Math 8803: MCG

- Primer, Farb-M
- Flipped / Just in Time
- ~ 1 chapter / week
- Midterm: Read & summarize a paper on MCG

  Target: Oct 5

- Final: Attempt research

  Proposal Nov 2

  Target Nov 23

  Groups ok.

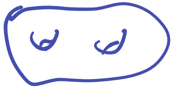
- Participation

  Teams: Q's for class

  Open q's.

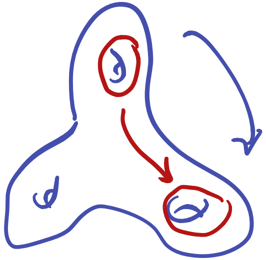
Also: This Wed 11:15 start.

Mapping Class Gps

$S =$ 

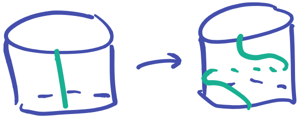
$$\mathrm{Mod}(S) = \pi_0 \, \mathrm{Homeo}^+(S)$$
$$= \mathrm{Homeo}^+(S) / \text{isotopy}$$

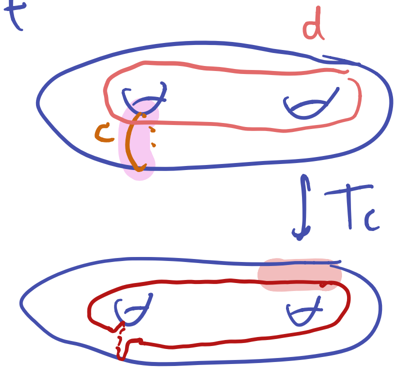
Sample elements

① 

S_3

② Dehn twist





3 Reasons

① Alg geometry.

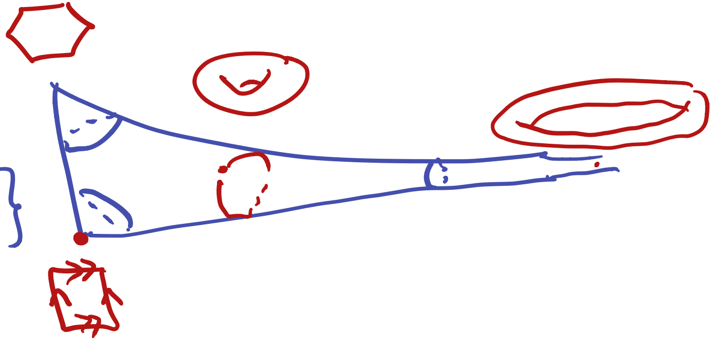
$\mathrm{Mod}(S_g) = \text{"}\pi_1\text{"}\, \mathcal{M}_g$

$\mathcal{M}_g$ = moduli space of alg. curves of genus g.

$\mathcal{M}_1$

$\shortparallel$

{Eucl. unit area tori}



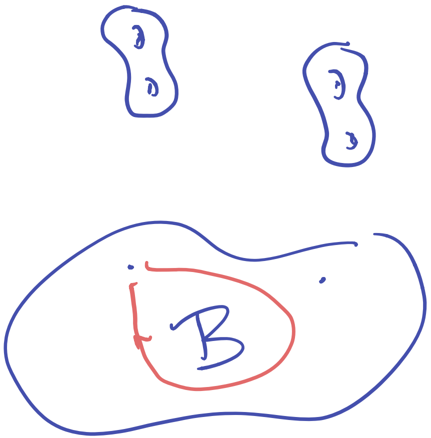
② Topology

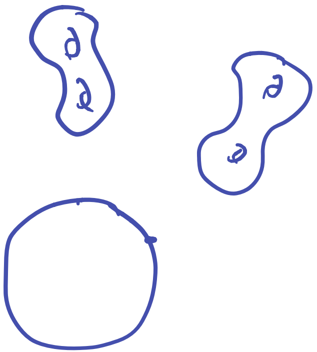
$$\{S\text{-bundles over } B\} \longleftrightarrow \{\pi_1 B \to \mathrm{Mod}^{\pm}(S)\}$$

"monodromy"

Mobius band also

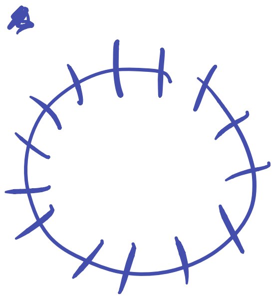
Annulus: $[0,1]$-bund over S^1





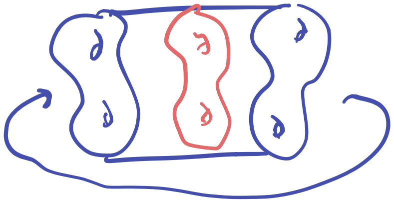
$B = S^1$

e.g. $S \times S^1$



Agol, Wise, Perelman, Thurston....

Essentially all 3-manifolds arise this way.



$f \in \mathrm{Mod}(S)$

$$S \times [0,1] \big/ (x,1) \sim (\varphi(x),0)$$

$$[\varphi] = f$$

Donaldson:

All symplectic 4-mans arise essentially this way.

Also: Contact topology: open bks

③ Geometric Group Thy

$\mathrm{Out}(G) \cong \mathrm{Aut}(G)/\mathrm{Inn}(G)$

Dehn - Nielsen - Baer thm

$$\mathrm{Mod}^{\pm}(S_g) \cong \mathrm{Out}\,\pi_1(S_g)$$

Topology. Algebra

Related topics

Number thy.

Group theory

Rep thy

Grap cohom.

Graph thy

Complex anal.

Hyp. geom

Alg top.

Dynamics

Combinatorics

Overview of Book/Class

Part I

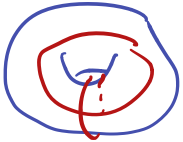
① Curves on surfaces – Wed.

homeos : linear maps ::

curves : vectors

② MCG basics

$\mathrm{Mod}(T^2) \xrightarrow{\cong} SL_2\mathbb{Z}$



Alexander Method

③ Dehn twists

Prop. $a \neq b$

alg. $T_a T_b T_a = T_b T_a T_b$

$\iff i(a,b) = 1.$ topol.

④ Generating MCG

Dehn: $\mathrm{Mod}(S_g) = \langle T_c \rangle$

⑤ Presentations of MCG

$$H_1(\mathrm{Mod}(S_g)) = 0$$

$$H_2(\mathrm{Mod}(S_g)) \cong \mathbb{Z}$$

$H_k(\mathrm{Mod}(S_g)) \longleftrightarrow$ characteristic classes for S_g-bundles

Super duper mysterious.

⑥ Symplectic rep.

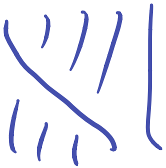
⑦ Torsion

In $\mathrm{Mod}(S_g)$, elements of order

1, 2, 3, 4, 6, 7, 8, 9, 12, 14.

⑧ DNB (see above)

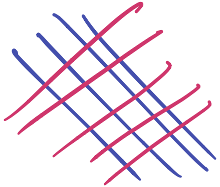
⑨ Braid gps



Parts II & III

Nielsen-Thurston Classification Thm: Any f in $\mathrm{Mod}(S)$ has a rep. φ that is

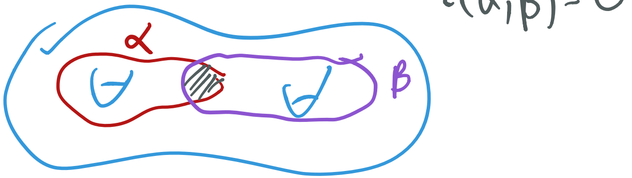
1. finite order
2. reducible: fixes a collection of disjoint curves.



3. pseudo-Anosov: like $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \circlearrowright \mathbb{R}^2$

Chapter 1 Highlights

① Geometric int



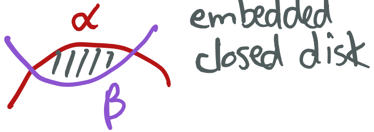
$$i(\alpha,\beta) = \min_{\substack{\alpha'\sim\alpha \\ \beta'\sim\beta}} |\alpha' \cap \beta'|$$

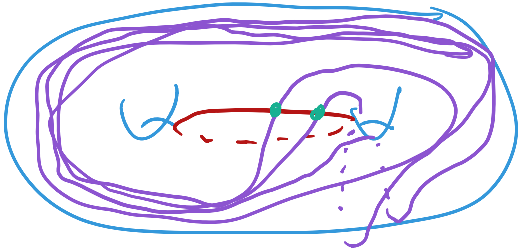
← function on pairs of homotopy classes.

② Bigon criterion

α,β are in minimal position (they realize $i(\alpha,\beta)$)

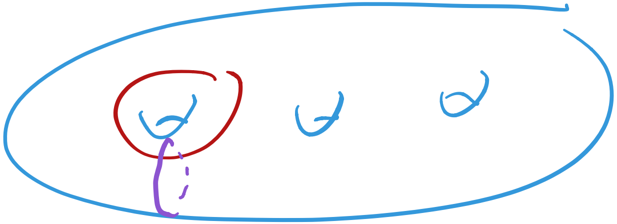
$\iff$ they do not form a bigon





③ Change of coordinates principle.

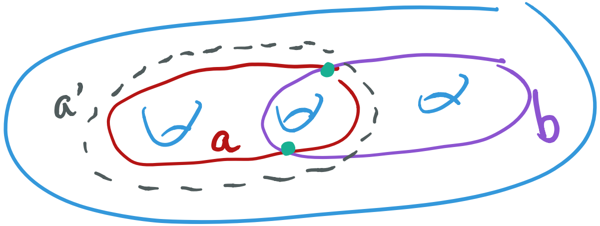
Example. if $i(a,b)=1$ then it's this pic



$T_a T_b T_a$
$= T_b T_a T_b$

Geometric intersection number

Observ. 1 $i(a,b) \neq |\hat{i}(a,b)|$



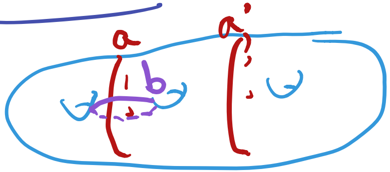
$i(a,b) = 2$

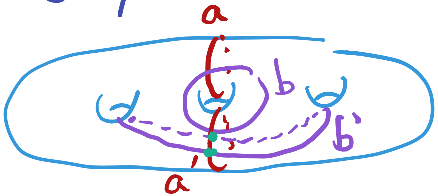
$\hat{i}(a,b) = 0$

Bigon crit $\Longrightarrow$ min. pos.

homology class

Observ. 2 $[a] = [a'] \not\Longrightarrow i(a,b) = i(a',b)$



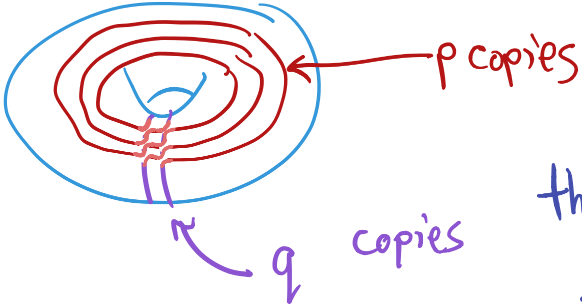


Fact. On T^2: $\left\{\begin{matrix}\text{hom. classes of}\\ \text{simple closed curves}\end{matrix}\right\} \longleftrightarrow$ primitive elts of $\mathbb{Z}^2 / \pm$

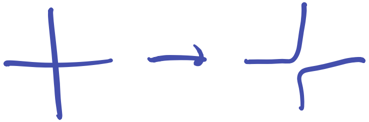
(not an integer multiple, so $(5,10)$ not prim.

The map $\longleftarrow$ is:

$(p,q) \longrightarrow$



then surger:



Fact. $i\big((p,q),(r,s)\big) = \left|\begin{matrix} p & r \\ q & s \end{matrix}\right|$

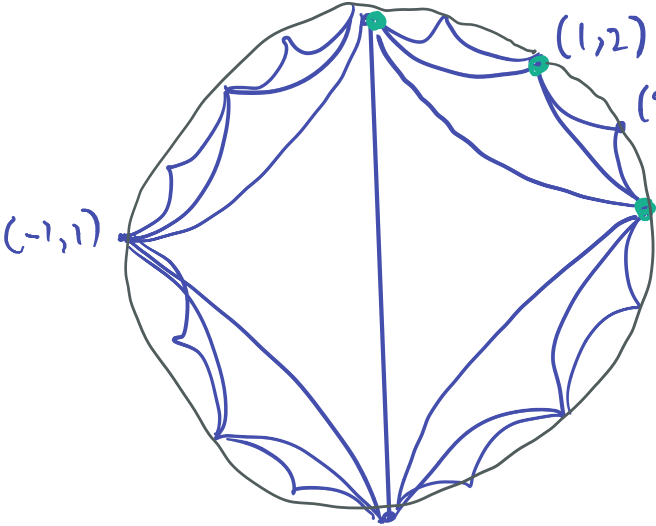
Pf. First check for $(p,q)=(1,0)$.
General case: apply $A \in SL_2\mathbb{Z}$ s.t. $A\binom{p}{q} = \binom{1}{0}$.

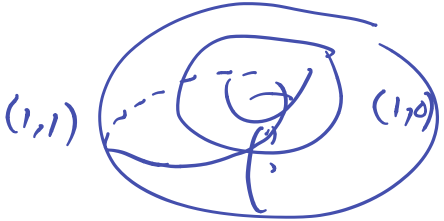
lin. map of T^2

Farey graph

vertices : s.c.c. on T^2 (hom. classes)

edges : $i = 1$





"complex of curves for T^2"

③ Change of Coords Principle

First example: $\alpha, \beta \subseteq S_g$ nonsep.

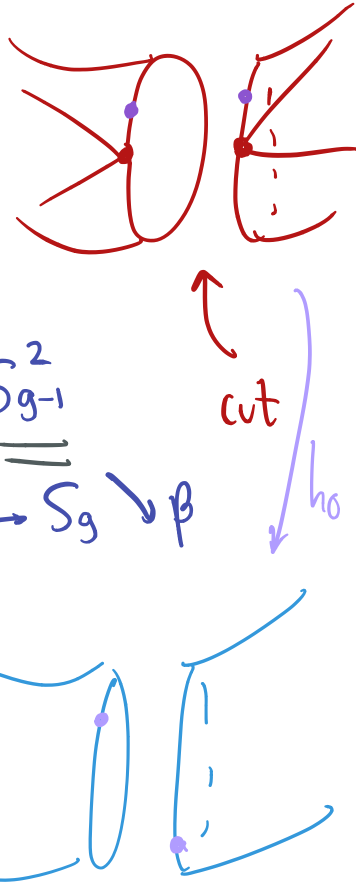
cut

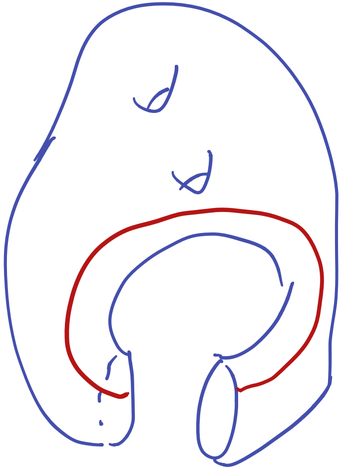
$\exists\; h \in \mathrm{Homeo}(S_g)$ st $h(\alpha) = \beta$.

Pf. $S_g \setminus \alpha$ & $S_g \setminus \beta$ are both $\underline{\underline{S_{g-1}^2}}$

Class. of surf's $\rightsquigarrow h_0 : S_g \setminus \alpha \longrightarrow S_g \setminus \beta$

$\rightsquigarrow h$.





