) the boundary sum of all these

\[ C_n' = \ldots \]

we could alternately construct the \( R_n \) and \( C_n, C_n' \)

by doing the above construction to

\( \#_n K \) and \( \#_{3n} S^2 \times S^2 \)

**lemma:**

If \( X \) is any closed smooth spin 4-manifold, then

exists \( m > 0 \) st. for any \( n > m \), \( R_n \) and \( C_n \)

cannot be smoothly embedded in \( X \)

**Proof:**

as noted above \( I_X = \bigoplus_{2k} E_8 \bigoplus H \)

let \( m \) be any integer with \( 2m > l - 2k \)

if \( n \geq m \) then \( R_n \) cannot be embedded in \( X \)

to see this assume it does
So we can glue $X - C_n$ to a piece of $\mathbb{R}^n$ cut along $\partial C_n'$ to get $Z$ with $I_Z = \oplus 2n+2e G e \oplus \ell H$

but $\ell < 2k+2n \notin \text{Furuta!}$

now all $R_n$ are distinct since by construction $R_n$ embeds in $\#_{3n} S^2 \times S^2$

but by lemma not in $\#_{2n} S^2 \times S^2$

from this it is clear that infinitely many of the $R_n$ are different

but now assume $R_n = R_m$ for $n < m$

this implies for any $k > m$, $R_k = R_k$ for $n = l \leq m$

$\therefore R_n \neq R_m$ for $n \neq m$

**exercise:** $R_n$ can't embed in any neg. definite mfd or any spin mfd.

Infinitely many smooth structures on $M^3 \times \mathbb{R}$

let $M$ be a compact closed smooth
orientable 3-manifold

Fact: \( \exists n \text{ s.t. } M \text{ smoothly embeds in } \#_n S^2 \times S^2 \) 
(\( \therefore M \times \mathbb{R} \text{ does too} \))

Idea: \( M \) is obtained from \( S^3 \) by Dehn surgery on a link with even integer coeff

\[ \text{e.g. } \begin{array}{c}
\includegraphics[width=3cm]{diagram1.png} \\
\end{array} \]

so \( M = \partial X \) where \( X \) is 4-manifold obtained from \( B^4 \) by attaching 2-handles

\[ \partial(X) = \partial( X \times [-1,1] ) \text{ has handle diagram} \]

\[ \begin{array}{c}
\text{handle slides give} \quad \includegraphics[width=3cm]{diagram2.png} \\
\text{this is } \#_n S^2 \times S^2
\end{array} \]

now \( (M \times \mathbb{R}) \# \mathbb{R}_{n+1} \) is not diffeo to \( M \times \mathbb{R} \)
since it contains a set that can't be embedded in \( \#_n S^2 \times S^2 \)
similarly infinitely many of \( (M \times \mathbb{R}) \# \mathbb{R}_k \) must be different \( \checkmark \)
If $M$ has boundary, but is orientable, then $D(M) = \tau(M \times I)$ is closed and as above embeds in $\#_n S^2 \times S^2$ so $M$ does too.

Now same argument $\Rightarrow M \times \mathbb{R}$ has infinitely many smooth strs.

If $M$ is non oriented, then let $\tilde{M}$ be its orientation double cover.

Note the double cover of $(M \times \mathbb{R}) \times \mathbb{R}^n$ is

$(\tilde{M} \times \mathbb{R}) \times \mathbb{R}^n$ and a diffeo of $(M \times \mathbb{R}) \times \mathbb{R}^n$

with $(\tilde{M} \times \mathbb{R}) \times \mathbb{R}^n$ will lift to a diffeo of

$(\tilde{M} \times \mathbb{R}) \times \mathbb{R}^n$ to $(\tilde{M} \times \mathbb{R}) \times \mathbb{R}^n$

i.e., infinitely many of $(M \times \mathbb{R}) \times \mathbb{R}^n$ must be different.