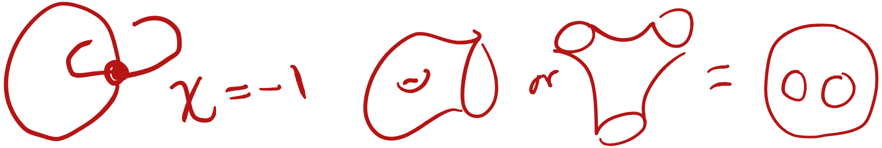
<u>Example</u> If $i(\alpha,\beta)=1$ & $i(\gamma,\delta)=1$

then $\exists\ h \in \mathrm{Homeo}(S_g)$ s.t. $h(\alpha,\beta)=(\gamma,\delta)$

Same proof: Cut, use class. of surf.

$$S_g \setminus (\alpha\cup\beta) = S_{g-1}^1$$

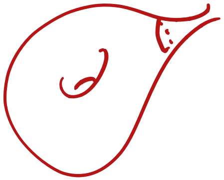
$$\chi(S_g)=2-2g \qquad \chi(S_{g-1}^1) = 2-2(g-1)-1$$
$$= 3-2g$$



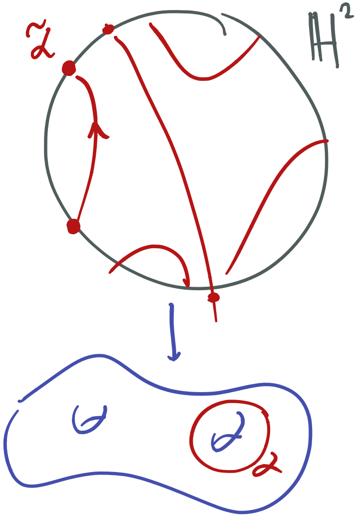
Extra time

Fact. $1 \neq \alpha \in \pi_1(S_g)$ $\quad g > 1$

$$\Rightarrow C(\alpha) \cong \mathbb{Z} = \langle \alpha_0 \rangle \qquad \alpha_0 = \text{root of } \alpha.$$



Pf.



Alg. top: $\pi_1(S) \longrightarrow \mathrm{Homeo}(\tilde{S})$
(deck trans)

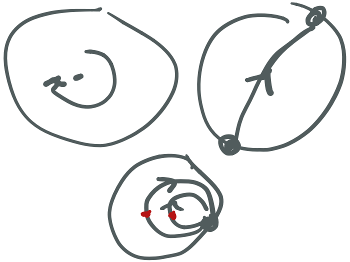
Here: $\pi_1(S) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$

image discrete.

Fact 1 $\alpha \longrightarrow$ hyp/lox isometry
(i.e. translates along axis)

Fact 2 In $\mathrm{Isom}^+(\mathbb{H}^2)$
$C(\text{hyp isom}) \cong \mathbb{R}$ = translation along axis.

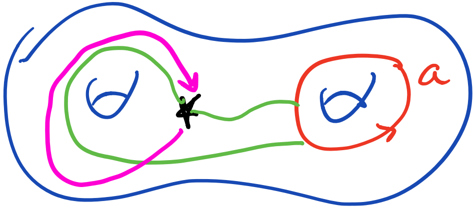
Classifi. of $\mathrm{Isom}^+ \mathbb{H}^2$



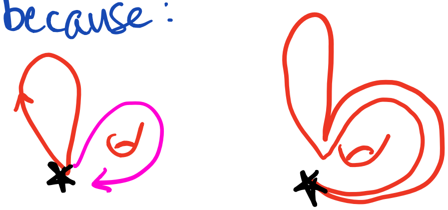
1.2.1 Closed curves & geodesics

$$\left\{ \begin{array}{c} \text{conj classes} \\ \text{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{free hom. classes} \\ \text{of oriented } \cancel{s}\text{.c.c.} \end{array} \right\}$$

$\boxed{\leftarrow}$

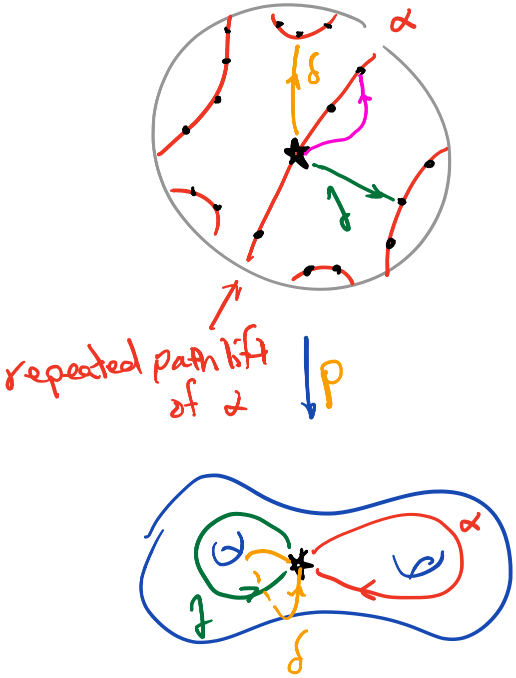


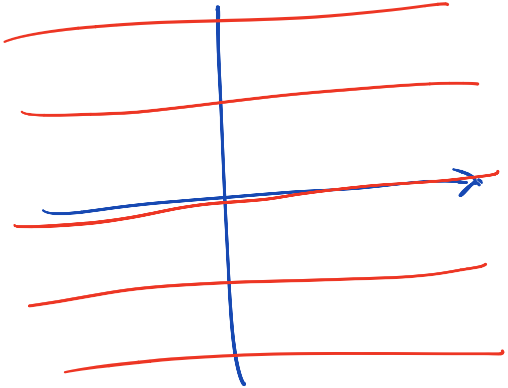
The two elts of π_1 you get differ by a point push $\longleftrightarrow$ conj. because:

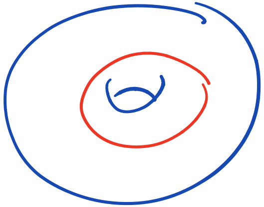


$$\left\{\begin{matrix}\text{elts of conj} \\ \text{class } a\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{lifts to } \mathbb{H}^2 \\ \text{of } a\end{matrix}\right\}$$

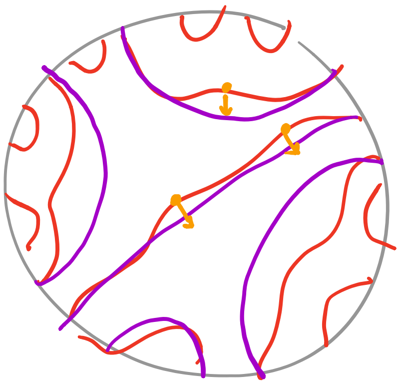
lift = component of $p^{-1}(\alpha)$





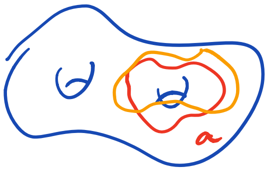


$$\left\{ \begin{array}{c} \text{free hom. classes} \\ \text{of (simple) curves} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(simple)} \\ \text{geodesics} \end{array} \right\}$$

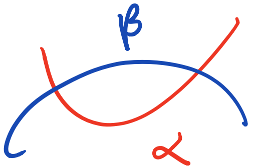


straight line homotopy to closest projection

injectivity: homotopies can't change endpts at $\partial_\infty \mathbb{H}^2$.



Bigon criterion α, β are in min pos. $\Longleftrightarrow$ α, β form no bigons.



Bigon.

$\boxed{\Rightarrow}$ easy.

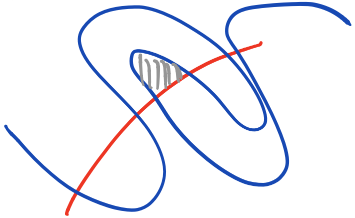
$\boxed{\Leftarrow}$ two proofs.

Lemma. α, β form no bigons

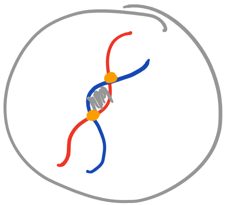
$\Longleftrightarrow$ any two lifts intersect 0,1 times.

Pf of Lemma. $\boxed{\Rightarrow}$ Lift the bigon to $\mathbb{H}^2$ (lifting criterion)

$\pi_1(\text{bigon}) = 1$,



$\boxed{\Leftarrow}$

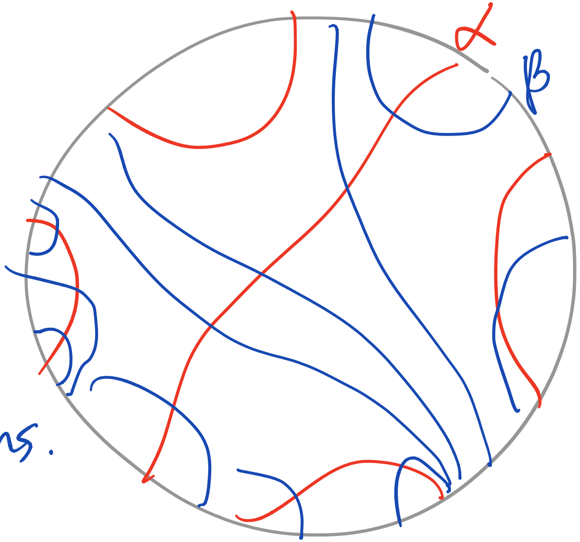


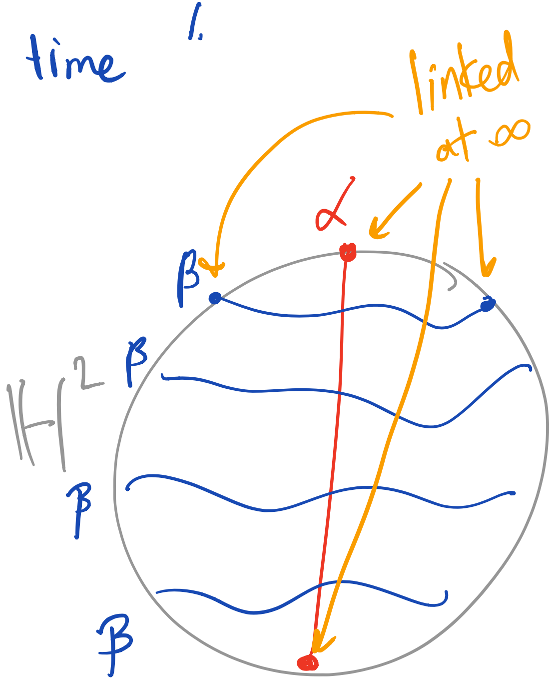
Check this bigon in $\mathbb{H}^2$ descends to bigon in S. (check inj).

To prove Big. Crit, need to show:

If all lifts of α, β intersect ≤ 1 time then α, β min. pos.

Homotopies is S can't change linking at ∞ and so can't remove intersections.

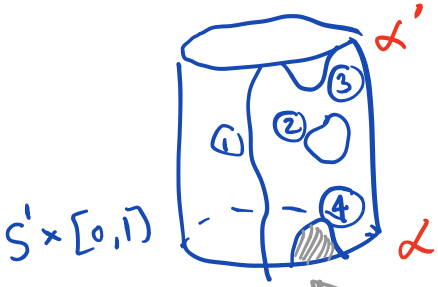




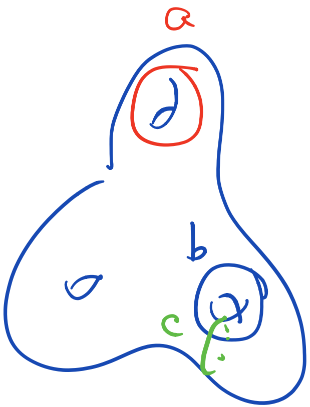
Proof #2. Suppose α, β not in min. pos.

Want to find a bigon.

Let $H: S^1 \times [0,1] \to S$ be a homotopy of α that reduces intersection.



maps to bigon in S.



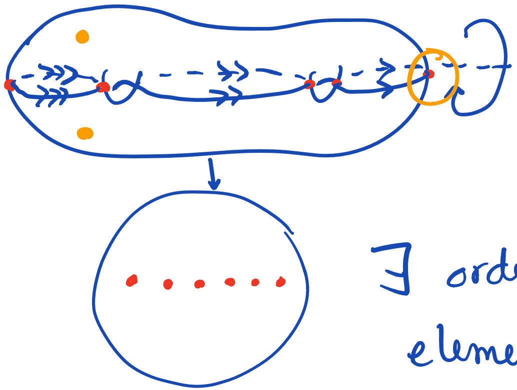
Chapter 2

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

& marked pts fixed as a set.

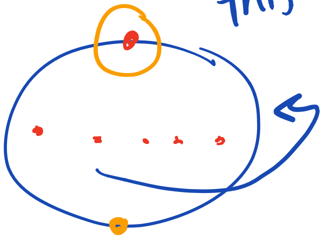
$$\cong \mathrm{Homeo}^+(S, \partial S) \,/\, \text{homotopy}.$$

<u>example</u> order 5 elt in $\mathrm{Mod}(S_2)$



hyp inv.

$\exists$ order 5 element!



By lifting on't this lifts.

order 5 or 10 in $\mathrm{Mod}(S_2)$?

Basic examples

D^2, $D^2 \setminus pt$, $S_{0,1} \cong \mathbb{R}^2$, $S_{0,0} \cong S^2$, $S_{0,3}$ maybe

A = annulus, $S_{1,0} = T^2$

marked pt

genus

Alexander Lemma

Prop. $\mathrm{Mod}(D^2) = 1$.

Pf. $\varphi \in \mathrm{Homeo}^+(D^2, \partial D^2)$

$\varphi_0 = \varphi$

$\varphi_1 = \mathrm{id}$

φ_t :

id

$1-t$

φ here

(really φ conj by scaling by $1-t$)

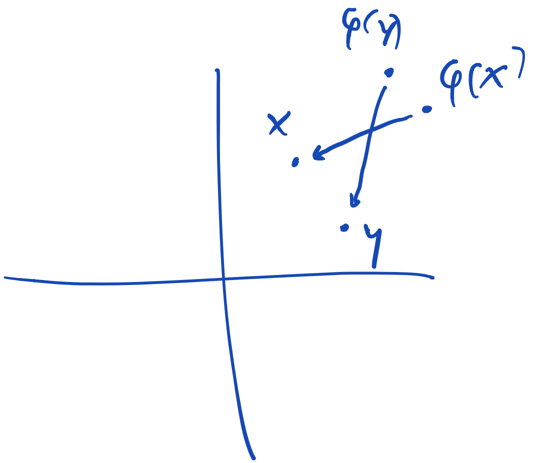
Cor of Proof

$\mathrm{Mod}(D^2 \setminus pt) = 1$.

Prop. $\mathrm{Mod}(S_{0,1}) = \mathrm{Mod}(\mathbb{R}^2) = 1$

Pf. Straight line homotopy.

$\varphi \in \mathrm{Homeo}^+(\mathbb{R}^2)$



Prop. $\mathrm{Mod}(S_{0,0}) = \mathrm{Mod}(S^2) = 1.$

Pf. First homotope so $\varphi(\text{north pole}) = \text{north pole}$

Apply prev. Prop.

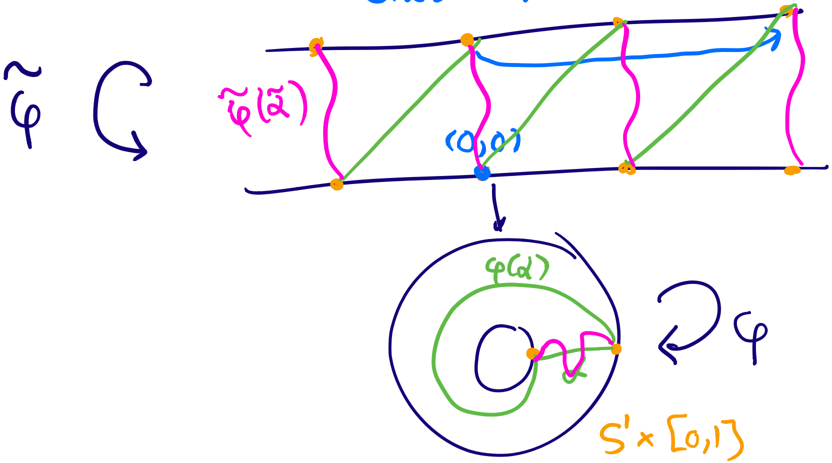
Prop. $\mathrm{Mod}(A) \cong \mathbb{Z}$.