Uncountably many large $\mathbb{R}^n$'s

For this we need: an end $E$ of an open manifold $X$

is called periodic if $\exists$ a shift map $\phi : E \to E$

st. $\phi : E \to \phi(E)$ is a diffeomorphism

and $\phi^n(E)$ exits any compact set for some $n$
**Example:** let $X$ = open 4-manifold with a compact set $K$ s.t. $X - K$ has 2 components $B$ and $E$ as shown

\[ \begin{array}{c}
B & | & K & | & E \\
\end{array} \]

and $\phi : B \to E$ a diffeomorphism

st. "\infty" in $B$ maps to "\infty" in $E$

now let $X_\infty = \bigcup X_i / \sim$ where $X_i = X$

and $B$ in $X_i$ glued to $E$ in $X_{i-1}$

\[ \cdots \]

clearly end periodic

**Thm (Taubes):**

Let $X$ be a smooth open simply connected 4-manifold with one end.

If $X$ is end periodic and $I_X$ is definite

then $I_X$ is $\Theta_n(1)$ or $\Theta_n(-1)$
now let $R$ be the first example constructed above
let $f: R \to [0, \infty)$ be a topological radial function

\exists \text{ some } A \text{ st: } C' \subset f^{-1}(\Sigma_0, A))

let $R_t = f^{-1}(\Sigma_0, t))$ for $t > A$

Claim: $R_t \neq R_s$ for $t \neq s$

if not let $\psi$ be a diffeo $R_t \to R_s \quad t < s$

\exists \varepsilon > 0 \text{ st } \psi(R_t - R_t - \varepsilon) < R_s - R_t$

now consider the component of $K - f^{-1}(\Sigma_0, s))$

$I = \emptyset \subset E$

we can now glue $\varepsilon$ copies of $(R_s - R_t - \varepsilon)$ to this using $\phi^0 \psi$ to get $\hat{X}$ an open mfd
II) Topologically slice not smoothly slice knots
(more large exotic $\mathbb{R}^4$'s)

given a knot $K \subset S^3$

let $X(K) = B^4 \cup 2$-handle attached to $K$
with framing 0

ie glue $D^2 \times D^2$ to $B^4$ along $S^1 \times D^2$
by an embedding $\Phi : S^1 \times D^2 \to S^3$

sending $\Phi(s \times \{0\})$ to $K$ and
$\Phi(s \times \{1 \times 0\})$ to a copy
of $K$ linking 0 times w/ $K$

$X(K)$ is called the zero trace of $K$

**Lemma (Trace Embedding Lemma):**

$K$ is smoothly (resp. topologically) slice in $B^4$
$\iff$
$X(K)$ smoothly (resp. topologically) embeds in $S^4$,
or $\mathbb{R}^4$

recall $K$ is slice in $B^4$ if $\exists$ an
embedded disk \( D^2 \subset B^4 \) s.t. \( 2D^2 = K \)

it is smooth/top slice of the embedding

is smooth/topological (locally flat).

**Proof:** \( \Rightarrow \) \( K \) slice means we have

\[
\begin{array}{c}
\text{glue } B^4 \text{ to this } B^4 \text{ to get}
\end{array}
\]

a nbhd of \( D^2 \) in \( B^4 \) is \( D^2 \times D^2 \) attached to the other \( B^4 \) along \( S^1 \times D^2 \)

ie. \( B^4 \cup D^2 \times D^2 \) is result of a 2-handle attachment to \( K \)

if framing not zero then

\( D^2 \cup \text{Seifert surface for } K \)

would give a non-trivial
homology class in $H_2(S^4) = 0$
(since self-intersection $\neq 0$)

i.e. we have $X(K)$ embedded in $S^4$

(⇐) if $X(K)$ embeds we see

![Diagram showing $S^4$ and $X(K)$]