Pf. Define $L : \mathrm{Mod}(A) \to \mathbb{Z}$.

Let $[\varphi] \in \mathrm{Mod}(A)$

Restrict $\tilde{\varphi}$ to $\mathbb{R} \times \{1\}$ in $\mathbb{R} \times [0,1]$ ← univ. cover of A

$\tilde{\varphi}$ ← chosen to fix $(0,0)$



Surjectivity : Dehn twist $\longrightarrow \pm 1$

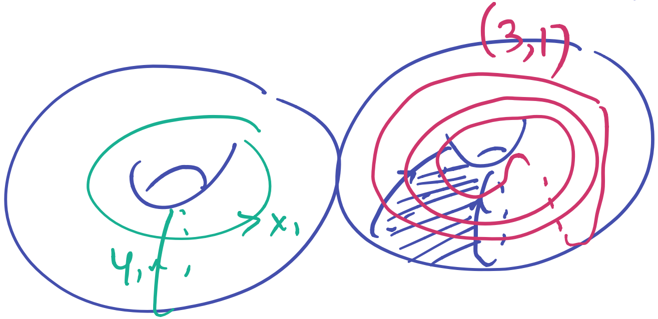
Injectivity : Straight line homotopy $\tilde{\varphi} \to \mathrm{id}$.

The Torus

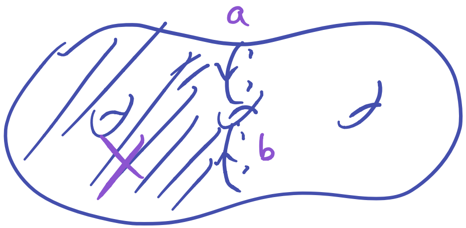
not GL because $\hat{i}$.
and $\hat{i} \longleftrightarrow \det$.

Prop. The map $\mathrm{Mod}(T^2) \to SL_2\mathbb{Z}$ given by action on $H_1(T^2;\mathbb{Z})$ is an $\cong$.

homology.

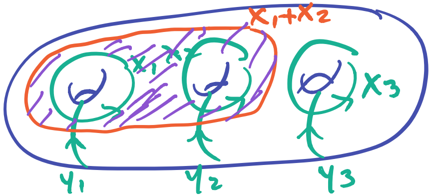


$a - b = \partial X$

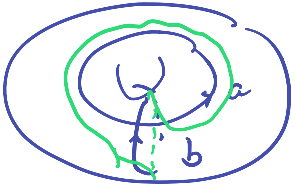


$\mathrm{Mod}(S_g) \to GL_{2g}\mathbb{Z}$

$H_1(S_g;\mathbb{Z}) = \mathbb{Z}^{2g}$



Pf. $\boxed{\text{Surjectivity}}$



Pf #1

$$T_a \longmapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad T_b \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

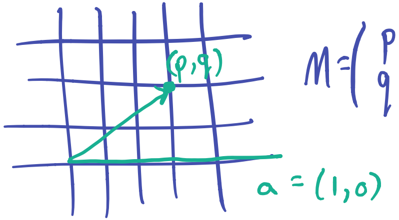
Pf # 2

Let $M \in SL_2\mathbb{Z}$, thought of as lin map $\mathbb{R}^2 \to \mathbb{R}^2$

M descends to $\varphi \in \mathrm{Homeo}^+(T^2)$

and $\varphi_* = M$

action on H_1.



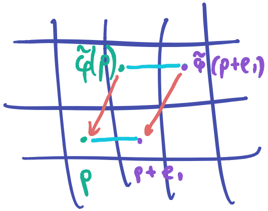
$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$

Injectivity

Pf #1 K(G,1) theory

$$\left\{ \substack{\text{based maps}\\ T^2 \to T^2} \right\} \Big/ \sim \quad \longleftrightarrow \quad \left\{ \substack{\text{homoms}\\ \mathbb{Z}^2 \to \mathbb{Z}^2} \right\}$$

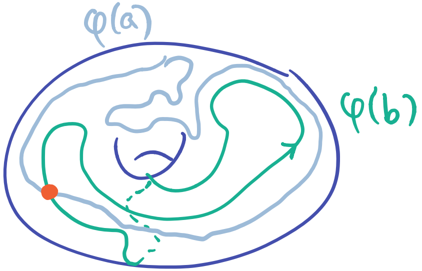
Pf #2 Straight-line homotopy.



$\varphi \in$ kernel

$\Longrightarrow$ S.L.H. equivariant w.r.t. deck trans.

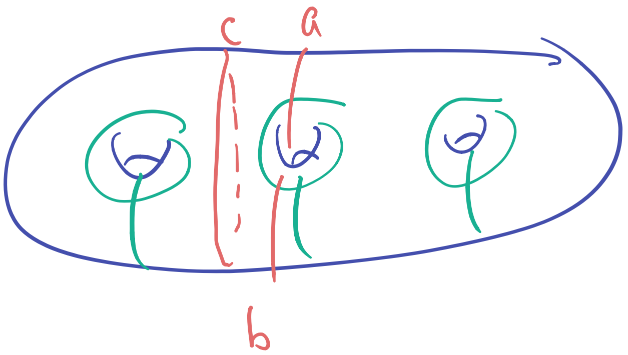
Pf #3



What about higher genus?

$\mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2g})$ has a (big) kernel!

Torelli gp.



T_c

$T_a T_b^{-1}$

See Chap. 6.

Proposition 2.8 (Alexander method) *Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties.*

1. *The γ_i are pairwise in minimal position.*
2. *The γ_i are pairwise nonisotopic.*
3. *For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, or $\gamma_j \cap \gamma_k$ is empty.*

(1) If there is a permutation σ of $\{1, \ldots, n\}$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

If we regard $\cup\gamma_i$ as a (possibly disconnected) graph Γ in S, with vertices at the intersection points and at the endpoints of arcs, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of Γ.

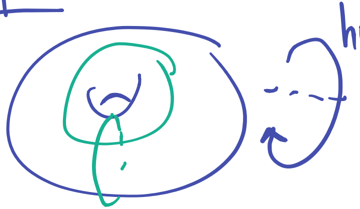
(2) Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.

Morally: A mapping class is determined by its action on (finitely many) curves.

Q. Is there a version without hypoth. ?

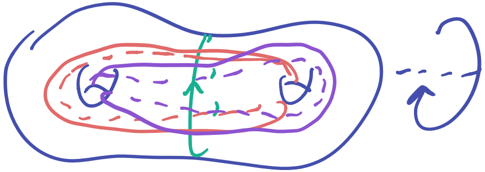
Examples.

1



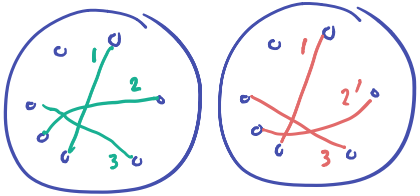
hyp inv

2



Q. Is there a similar example satisfying 3. in Prop 2.8?

3



Is there a notion of canonical pos. for curves failing 3?

Proposition 2.8 (Alexander method) *Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties.*

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(2) Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.

Pf by induction on n.

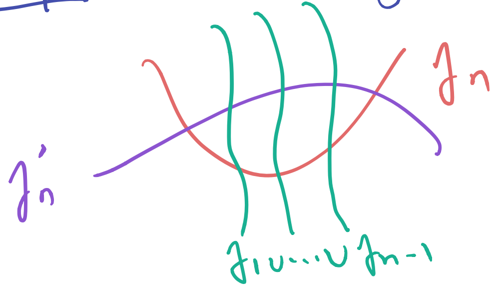
Say we modified φ by homotopy

so $\varphi(\gamma_1' \cup \cdots \cup \gamma_{n-1}') = \gamma_1 \cup \cdots \cup \gamma_{n-1}$

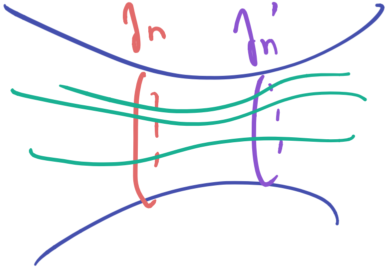
Want to isotope φ s.t. $\gamma_n' \to \gamma_n$

& we $\boxed{\text{fix } \gamma_1 \cup \cdots \cup \gamma_{n-1}}$.

Step 1. Remove bigons



Step 2. Remove annulus



<u>Cor.</u> If $c_i = [\gamma_i]$ as in Prop.

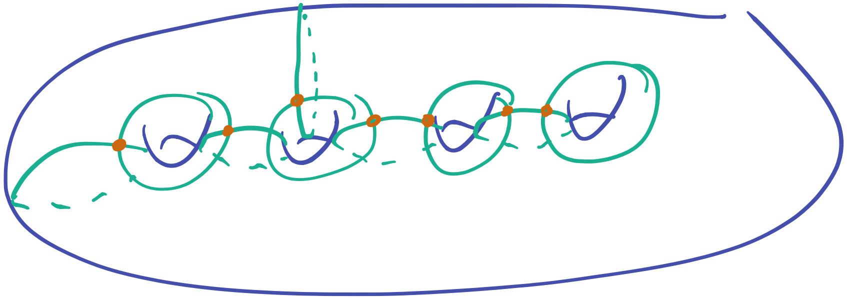
& $f \in \mathrm{Mod}(S)$ fixes $\{c_i\}$

Then f has finite order.

Moreover, f is det. by induced action on

$\cup \gamma_i$, thought of as a graph.

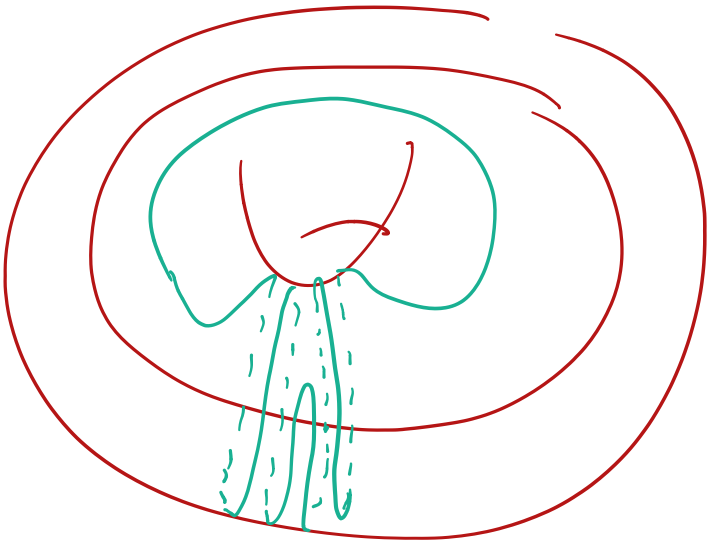
A good Alexander system:



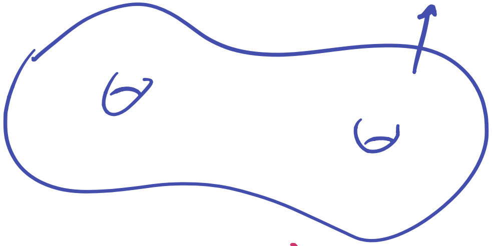
This graph has no nontrivial automorphisms

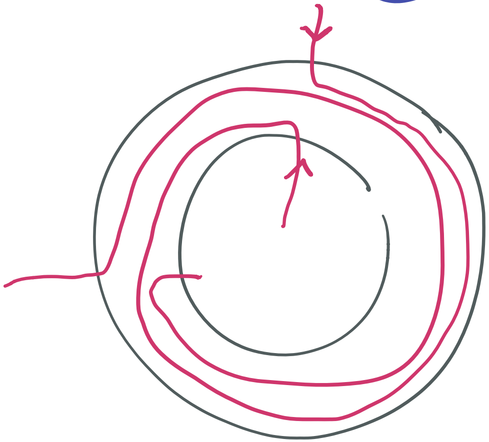
So if f fixes ~~each curve~~ the $\{c_i\}$ as a set then $f = id$ in $Mod(S_4)$

filling, but not really.



Dehn twists



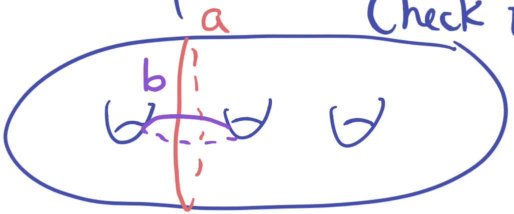


Prop. Dehn twist have ∞ order.

If a nonsep the T_a^k acts nontrivially on $H_1(S_g)$ $k \neq 0$ hence $T_a^k \neq id$.

But for a sep. T_a^k acts trivially. Can draw $T_a^k(b)$.

Check $i(T_a^k(b), b) \neq 0$



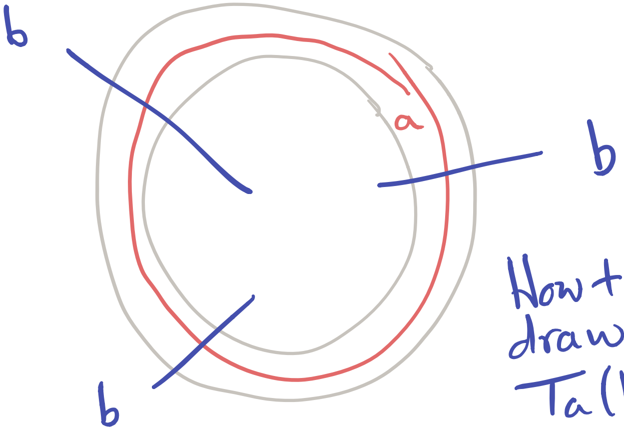
$\Rightarrow T_a^k(b) \neq b$

$\Rightarrow T_a^k \neq id$

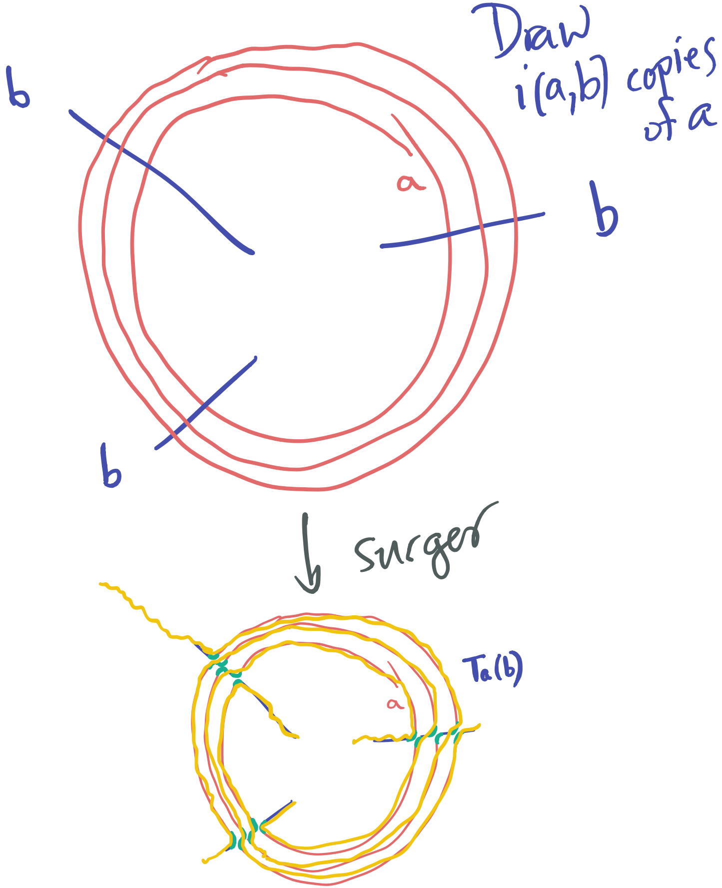
Prop. $i(T_a^k(b), b) = |k|\, i(a,b)^2$

Cor. $|T_a| = \infty$.

Need: Surgery description of Dehn twists

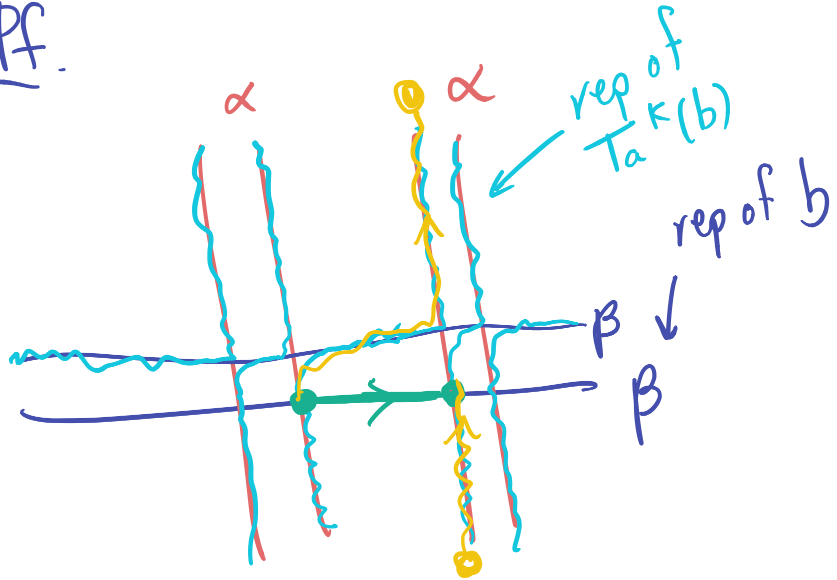


How to draw $T_a(b)$?



Prop. $i(T_a^k(b), b) = |k|\, i(a,b)^2$

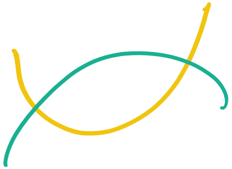
Pf.



$i(a,b) = 2 \quad k = 1.$

Our rep of $T_a^k(b)$ intersects β $|k|\, i(a,b)^2$ times.

Remains to check: No bigons.



$\blacksquare$

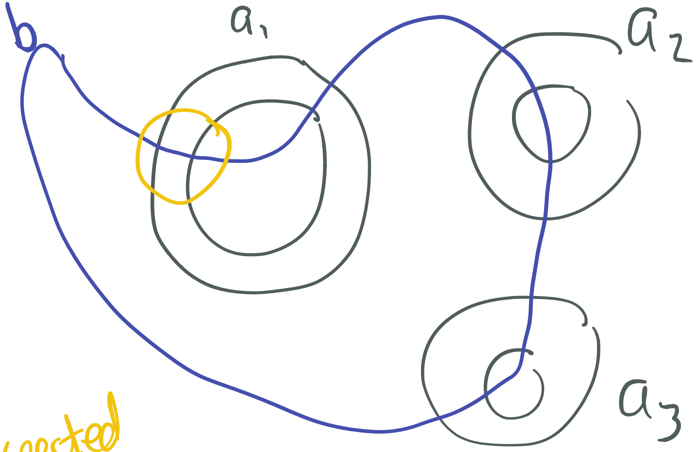
Prop. $a_1, \ldots, a_n$ $\quad i(a_i, a_j) = 0$.

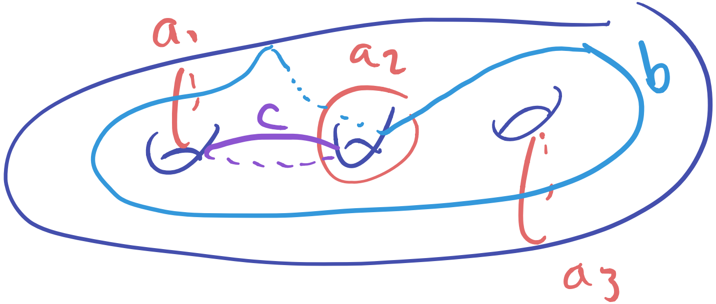
$e_i \geq 0$.

$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \boxed{\sum_{i=1}^{n} e_i \, i(a_i, b) i(a_i, c)} \right| \leq i(b, c)$$

expected # of intersections





Note: for $b = c$ and $n = 1$ get last prop.

Q Example where expected value is not right.

Prop. $a_1, \dots, a_n$ $\quad i(a_i, a_j) = 0$.

$e_i \geq 0$.

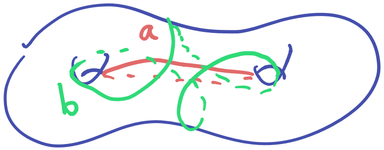
$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \boxed{\sum_{i=1}^{n} e_i \, i(a_i, b) \, i(a_i, c)} \right| \leq i(b, c)$$

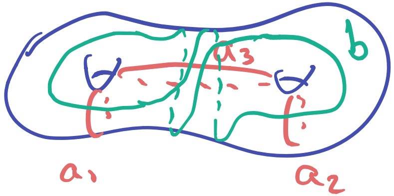
Cor. $\exists$ pair of filling curves on any S. with $\chi(S) < 0$.

$\{a, b\}$ filling if $\max\{i(a,c), i(b,c)\} > 0$ $\forall\, c$

example



Pf of Cor. Choose pants decomp. of S



* Find a b s.t. $i(a_i, b) > 0$ $\forall\, i$

Let $a = \left(\prod T_{a_i}\right)(b)$

By prop, a & b are filling.

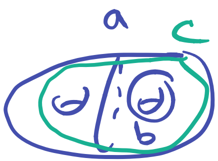
Basic Facts

Fact 1 $T_a = T_b \iff a = b$

Pf. Find c s.t. $i(a,c) \neq 0$
$i(b,c) = 0$

Then $i(T_a(c), c) = i(a,c)^2 \neq 0$
$i(T_b(c), c) = i(c,c)^2 = 0$.

How to find c?



Case 1. $i(a,b) > 0$ take $c = b$

Case 2 $i(a,b) = 0$. Use change of coords.

Fact 2 $f T_a f^{-1} = T_{f(a)}$

commutes

Fact 3 $(f \leftrightarrow T_a) \iff f(a) = a$.

Pf. $\Rightarrow$ Fact 1 + Fact 2.
$\Leftarrow$ Fact 2

Fact 4. a, b non sep
Then T_a conj to T_b in MCG

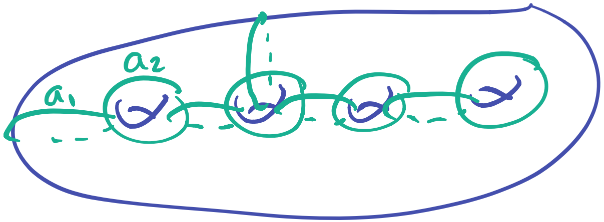
Pf. Fact 2 + Change of coords.

Fact 5. $i(a,b) = 0 \iff$
$(T_a \leftrightarrow T_b)$

Pf. Use first Prop & Fact 2

Thm. For $g \geq 3$, $Z(\mathrm{Mod}(S_g)) = 1$.

Pf. Use the Alex. system



$f \in Z(\mathrm{Mod}(S_g))$

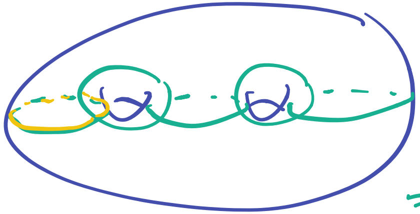
$\Longrightarrow f(a_i) = a_i \ \forall i$

(Fact 3)

The graph $\Gamma = \cup a_i$ has no nontrivial autos.

So Alex Meth $\Longrightarrow f = \mathrm{id}$.

What about $g = 1, 2$?



These generate.

$\Longrightarrow$ hyp inv. in central.

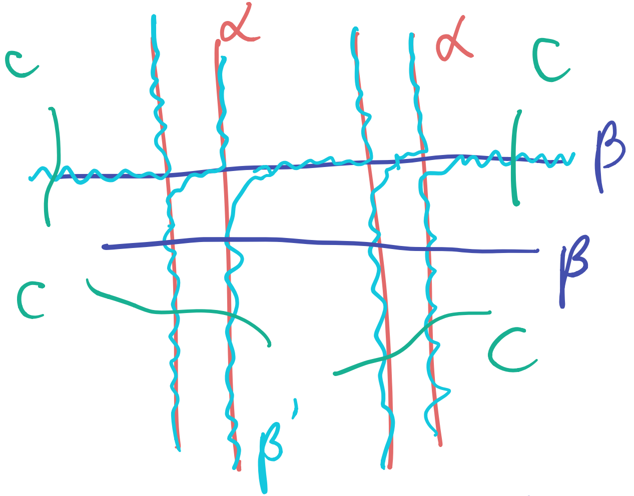
$Z(\mathrm{Mod}(S_2)) = \mathbb{Z}/2$

Prop. $a_1, \dots, a_n$ $\quad i(a_i, a_j) = 0$.

$e_i \geq 0$.

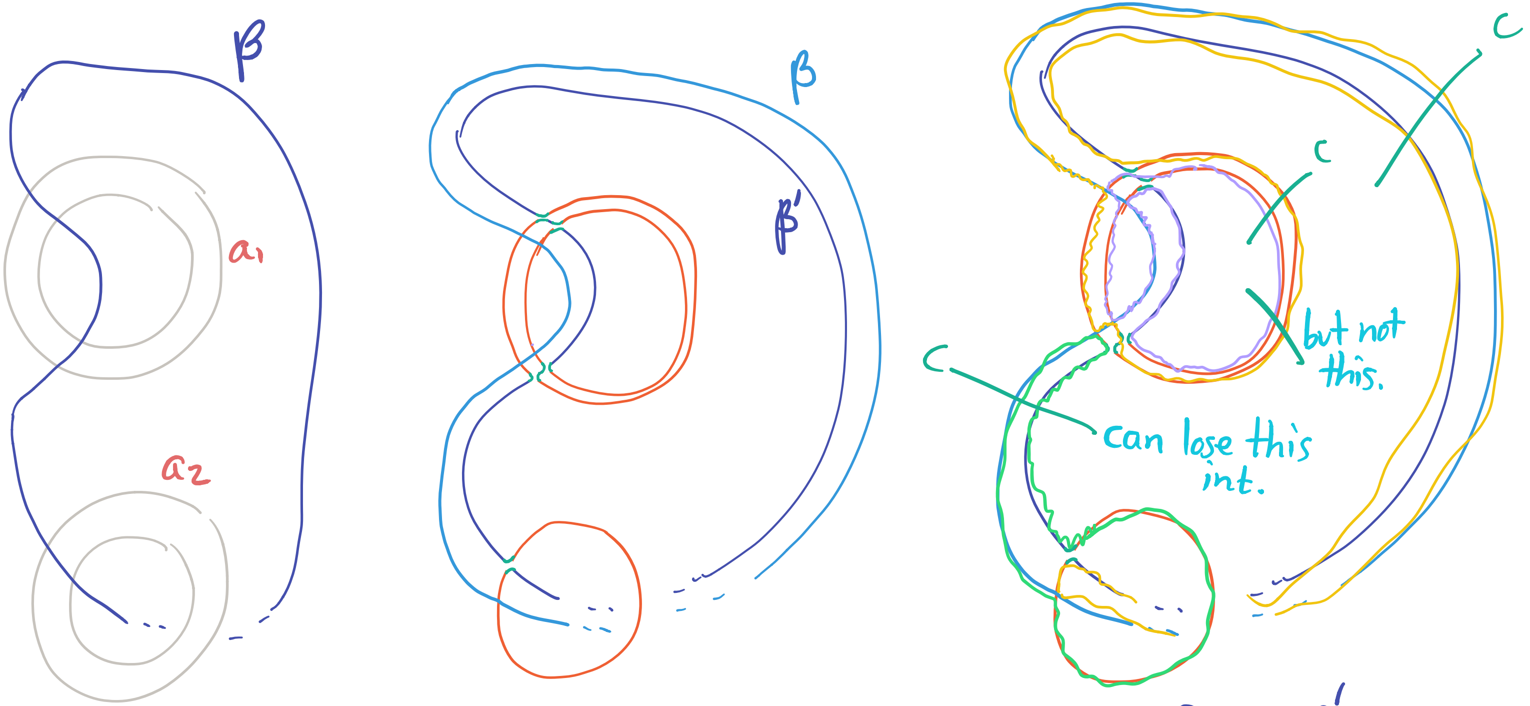
$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \sum_{i=1}^{n} e_i \, i(a_i, b) i(a_i, c) \right| \leq i(b, c)$$



Pf. Make a rep β' of $M(b)$ as before:

Key obs: $\beta \cup \beta'$ can be decomp. as $\sum e_i \, i(a_i, b)$ copies of each a_i



Zig-zag: Turn left on β'
right on β

As above: $\boxed{\beta \cup \beta' \text{ is a bunch of copies of } a_i :}$

$\forall i$: $e_i \, i(a_i, b)$ copies of a_i

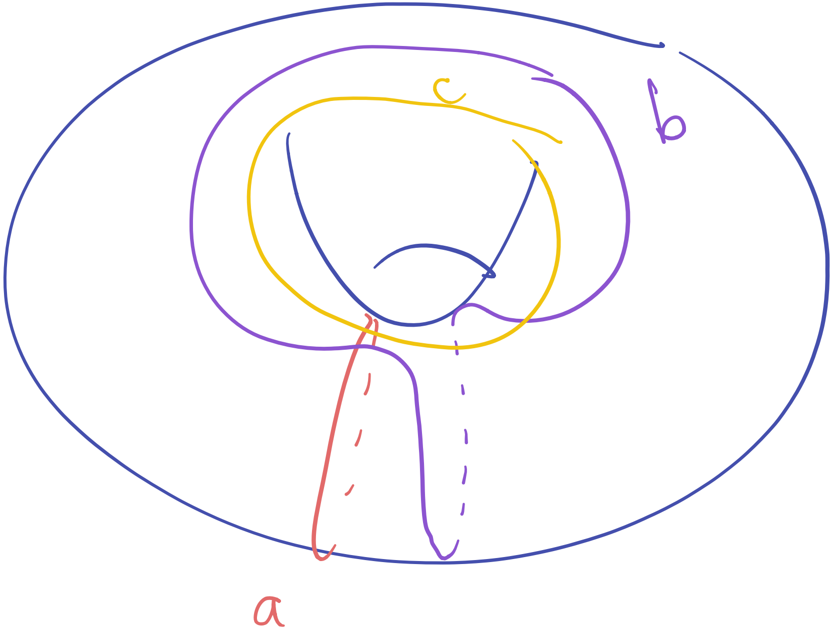
rep of c.

$$\sum e_i \, i(a_i, b) \, i(a_i, c) \le |(\beta \cup \beta') \cap \gamma|$$

$$= \underbrace{i(M(b), c) + i(b, c)}_{\#\text{ of int's you see in pic.}}$$

by fact at top

Need to prove other ineq.