let $B_0^4 = \overline{S^4 - B^4}$ in $X(K)$

So $B_0^4$ is a 4-ball

and $D^2 \times \{0\}$ in 2-handle
gives slice disk for $K$ in $B_0^4$

**Fact:** There are topologically slice knots that are not smoothly slice.

to see this need

Freedman: If $K \subset S^3$ has Alexander polynomial $1$, then $K$ is topologically slice
Given a top slice, but not smoothly slice $K \subset S^3$

we can construct a large exotic $\mathbb{R}^4$

Since $K$ is top slice, lemma above says $\exists$ a topological embedding

$$\phi: X(K) \to \mathbb{R}^4$$

let $\mathcal{C} = \mathbb{R}^4 - \phi(\text{int } X(K))$

Quinn proved that any open 4-manifold has a smooth structure
so we can put a smooth str on \( C \)
\[ \partial C = - \partial X(K) \] and these are smooth 3-manifolds so they are diffeomorphic
\[ \Psi: \partial C \rightarrow -\partial X(K) \]

let \( P = X(K) \cup_4 C \)

by Freedman’s work discussed above we know \( P \) is homeomorphic to \( \mathbb{R}^4 \)

but \( X(K) \) smoothly embeds in \( P \) so \( P \) can’t be \( \mathbb{R}^4 \) or \( K \) would be smoothly slice by Trace Embedding Lemma

**Question:**

Can you construct more than one exotic \( \mathbb{R}^4 \) using such knots?

almost certainly yes, but how do you distinguish them?
Constructing small exotic $\mathbb{R}^4$ using the failure of the smooth 5D $h$-cobordism theorem

an $h$-cobordism from $M_0^5$ to $M_1^5$ is a compact $(n+1)$-manifold $W$ such that

$$\partial W = -M_0 \cup M_1$$

and the inclusions $i_j : M_j \to W$ are homotopy equiv.

Fact: if $M_0$ and $M_1$ are homotopy equivalent then they are $h$-cobordant (Novikov/Wall)

Facts about $h$-cobordisms & handlebodies:

"recall" an $n$-dimensional $k$-handle is

$$h^k = D^k \times D^{n-k}$$

$$2 \cdot h^k = (\partial D^k) \times D^{n-k} = S^{k-1} \times D^{n-k}$$

$h^k$ is attached to the $\partial$ of an $n$-manifold $X$ by an embedding $\phi : \partial h^k \to \partial X$

so attaching $h^k$ to $X$ is

$$X \cup_{\phi} h^k = \frac{X \cup h^k / \sim \subset h^k \cup \phi(x) \in \partial X}{x \in \partial h^k \cap \phi(x) \in \partial X}$$

Example:

a 0-handle is just $D^n$ attached along $\partial$

so attaching 0-handle is just
a handlebody is a manifold $X^n$ built from $\varnothing$ or $M^{n-1} \times \{0,1\}$ by a sequence of handle attachments

**Example:**

$\varnothing \rightarrow \quad \Rightarrow \quad \Rightarrow \quad \cong \quad \Rightarrow$

$\rightarrow \quad \Rightarrow \quad \cong \quad \Rightarrow$

**Facts about handlebodies:**

1) any compact smooth manifold, or cobordism, has structure of a handlebody

2) handles can be attached with increasing index
3) if $h^k$ and $h^{k+1}$ attached to $\partial M$ so that attaching sphere $h^{k+1} \cap$ belt sphere $h^k$ exactly once (and transversely) then

$$M \cup h^k \cup h^{k+1} \cong M$$

4) if $M^n$ connected and $\partial \neq \emptyset$ then can assume no 0-handles

$$\partial \neq 0 \text{ then no } n \text{-handles}$$

(just cancel as above)

5) if $X$ is a cobordism, $\pi_1(X) = 1$, and $n \geq 5$, then can assume there are no $l$ or $(n-1)$-handles (and no 0 or $n$-handles, by 4))

Now suppose $M$ and $M'$ are homeomorphic non-diffeo. 4-manifolds (such examples exist due to
From above, find a cobordism $X$ with

$$\partial X = -M \cup M'$$

and we can assume there are no 0, 1, 4, and 5 handles.