$i(T_a(b), c) = 0.$

<u>expected</u>

$1 \cdot 1 \cdot 1 = 1.$

Relations b/w 2 Dehn twists

Prop. $i(a,b)=1 \overset{\text{top.}}{\Longrightarrow}$ alg.

$$T_a T_b T_a = T_b T_a T_b$$

"braid relation"

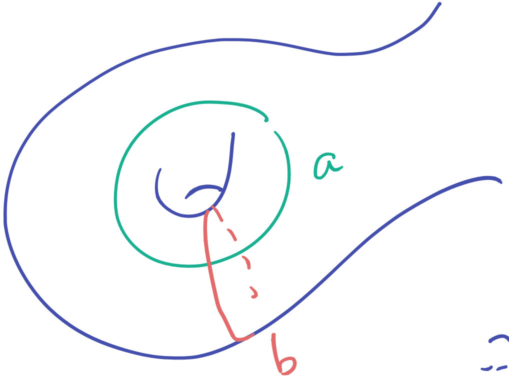
Pf. $(T_a T_b) T_a = T_b (T_a T_b)$

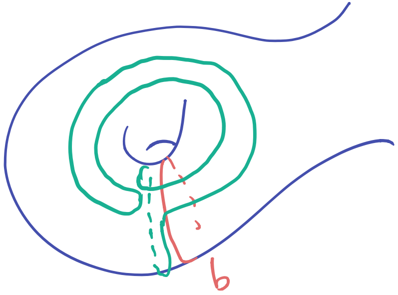
$\Longleftrightarrow (T_a T_b) T_a (T_a T_b)^{-1} = T_b$

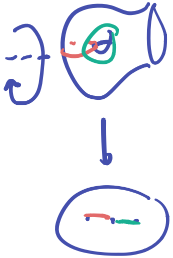
$\Longleftrightarrow T_{T_a T_b(a)} = T_b$

$\Longleftrightarrow T_a T_b(a) = b$

Change of coords







Converse!

Prop. $T_a T_b T_a = T_b T_a T_b$, $a \neq b$

$$\implies i(a,b) = 1.$$

Pf. $T_a T_b T_a = T_b T_a T_b$

$$\implies T_a T_b(a) = b$$

(as above).

So: $i(a,b) = i(a, T_a T_b(a))$

$= i(a, T_b(a))$

$= i(a,b)^2$

$\implies i(a,b) = 0$ or $1 \ldots$ $\blacksquare$

Application

Given $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_g)$

If you can show

$$T_a \longrightarrow T_{a'}$$

Then curves $\longrightarrow$ curves

$$a \longmapsto a'$$

$$i(a,b) = 1 \longmapsto i(a',b') = 1.$$

Next: $\langle T_a, T_b \rangle$ $\forall a,b$.

Ping Pong Lemma

$G \circlearrowright X$ = set.

$g_1, g_2 \in G$

$X_1, X_2 \subseteq X$ nonempty disj.

$g_i^k(X_j) \subseteq X_i \quad i \neq j \quad k \neq 0.$

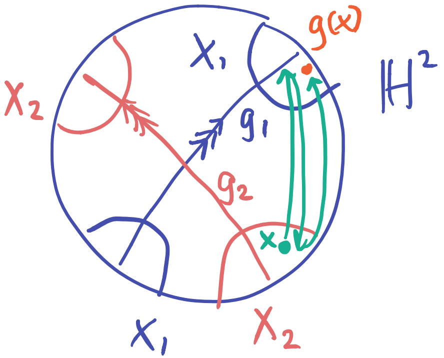
$\Longrightarrow \langle g_1, g_2 \rangle \cong F_2$

Pf. Let $g \in \langle g_1, g_2 \rangle$

WLOG (by conj)

$g = g_1^* g_2^* g_1^* \cdots g_2^* g_1^*$

Original source/application

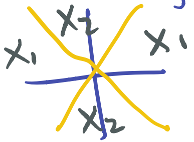


Second application:

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \cong F_2$$

$X_1 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : a > b \right\}$

X_2 similar.



Prop. $i(a,b) > 1$

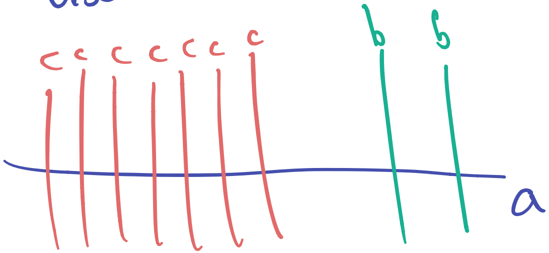
$$\Rightarrow \langle T_a, T_b \rangle \cong F_2.$$

Pf. Ping pong.

$X_1 = \{c : i(c,b) > i(c,a)\}$

X_2 similar.

Use our i-num. formulas.



In general: $j,k \neq 0$

$$\langle T_a^j, T_b^k \rangle \cong F_2$$

unless

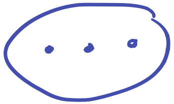
$i(a,b) = 1$ and

$\{j,k\}$ is $\{1,1\}$ $\{1,3\}$ $\{1,2\}$

$a = \sigma_1^2$, $b = \sigma_2$

$112112 = 211211$

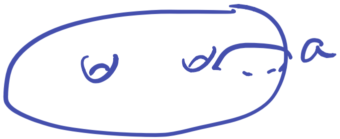
$abab = baba$



More Dehn twists: J. Mortada

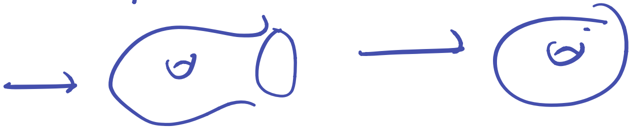
Cutting, capping, including

Later: want to prove things by induction, hence understand $\mathrm{Mod}(S_g, a)$



Cut along a:

Cap:

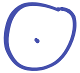


Including

Prop $S \subseteq S'$ no compl. disks

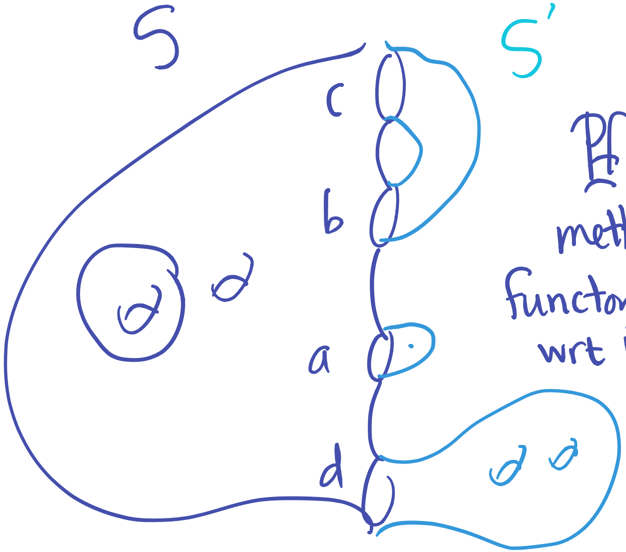
S closed in S'

$S \neq A$

a_i comp's of ∂S bounding 

$\{b_i, c_i\}$ bound 

$$\ker(\mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S')) = \langle T_{a_i}, T_{b_i} T_{c_i}^{-1} \rangle \cong \mathbb{Z}^k$$

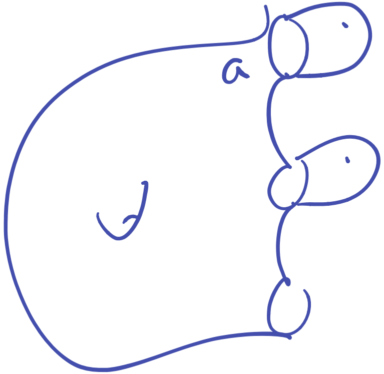


Pf. Alex. method is functorial wrt inclusion of surfaces.

Kernel:

$$\langle T_a, T_b T_c^{-1} \rangle \cong \mathbb{Z}^2$$

Capping. Special case where $S \setminus S' =$  $\quad S \neq A$

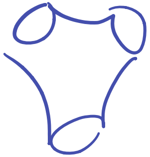


$\ker = \langle T_{a_i} \rangle \cong \mathbb{Z}^k$

$P \hookrightarrow S_{0,3} =$ 

$\to \mathrm{Mod}(P) \longrightarrow \mathrm{PMod}(S_{0,3}) = 1$

$\ker \; \mathbb{Z}^3$

$P =$ 

Applications

$\mathrm{Mod}(P) \cong \mathbb{Z}^3$

$\mathrm{Mod}(S_1^1) \cong \langle a, b \mid aba = bab \rangle$

$\cong \pi_1(S^3 \setminus$  $) \quad \cong B_3$

$\cong \widetilde{SL_2 \mathbb{Z}}$

Cutting

$S = S_{g,n}$

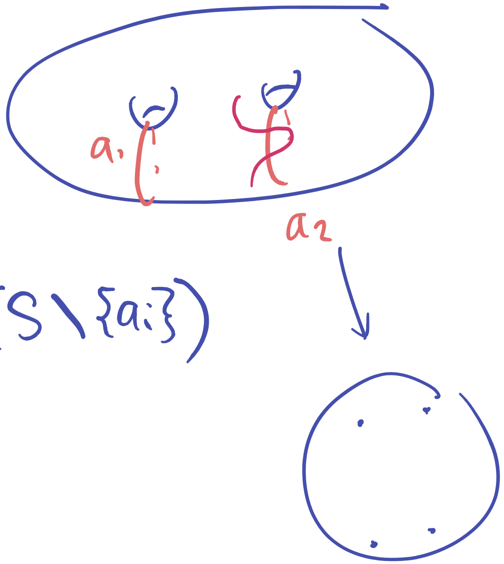
$a_1, \ldots, a_k$ distinct, disjoint

There is a well-def map

Stab of $\{a_i\}$

$$\mathrm{Mod}(S, \{a_i\}) \longrightarrow \mathrm{Mod}(S \setminus \{a_i\})$$

with kernel $\langle T_{a_i} \rangle$



roughly

Pf. Apply inclusion homom to

$$S - \mathrm{Nbd}(\cup a_i) \hookrightarrow S$$

Q. Given $a_1, \ldots, a_k$

When is $\langle T_{a_1}^{e_1}, \ldots, T_{a_k}^{e_k} \rangle$ free?

(Hamidi-Tehrani)

When is it a RAAG? (Runnels + refs)

Q (Afton) For which $G \leq MCG$

$\exists\ c, k$ s.t. $\langle G, T_c^k \rangle \cong G * \mathbb{Z}$

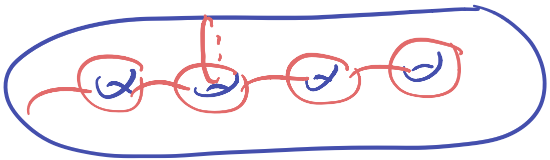
Q. When is it $= MCG \ldots$

Chap 4. Generation.

fixing marked pts

Thm. $\mathrm{PMod}(S_{g,n})$ is finitely gen. by Dehn twists about nonsep curves.

Humphries:



$2g+1$

(minimal)

Application (later today?):

Every closed, orientable M^3 obtained from S^3 by Dehn surgery.

Application (next week?)

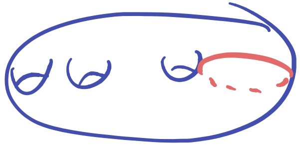
$$H_1(\mathrm{Mod}(S_g)) = 0$$

Thm. $PMod(S_{g,n})$ is finitely gen. by Dehn twists about nonsep curves.

Proof strategy

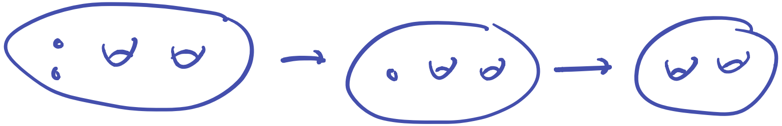
① Induction on genus:

Mod(S_g) is gen. by stabilizers of nonsep curves



"complex of curves"

② Induction on punctures.



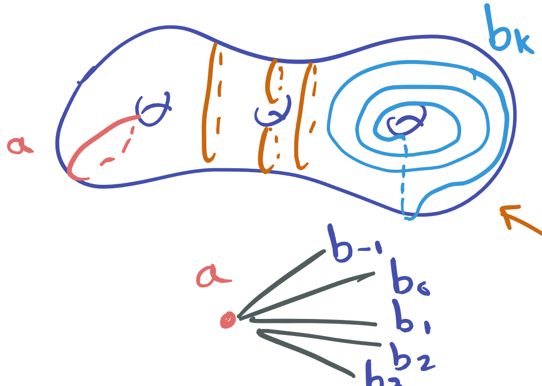
"Birman exact sequence"

Complex of curves (Harvey)

$C(S)$ has

vertices: isotopy classes of ess. s.c.c. in S

edges: disjointness.



5-clique

Facts

1. locally infinite
2. connected (next!)
3. $\mathrm{Mod}^{\pm}(S) \xrightarrow{\cong} \mathrm{Aut}(C(S))$ (Ivanov)

   applications...

   $\mathrm{Aut}\,\mathrm{Mod}(S_g) \cong \mathrm{Mod}^{\pm}(S_g)$
   $\cong \mathrm{Isom}\,\mathrm{Teich}(S_g)$
4. $C(S)$ is hyperbolic

   many applications...

& ∞-diameter.

exercise: find vertices of distance 3, 4, ...

Thm. $3g+n>5$

$\mathcal{C}(S_{g,n})$ is connected.

Pf. Induct on $i(a,b)$.

(Say $n=0$)

Base cases:

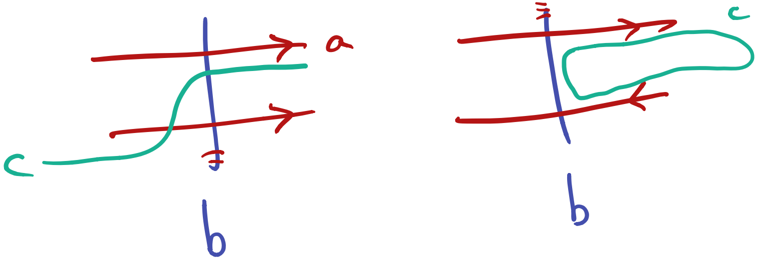
$i(a,b)=0$ ✓

$i(a,b)=1$ ✓ change of coords.

Assume $i(a,b)\geq 2$.

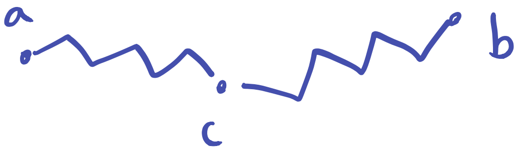
Orient a.

Two pictures:



Check: ① c essential

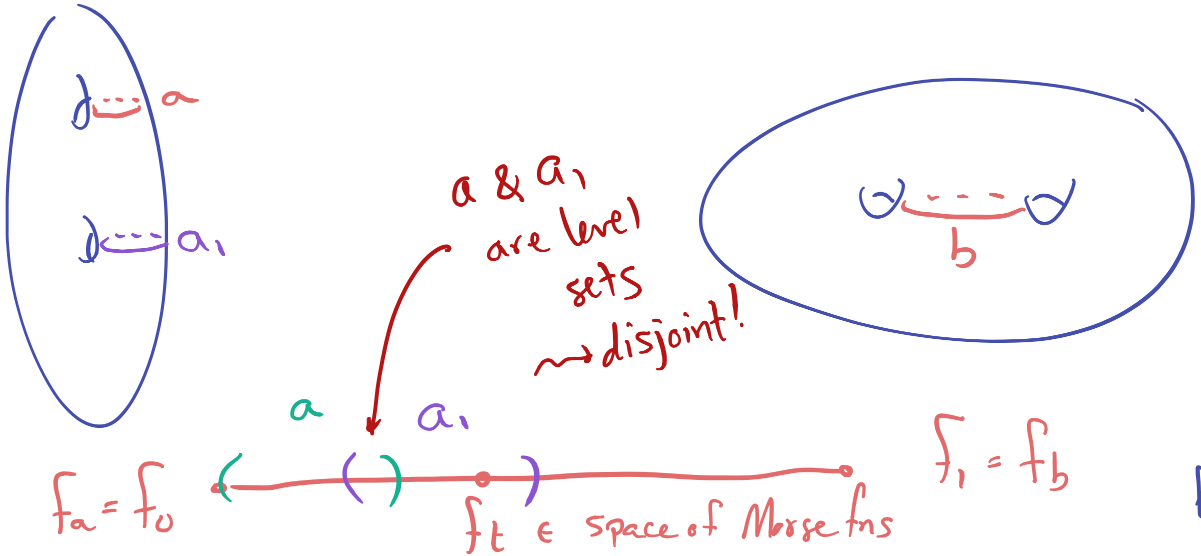
② $i(a,c), i(b,c) < i(a,b)$



$\blacksquare$

Cerf theory proof (Ivanov)

Given a, b. Choose Morse fns f_a, f_b s.t. a, b level sets on S.



∎

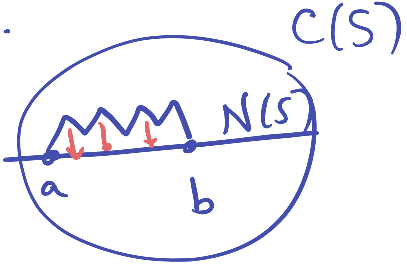
Complex of Nonsep curves

$N(S)$ = subcomplex of $C(S)$ spanned by nonseps.

Thm $N(S_g)$ connected $g > 1$.

Note. $N(S_{1,n})$ not connected!

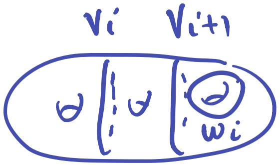
Pf of Thm.

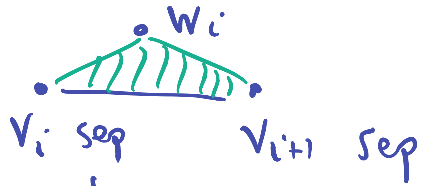


$a, b \in N(S)$

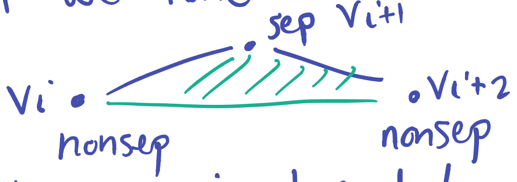
Connect by path v_i in $C(S)$.

Can assume no consec. v_i are sep.





If we have



either: sep is not needed.
or can replace with a nonsep.

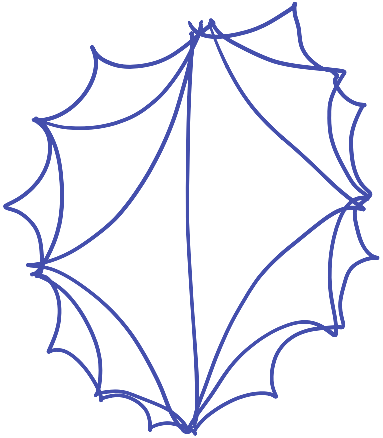
Modified complex $\hat{N}(S)$

Same vertices as $N(S)$

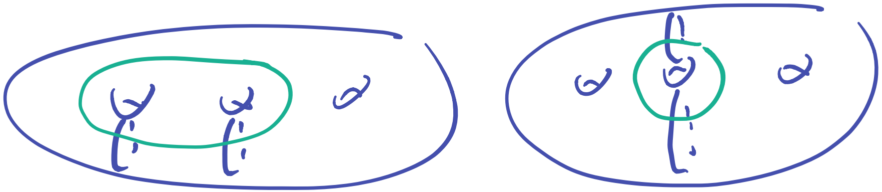
edges: $i(a,b) = 1$.

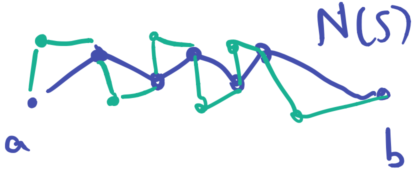
Thm. $\hat{N}(S_g)$ connected $g \geq 1$.

$g=1$



Pf of Thm





∎

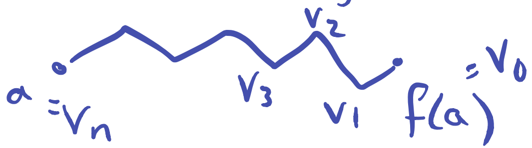
Prop. $\mathrm{Mod}(S)$ is gen. by stabilizers of (oriented) nonsep. s.c.c.

(Induction on genus).

Pf. Let $f \in \mathrm{Mod}(S_g)$

a = nonsep curve.

$\hat{N}(S_g)$



For each i :

$$T_{v_i} T_{v_{i+1}}(v_i) = v_{i+1} \quad \text{(braid reln)}$$

So $\left(\prod T_{v_{i_j}}\right) f = \bar{f} \in \mathrm{Stab}(a)$

all twists stabilize some nonsep curve.

$\Rightarrow f \in \langle$ stabilizers of nonsep curves $\rangle$

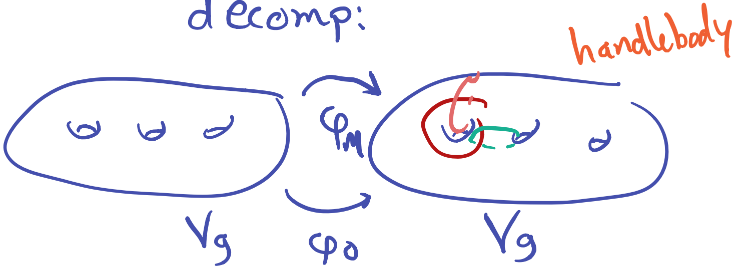
$\in \langle$ Dehn twists about nonseps, $\mathrm{Stab}(a) \rangle$

Thm (Waldhausen) M^3 = closed, oriented 3-man

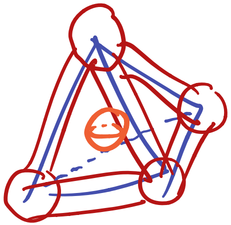
Then M^3 obtained from S^3 by $\underbrace{\text{Dehn surgery}}$

↳ remove disjoint collection of solid tori, reglue.

Pf. **Step 1**. M^3 has a Heegaard decomp:



Why? Triangulate M^3.
Thicken 1-skeleton.
That's one V_g.
The complement is other.



Step 2. Use fact that $\mathrm{Mod}(S_g)$ is gen by Dehn twists

M^3 has Heeg. decomp with φ_M
S^3 has - - - - with φ_0

$\varphi_M \varphi_0^{-1} \in \mathrm{Homeo}(S_g)$
$\shortparallel$
product of T_a

Thm (Dehn '22)

$\mathrm{PMod}(S_{g,n})$ is fin. gen. by Dehn twists.

Lickorish '60s: nonsep. curves.

Humphries '70s: $2g+1$ curves. (minimal)

Pf sketch. Let $f \in \mathrm{PMod}(S_{g,n})$

Choose some curve a.

Step 1 Find $\prod T_{c_i}$ s.t.

$$\prod T_{c_i} f(a) = a$$

(with orientation)

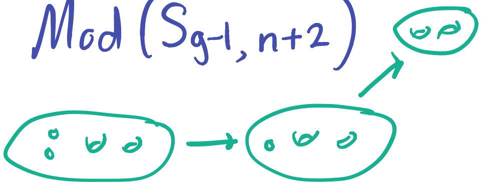
how to get fin. gen. here?

$C(S)$ is conn.

Step 2. $\mathrm{Stab}(a)$ fin. gen. by Dehn tw's.

cutting homom. $\hookrightarrow \mathrm{Mod}(S_{g-1,n+2})$

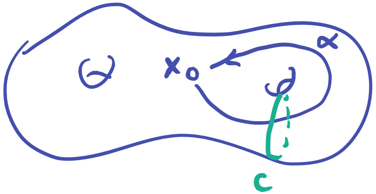
Birman exact seq.



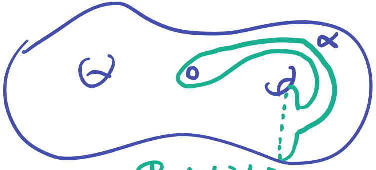
Towards Birman ex. seq

Push map: $\pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x)$

isotopy extension



$\xrightarrow{\mathrm{Push}(\alpha)}$



Not obviously well def.

Forgetful map: $\mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S)$

Note: $\mathrm{Push}(\pi_1(S,x)) \subseteq \ker(\mathrm{Forget})$

Birman: this is $=$

$\ker(\mathrm{Forget}) \longrightarrow \pi_1(S,x)$

Given φ choose a homotopy to id, so x traces a loop.

Thm (Birman '69) $\chi(S)<0$ This is exact:

$$1 \longrightarrow \pi_1(S,x) \xrightarrow{\text{Push}} \mathrm{Mod}(S,x) \xrightarrow{\text{Forget}} \mathrm{Mod}(S) \longrightarrow 1$$

Pf. This is a fiber bundle:

$$\begin{array}{c} \mathrm{Homeo}^+(S,x) \longrightarrow \mathrm{Homeo}^+(S) \\ \qquad\qquad\qquad\qquad \downarrow \varepsilon = \text{eval at } x \\ \qquad\qquad\qquad\qquad U \subseteq S \end{array}$$

Choose $U \in S$, $x \in U$

$\forall\, u \in U$, choose φ_u s.t. $\varphi_u(x) = u$

vary continuously wrt u.

$$U \times \mathrm{Homeo}^+(S,x) \longrightarrow \varepsilon^{-1}(U)$$
$$(u, \psi) \longmapsto \varphi_u \circ \psi$$

What is a fiber bundle?

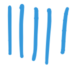
$$\begin{array}{ccl} F & \rightarrow & E = \text{total sp.} \\ & & \downarrow p \\ \text{"fiber"} & & U \subset B = \text{base sp.} \end{array}$$

Locally: $p^{-1}(U) = U \times F$

examples:

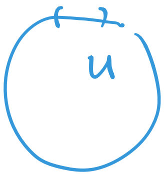
E = cylinder or Möbius band

$B = S^1$, $F = I$



(u,x)

$x \in I$



S^1

For E = cov space, F = discrete set

This is a fiber bundle:

$$\begin{tikzcd} \mathrm{Homeo}^+(S,x) \arrow[r] & \mathrm{Homeo}^+(S) \arrow[d, "\varepsilon \,=\, \text{eval at } x"] \\ & S \end{tikzcd}$$

$\rightsquigarrow$ LES for fiber bundles.

(Hamstrom) $\chi(S) < 0$.

$$\cdots \longrightarrow \pi_1 \mathrm{Homeo}^+(S,x) \longrightarrow \cancel{\pi_1 \mathrm{Homeo}^+(S)} \nearrow 1 \longrightarrow \pi_1 S \checkmark$$

this is the push map

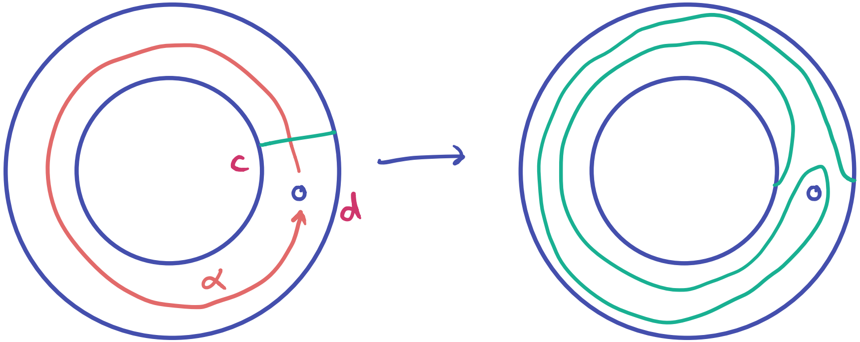
$$\longrightarrow \pi_0 \mathrm{Homeo}^+(S,x) \longrightarrow \pi_0 \mathrm{Homeo}^+(S) \longrightarrow \cancel{\pi_0 S} \nearrow 1$$

$$\pi_0 \mathrm{Homeo}^+(S,x) = \mathrm{Mod}(S,x), \qquad \pi_0 \mathrm{Homeo}^+(S) = \mathrm{Mod}(S)$$

this is the forgetful map.

$\square$

Push maps in terms of Dehn twists

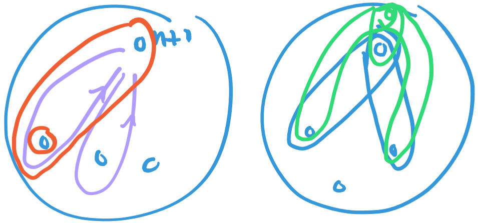


$$\mathrm{Push}(\alpha) = T_c T_d^{-1}$$

Helps because $\pi_1(S)$ is generated by simple loops.

Special case. $\mathrm{PMod}(S_{0,n})$ is fin. gen. by Dehn twists.

$\binom{n}{2}$ gens.



Pf. Ind. on n.

Base cases: $\mathrm{PMod}(S_{0,n}) = 1 \quad n \leq 3$.

Ind step:

fin gen

$$1 \longrightarrow \pi_1(S_{0,n}) \xrightarrow{\text{Push}} \mathrm{PMod}(S_{0,n+1}) \longrightarrow \mathrm{PMod}(S_{0,n}) \to 1$$

(image of) each gen is a product of ~~2~~ 1 Dehn twists

Fin. Gen by Dehn twists by induction. Each has a lift that is a Dehn twist

$\square$

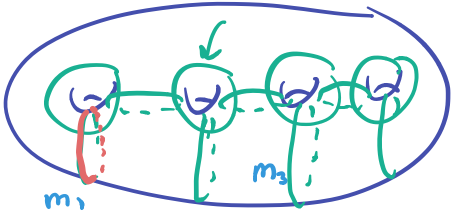
Same argument gives step ② for Dehn's thm

In a short ex. seq, middle gp is gen by: gens on left & lifts of gen on right

Explicit sets of gens

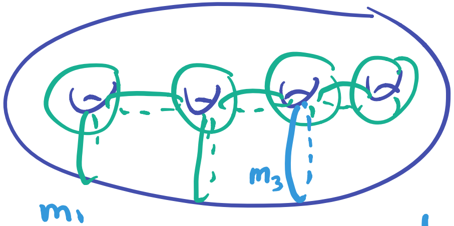
$\underbrace{h\,\mathrm{Push}(\alpha)h^{-1}} = \mathrm{Push}(h\alpha)$

Lickorish:



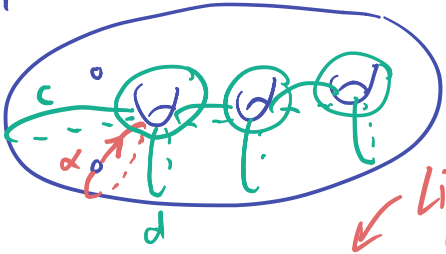
$3g-1$

Humphries:



$2g+1$

Ind step:



Lick. gens!

$\mathrm{Push}(\alpha) = T_c T_d^{-1}$

Use Lick gens to take α to other gens for π_1

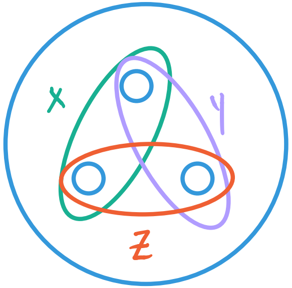
Need to find product h of Hump. gens taking m_1 to m_3

$hT_{m_1}h^{-1} = T_{m_3}$

Chap 5. Presentations & H_1, H_2

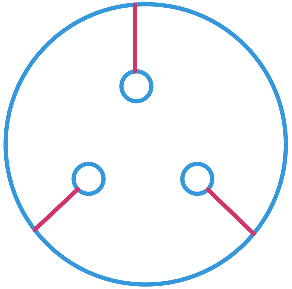
Lantern Relation $\quad S_{0,4} \subseteq S$

$$T_x T_y T_z = \prod T_{\partial_i}$$

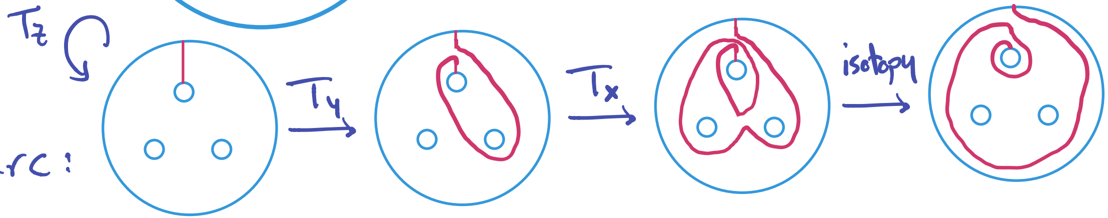


Pf #1 Alex Method:

Check relation on 3 arcs.