So $X = M \times [0, 1] \cup 2$-h's $\cup 3$-h's.

The CW-chain complex $C_\ast(X, M)$ is generated by $k$-handles, and

$$\partial_X h^k = \sum_i \langle h_i^k, h_i^{k-1} \rangle h_i^{k-1}$$

where $\langle h_i^k, h_i^{k-1} \rangle$ is the algebraic intersection of the attaching sphere of $h_i^k$ and the belt sphere of $h_i^{k-1}$.

Since $H_2(X, M) = H_3(X, M) = 0$ (since $M \to X$ a homotopy equiv),

we know $\partial_3$ is an isomorphism (in particular

# $2$-h = # $3$-h).

After "sliding handles," we can assume

(attach sphere $h_i^3$) · (belt sphere $h_j^2$) = $\delta_{ij}$.

If the geometric $\Lambda = \delta_{ij}$, then we could cancel all handles and $X = M \times [0, 1]$.

So $M' = M \times \{1\}$ by choice of $M, M'$.

From now on assume only one 2 and 3-handles.

(argueent same if more, and 3 examples like this)
we can find Casson handles in $X_{1/2}$ to cancel extra intersections between $A = \text{attaching sphere of } h^3$ and $B = \text{belt sphere of } h^2$

and arrange that $N = \text{open nbhd of } A \cup B \cup$ Casson handles is homeomorphic (by Freedman) to $S^2 \times S^2 - B$.

let $U = \text{everything above and below } N$

set $R_\sim = M \setminus U$ and $R_+ = M' \setminus U$

note: $R_\sim$ is obtained from $N$ by "surgering $B"$

so is topologically $R^4$

similarly for $R_+$
Let $K =$ union of all cores and cocores of handles together with points above and below $A \cup B$

$U-K$ is a trivial cobordism from $R_-(K \cap R_-)$ to $R_+-(K \cap R_+)$

**Claim:** $R_+ \subset R^4$

Indeed, we can build $S^4 \times [0,1]$ from $S^4 \times [0,1]$ by attaching a cancelling pair of 2 and 3-handles.

Now add double points to attaching & belt spheres

Now since we added the double pts $\exists$ embedded disks to cancel them and a nbd of these disks are 2-handles

Recall any Casson handle embeds in a real
2-handle
so we can find \( N \) in \((S^4 \times [0,1])_{1/2}\) and \( U \) in \( S^4 \times [0,1]\)
now \( U \cap (S^4 \times [0,1]) \) is \( R_- \) an is clearly contained in \( M = S^4 - pt \)
same for \( R_+ \)

**Claim:** \( R_+ \) not diffeomorphic to \( R^4 \)

If \( R_- \) is diffeomorphic to \( R^4 \) then \( \exists \) a 4-ball \( D_- \)
in \( R_- \) s.t. \( K_- \subset D_- \)
since \( U - K \) is a product, above \( D_- \) is an \( S^3 \) in \( R_+ \)
bounding some compact set \( D_+ \) in \( R_+ \) (and \( K_+ \subset D_+ \))
so we see

\[
\begin{align*}
M - \text{int}(D_-) &\cong M' - \text{int}(D_+) \\
\text{and} \quad S^4 - \text{int}(D_-) &\cong S^4 - \text{int}(D_+)
\end{align*}
\]

now \( D^4 \) = \( S^4 - \text{int}(D_-) \cong S^4 - \text{int}(D_+) \)

\( \text{recall} \cong D^4 \)
so \( D_+ \) also a 4-ball

\[
\therefore M = (M - \text{int}(D_-)) \cup 4\text{-handle}
\]
\[
M' = (M' - \text{int}(D_+)) \cup 4\text{-handle}
\]
So \( M - \text{int}(D) \equiv M' - \text{int}(D') \Rightarrow M \cong M' \) 