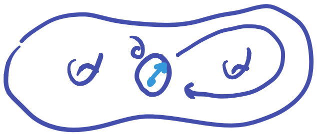


We'll do one arc:



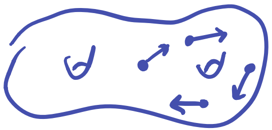
Pf #2 Boundary pushing $\chi(S) < 0$ $S^\circ = S \setminus$ open disk.

$$\text{Push} : \pi_1 UT(S) \longrightarrow \text{Mod}(S^\circ)$$



gen. of $\pi_1(\text{fiber}) \longmapsto T_\partial$

$\underline{\pi_1 UT(S)}$

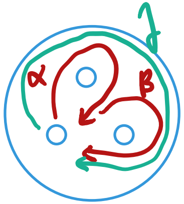


$$\begin{array}{c} S^1 \to UT(S) \\ \downarrow \\ S \end{array}$$

The lantern relation is:

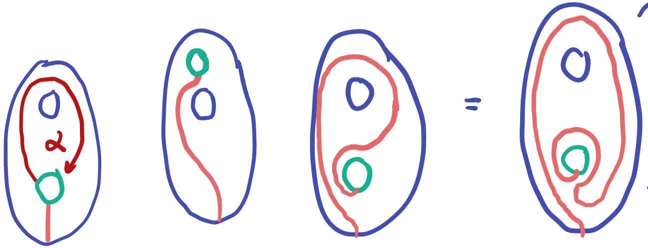
$\text{Push}(\beta)\text{Push}(\alpha) = \text{Push}\,\gamma$

push w/o rotating



$\beta\alpha = \gamma$ in $\pi_1(UT(S))$.

Why is this lantern relation?



$\text{Push}\,\alpha = T_x T_{\partial_1}^{-1} T_{\partial_2}^{-1}$

Thm $H_1(\mathrm{Mod}(S_g)) = 1$
$g \geq 3$.

$H_1(G) \cong G^{\text{abel.}} = G/[G,G]$

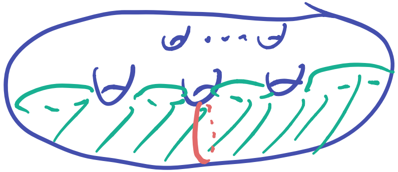
So: no char. classes for S_g-bundles over S^1.

Pf. **Fact 1.** $\mathrm{Mod}(S_g)$ gen by T_c, c nonsep (Dehn-Lick.)

Harer + Mumford

Fact 2. Such T_c are conjugate (Change of coords)

Fact 3. $\exists$ lantern reln in S_g w/ all 7 curves nonsep.



(?!)

Given $\mathrm{Mod}(S_g) \longrightarrow A$
gens $T_{c_i} \longmapsto t$ by fact 2

by fact 1, Image is $\langle t \rangle$ (cyclic)

Fact 3: $t^3 = t^4 \Rightarrow t = 1$.

Presentations

We have them (see book)

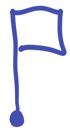
Next goal: proof of fin. presentability.

fin generation $\longleftrightarrow$ action on connected complex with finite quotient.

$H_2(G)$ $\longleftrightarrow$ fin presentability $\longleftrightarrow$ . . . . . simply connected . . .

(abelianized version)

Arc complex $\mathcal{A}(S)$

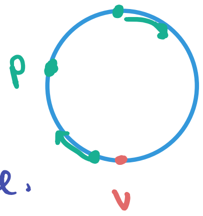


vertices : arcs $/\sim$

edges : disjointness

k-simplices: $(k+1)$ pairwise disjoint arcs.

(flag complex)



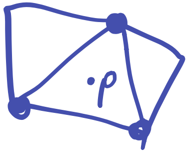
Thm. $\mathcal{A}(S_{g,n})$ contractible.

Pf (Hatcher) v = any vertex

Goal: homotope $\mathcal{A}(S_{g,n})$ into

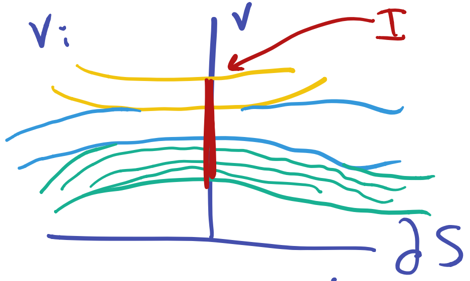
union of all simp. cont. v $\longrightarrow$ Star$(v) \simeq *$ so paths vary contin.

Let $p \in |\mathcal{A}(S_{g,n})|$

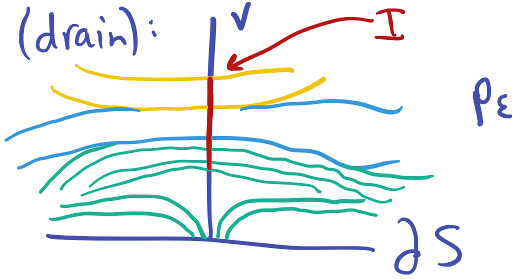


p = weighted sum of disjoint arcs

thicken arcs to bands, push together at v:



Homotopy (drain):



Prop. Say $G \curvearrowright X \simeq *$ w/o rotations 

$\Rightarrow$ $\mathrm{Stab}(e) \subseteq \mathrm{Stab}(v)$

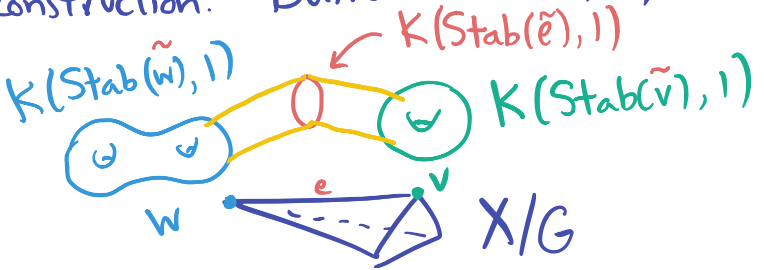
& ① X/G finite

② vertex stabs f.p.

③ edge stabs f.g.

Then G is f.p.

Pf idea Borel construction: Build a $K(G,1)$ for G.



Thm. $\mathrm{Mod}(S_{g,n})$ fin pres.

Pf for $n>0$

Apply Prop.

Stab's are MCG's of simpler surfaces.

$\rightsquigarrow$ induction! $\square$

For $n=0$:

$$1 \to \pi_1(S_g) \to \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g) \to 1$$

Quotient of a fp gp by a fg gp is f.p.

Also, alg. geom. proof: $\mathcal{M}_{g,n}$ is a quasi-proj variety.

Q. What presentation do you get from this proof?

Last time: $H_1(\mathrm{Mod}(S_g);\mathbb{Z}) = 0$. $g \geq 3$

Today (and next time?):

$$H_2(\mathrm{Mod}(S_g);\mathbb{Z}) = \mathbb{Z}$$

$$H_2(\mathrm{Mod}(S_g^1);\mathbb{Z}) = \mathbb{Z}$$

$$H_2(\mathrm{Mod}(S_{g,1});\mathbb{Z}) = \mathbb{Z}^2$$

$g \geq 4$

surface bundles over surfaces

Upshot: $\exists$ alg. top which tells us a surf. bundle over surf is nontrivial.

Univ. coeff thm: Same answers for H^2 since

$$1 \to \cancel{\mathrm{Ext}(H_1(\mathrm{Mod}(S_g)))}^{\,0} \to H^2(\mathrm{Mod}(S_g)) \to \mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z}) \to 1$$

$$\mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z}) = H_2(\mathrm{Mod}(S_g);\mathbb{Z})/\text{torsion}$$

Overall Strategy

1. Upper bounds on H_2 using Hopf formula à la Pitsch
2. Lower bounds on H^2 by constructing two indep. classes: Meyer sig cocycle, Euler class.

Hopf Formula

Recall: $H_1(G) = G/[G,G]$ $H_2(G) = H_2(K(G,1))$ by defn

$G = \langle F \mid R \rangle \cong F/K$ $K = \langle\langle R \rangle\rangle$

$\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle$

$$H_2(G;\mathbb{Z}) \cong \underbrace{K \cap [F,F]}_{} / [K,F]$$

relns that are prod's of commutators

conjugate relations are equivalent.

surfaces

e.g. commuting elts $\leftrightarrow T^2$

Given

$\pi_1(S_g) \longrightarrow G$

$\rightsquigarrow S_g \longrightarrow K(G,1)$

" $K(\pi_1(S_g),1)$

So: $H_2(G) \leq K/[K,F]$ ← abelian, gen. by relations R.

So: an elt of $H_2(G)$ looks like $r_1^{n_1} r_2^{n_2} \cdots r_N^{n_N}$

Pitsch: For $G = MCG$, at most one choice of $(n_1, \ldots, n_N)$.

Hopf formula and MCG

For $\mathrm{Mod}(S_g^1)$ an elt of H_2 is of form

$$\left(\prod D_{ij}^{n_{ij}}\right)\left(\prod B_i^{n_i}\right) C^{n_0} L^{n_L}$$

disjointness braid chain lantern.

Will show: $n_{ij} = 0$, $n_i = 0$ i large

Hopf formula & commuting elts

For $g, h \in G$ $\quad g \leftrightarrow h$

$\rightsquigarrow \{g,h\}$ = class of $[g,h]$ in H_2 (think torus)

Fact 1. If $g \leftrightarrow h, k$ then

$$\{g, hk\} = \{g,h\} + \{g,k\}$$

since $[x,yz] = [x,y][x,z]^y$ ← conj. by y.

Fact 2. $\{g, h^{-1}\} = -\{g,h\}$

Back to MCG

Lemma. $T_a \leftrightarrow T_b$

$\Rightarrow \{T_a, T_b\} = 0$ in $H_2(\mathrm{MCG})$

Pf. Cut S along a

$H_1(\mathrm{Mod}(S \setminus a)) = 0$,

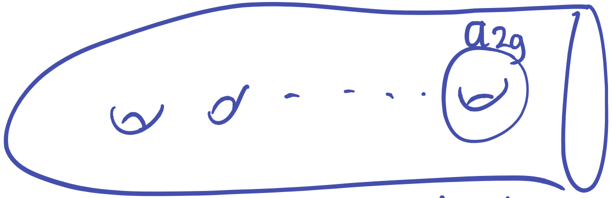
So $T_b = \prod [x_i, y_i]$

with $x_i, y_i \leftrightarrow T_a$ in $\mathrm{Mod}(S_g)$

Apply Facts 1 & 2:

$\{T_a, \prod [x_i,y_i]\} = 0$. $\square$

Eliminating more relations



The MCG gen $T_{a_{2g}}$ only appears in ~~disjointness relns &~~ one braid rel.

In that braid reln it appears with exponent sum = 1.

Q. Can we show this class is nonzero?

But... elts of $[F,F]$, hence H_2, have all exp sums $= 0$.

So $n_{2g} = 0$.

Q. Can we show H_3 is stable using similar idea?

Now have a finite lin. alg problem involving chain reln, lantern reln, a few braid relns:

Which choices of $\underbrace{n_0, n_1, n_2, n_3, n_4}_{\text{exps on braid relns}}, n_c, n_L$ make it so each MCG gen appears with exp sum 0?

Answer: 1 choice!

Lower bound: Constructing nonzero elts of H^2

Fact. A short exact sequence

$$1 \to \mathbb{Z} \to \tilde{G} \to G \to 1$$

with $\mathbb{Z}$ central gives $e \in H^2(G;\mathbb{Z})$

and $e = 0 \iff$ seq. is split

$\iff \tilde{G} \cong G \times \mathbb{Z}$

But we have: $1 \longrightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{\langle T_\partial \rangle}} \longrightarrow \mathrm{Mod}(S_g^1) \xrightarrow[\longleftarrow]{\text{cap } \partial} \mathrm{Mod}(S_{g,1}) \longrightarrow 1$

Non-split since $\mathrm{Mod}(S_{g,1})$ has torsion $\rightsquigarrow$ $e \in H^2(\mathrm{Mod}(S_{g,1});\mathbb{Z})$.

Euler class

Meyer Signature Cocycle

Still need an elt of $H^2(\mathrm{Mod}(S_g);\mathbb{Z})$.

Q. What is an elt of H^2? A. $\mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z})$.

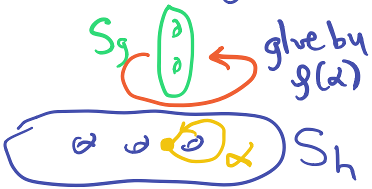
So, given elt of $H_2(\mathrm{Mod}(S_g))$, need a number.

Q. What is an elt of $H_2(\mathrm{Mod}(S_g))$? A. Surface in $K(\mathrm{Mod}(S_g),1)$

$$S_h \to K(\mathrm{Mod}(S_g),1)$$

$$\rho: \pi_1(S_h) \to \mathrm{Mod}(S_g).$$

The latter gives S_g-bundle over S_h (4-manifold):



4-manifolds have signature (describes intersection form on $H_2(M^4)$).

Signature is the desired number!

Ch 6. Symplectic rep.

$\hat{i} : H_1(S_g;\mathbb{Z}) \times H_1(S_g;\mathbb{Z}) \to \mathbb{Z}$

Can replace $\mathbb{Z}$ with $\mathbb{R}$

$\hat{i}$ is alternating, bilinear, nondegen.

$\forall\, x \neq 0\ \exists\, y$ s.t. $\hat{i}(x,y) \neq 0$.

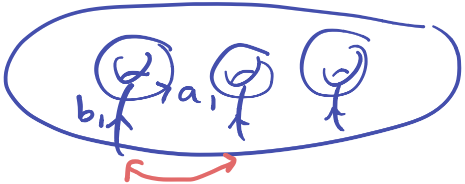
"symplectic"

Symplectic basis for $H_1(S_g;\mathbb{Z})$:

x_i, y_i

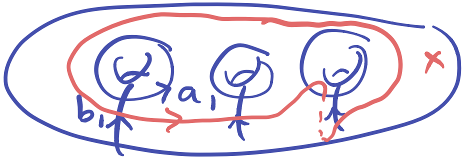
$\hat{i}(x_i,y_i) = 1$ all other $\hat{i}$'s 0.

A geometric symplectic basis in S_g is a set of oriented curves $\{a_i, b_i\}$ s.t. $i(a_i,b_i) = 1$, all other i's 0



and $\{[a_i],[b_i]\}$ is a sympl. basis for $H_1(S_g;\mathbb{Z})$.

Aside: computing homology classes



$$x = a_1 + a_2 + a_3 + b_3$$

Since $\hat{i}(x, b_1) = 1$, the coeff on a_1 is 1.

i.e. $a_i^* = b_i$

Euclidean alg. for curves

not a $\mathbb{Z}$ multiple

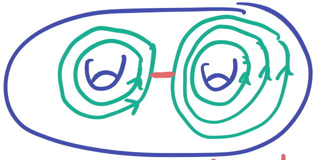
Prop. A nonzero elt of $H_1(S_g;\mathbb{Z})$ is rep by a scc $\iff$ it is primitive

Pf. ($\Rightarrow$) Change of coords.

($\Leftarrow$) Example. $(2,0,3,0)$ in $H_1(S_2;\mathbb{Z})$

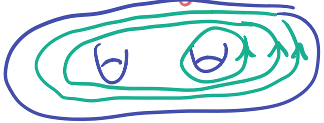
$$2x_1+3x_2$$

Start with:



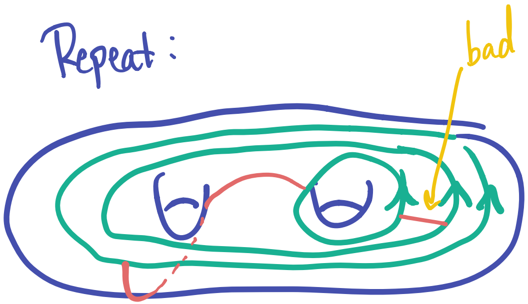
Choose arc connect right-hand sides

Surger



Same H_1 class!

Repeat:



\# curves in the two "bundles":

Step	1^{st} bundle	2^{nd} bundle
0	2	3
1	2	1
2	1	1
3	1	0

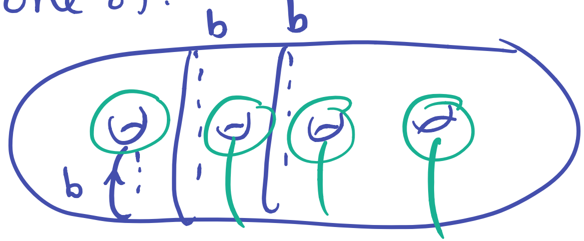
What we wanted!

The symplectic rep

$$\Psi : \mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z}$$
$$\qquad\qquad\qquad\qquad \shortparallel$$
$$\qquad\qquad\qquad \mathrm{Aut}(H_1(S_g;\mathbb{Z}); \hat{\imath})$$

__Prop.__ $\Psi(T_b^k)[a] = [a] + k\,\hat{\imath}(a,b)[b]$

__Pf.__ By change of coords, b is one of:



Choose a compatible geom. sympl. basis

Check the formula on the basis. $\square$

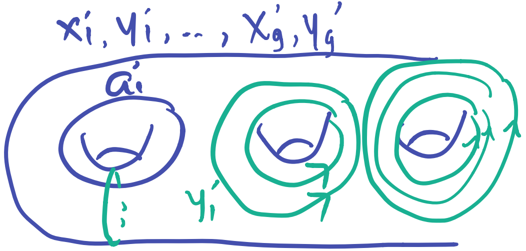
We see: Ψ has kernel.

e.g. T_b, b sep.

$\mathrm{Ker}\,\Psi$ called Torelli gp (Monday).

Surjectivity

$$\boxed{\psi : \mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z}}$$



Thm. ψ is surjective.

Pf #1. Hit the "elementary matrices"

Pf #4. Hit the transvections: $T_v(w) = w + \hat{i}(v,w)\,v$

$$Sp_{2g}\mathbb{Z} = \langle T_v : v \text{ prim} \rangle$$ fixed set codim 1.

Find T_c s.t. $\psi(T_c) = T_v$ using Eucl. alg.

Pf #3. Given $M \in Sp_{2g}\mathbb{Z}$, $M(\text{std basis}) = $ symplectic basis B.
Can soup up Euc. alg to get a <u>geom</u>. symp. basis $\tilde{B}$ representing B. By C of Coords $\exists\, f \in \mathrm{Mod}(S_g)$ $f\left(\begin{smallmatrix}\text{std}\\ \text{geom}\\ \text{basis}\end{smallmatrix}\right) = \tilde{B}$.

Residual finiteness

G is resid. fin if $\bigcap\limits_{\substack{\Gamma \leq G \\ \text{f.i.}}} \Gamma = 1$

or: $\forall f \overset{\neq 1}{\in} G \ \exists$ finite F, $\rho : G \to F$
s.t. $\rho(f) \neq \text{id}$.

Thm. $\text{Mod}(S_g)$ is resid. finite.

Pf. $g = 0, 1$ easy.
$\psi(f) \neq \text{id} \rightsquigarrow$ use rf'ness of $Sp_{2g}\mathbb{Z}$.
Remains to deal with $f \in \text{Torelli} = \ker(\psi)$.

Fact: $\ker\psi$ is torsion free.

Assume now $|f| = \infty$. Want finite F, $\rho : \mathrm{Mod}(S_g) \to F$, $\rho(f) \neq 1$.

Choose a hyp. metric on $S_g \rightsquigarrow \rho : \pi_1(S_g) \to PSL_2\mathbb{R} = \mathrm{Isom}^+\mathbb{H}^2$

$\mathrm{Im}\,\rho \leq PSL_2 A$ $\quad A$ = fin gen subring of $\mathbb{R}$.

Such A is res. finite. (black box)

length of curves $\longleftrightarrow$ traces of elts of $PSL_2\mathbb{R}$

$|f| = \infty \Rightarrow \exists\, \gamma \in \pi_1(S_g)$ s.t. $\ell(\gamma) \neq \ell(f(\gamma)) \in A$

A res. fin. $\Rightarrow \exists$ finite quotient Q st $\ell(\gamma) \neq \ell(f(\gamma))$ in Q.

Let $H = \ker\left(\pi_1(S_g) \to PSL_2 A \to PSL_2 Q\right)$. $H \overset{\text{f.i.}}{\leq} \pi_1(S_g)$

Take. $F = \mathrm{Out}\left(\pi_1(S_g)/H\right)$. $\square$

Torelli groups

$\psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}\mathbb{Z}$

$\mathcal{I}(S_g) = \ker \psi.$

- $\mathcal{I}(S_g)$ hard/non-linear part of MCG.

- All $\mathbb{Z}HS^3$ are:

$$H_g \coprod_{\varphi} H_g$$

$\varphi \in \mathcal{I}(S_g)$

- $\mathcal{I}(S_g) = \pi_1$ (Torelli space)

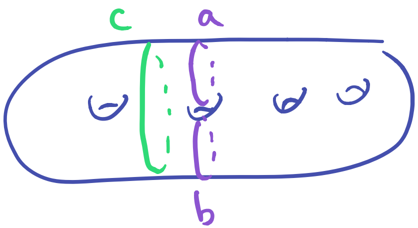
Space of Riem. surf's with homology framings

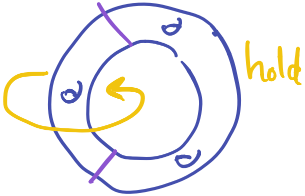
Examples of Elements

① T_c c sep.

② Bounding pair map

$T_a T_b^{-1}$ $[a]=[b]$
$i(a,b)=0$.





③ Fake bounding pair maps

$T_a T_b^{-1}$ $[a]=[b]$.

④ $[T_a, T_b]$ $\hat{i}(a,b)=0$.

special case of 3

$T_a(T_b T_a^{-1} T_b^{-1}) = T_a T_{T_b(a)}^{-1}$

hom. to a.

⑤ ~~Point~~ Boundary/handle pushes

Special case of 3

Generators

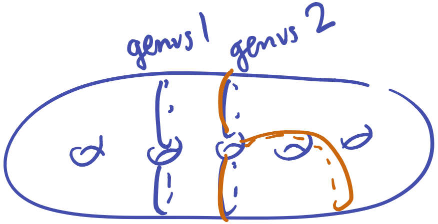
$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \xrightarrow{\Psi} \mathrm{Sp}_{2g}\mathbb{Z} \to 1$$

generators ⟵ relators

Birman: presentation for Sp.

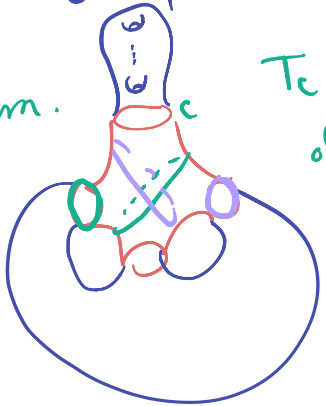
Powell ⇝ BP's & Sep. twists

Birman: Do these generate?



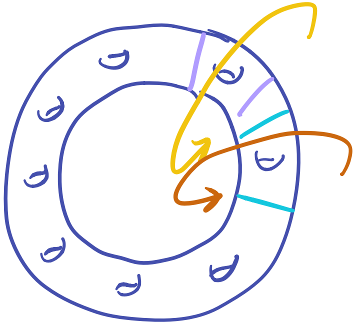
Johnson: ① Sep twists not needed

Lantern.



T_c = product of 3 BPs.

② Only BPs of genus 1 are needed.



So:

$\mathcal{I}(S_g)$ = normal closure in $\mathrm{Mod}(S_g)$ of a single BP of genus 1.

Johnson I: Finite generation.

Thm $g \geq 3$ $\mathcal{I}(S_g)$ f.g. by BPs of genus 1.

Pf idea. List $O(2^g)$ BPs $\{f_i\}$

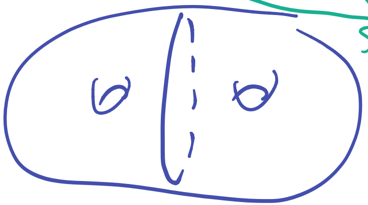
Check $\langle f_i \rangle \trianglelefteq \mathrm{Mod}(S_g)$

$\Longrightarrow \langle f_i \rangle = \mathcal{I}(S_g)$

$$\boxed{\begin{array}{c}\text{So:}\\ \mathcal{I}(S_g) = \text{normal closure in } \mathrm{Mod}(S_g) \\ \text{of a single BP of genus 1.}\end{array}}$$

$T_c T_d T_c^{-1} = T_{T_c(d)}$

Same H_1 splitting



Mess $\mathcal{I}(S_2) \cong F_\infty$

gen set $\longleftrightarrow$ H_1 splittings

Open Q. Explicit gen set.

Major Open Q. Is $\mathcal{I}(S_g)$ fin pres?

$H_2(G)$ ∞ gen $\Longrightarrow$ G not f.p.

Johnson Homomorphism

$\tau : \mathcal{I}(S_g) \longrightarrow \Lambda^3 H$

$H = H_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$

Issue: $\mathcal{I}(S_g)$ acts triv. on

$H = \pi/[\pi,\pi] \quad \pi = \pi_1(S_g)$

Remedy: Look at action of $\mathcal{I}(S_g)$ on

$\pi/[\pi,[\pi,\pi]]$ 2-step nilpotent "like abelian"

$\Lambda^3 G = \{\text{formal sums of } g_1 \wedge g_2 \wedge g_3\}/\sim$

G abel. $a \wedge b \wedge c = -b \wedge a \wedge c \Rightarrow a \wedge a \wedge b = 0.$

$(a+a') \wedge b \wedge c = a \wedge b \wedge c + a' \wedge b \wedge c.$

e.g. $H^k(T^n) = \Lambda^k \mathbb{Z}^n$

Lower central series of G

$G_1 = G$

$G_2 = [G,G]$

$G_3 = [G,[G,G]]$

$G_4 = [G,[G,[G,G]]]$

Probe G by understanding G/G_k.

Johnson Homomorphism

$\tau : \mathcal{I}(S_g) \twoheadrightarrow \Lambda^3 H$

$H = H_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$

Issue: $\mathcal{I}(S_g)$ acts triv. on

$H = \pi/[\pi,\pi] \quad \pi = \pi_1(S_g)$

Remedy: Look at action of $\mathcal{I}(S_g)$ on

$\pi/[\pi,[\pi,\pi]]$ 2-step nilpotent "like abelian"

Consider

$$1 \to \overset{N}{[\pi,\pi]/[\pi,[\pi,\pi]]} \to \overset{E}{\pi/[\pi,[\pi,\pi]]} \to \overset{H}{\pi/[\pi,\pi]} \to 1$$

$f(\tilde{x}) \quad \tilde{x} \qquad x$

Construct $\tau : \mathcal{I}(S_g) \to \underbrace{\mathrm{Hom}(H,N)}_{\text{abelian.}}$

Given $f \in \mathcal{I}(S_g)$

$x \in H$

need $\tau(f)(x) \in N$.

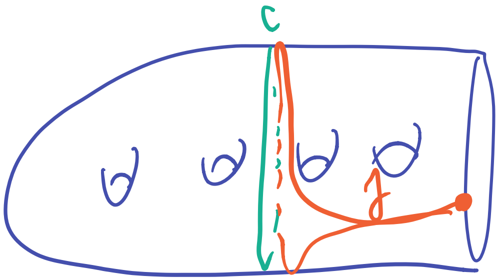
Lift x to $\tilde{x} \in E$.

$\rightsquigarrow f(\tilde{x})\tilde{x}^{-1} \in N$.

Image $\cong \Lambda^3 H$.

Computations

$\boxed{\tau(T_c) = 0}$ c sep.

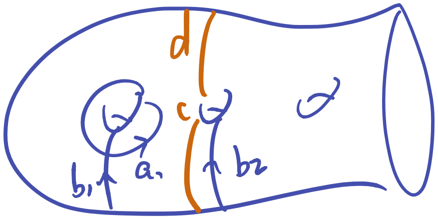


$T_c \longleftrightarrow$ conj. by $\gamma \in [\pi,\pi]$

$$f(\tilde{x})\ \tilde{x}^{-1}$$

$$\gamma \tilde{x} \gamma^{-1} \tilde{x}^{-1} = [y,z]\,\tilde{x}\,[y,z]^{-1}\,\tilde{x}^{-1} \in [\pi,[\pi,\pi]].$$

$$\tau(T_c T_d^{-1}) = a_1 \wedge b_1 \wedge b_2 \neq 0.$$



$\Longrightarrow \mathcal{I}(S_g)$ not gen by sep twists.

Topological interpretation #1

$$\alpha : \pi_1(S_g) \longrightarrow \mathbb{Z}^{2g} \quad \text{abelianization.}$$

$$\rightsquigarrow A : (S_g, *) \longrightarrow (T^{2g}, 0)$$

Consider $A \circ \psi$ $\quad [\psi] \in \mathcal{I}(S_g)$.

Since $[\psi] \in \mathcal{I}(S_g)$,

$$A \sim A \circ \psi.$$

The homotopy is a 3-man.

in $T^{2g} \rightsquigarrow \Lambda^3 H$.

$$\boxed{\text{An elt of } \Lambda^3 H \text{ is a sum of 3-}\cancel{\text{tori}}\text{ mnflds in } T^{2g}}$$

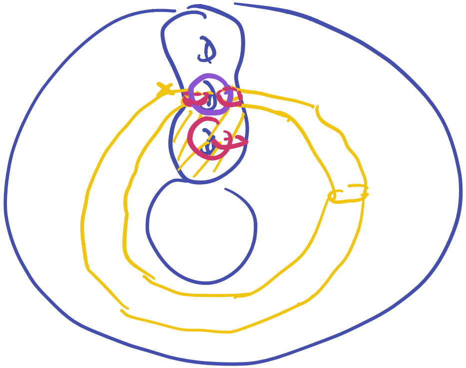
Top. interp #2

Given $f \in \mathcal{I}(S_g)$ need elt of $\Lambda^3 H$ or $(\Lambda^3 H)^* = \{\Lambda^3 H \to \mathbb{Z}\}$

$$5\, \underline{x \wedge y \wedge z} + 7\, \underline{a \wedge b \wedge c}$$

Given $f \in \mathcal{I}(S_g)$, $x \wedge y \wedge z \in \Lambda^3 H$ need a number:

Construct mapping torus M_f



$x \rightsquigarrow$ surface Σ_x in M_f

The desired number is

$$\hat{\iota}(\Sigma_x, \Sigma_y, \Sigma_z)$$

Chap 7. Torsion

<u>Thm.</u> (Fenchel–Nielsen)
Any fin. order $f \in \mathrm{Mod}(S_{g,n})$ has a rep $\varphi \in \mathrm{Homeo}^+(S_{g,n})$ of finite order

More: φ can be chosen to be isometry of a hyp./Eucl. metric.

<u>Pf.</u> Later chapter.

Same true for $G \leq \mathrm{Mod}(S_{g,n})$, $|G| < \infty$ much much harder.

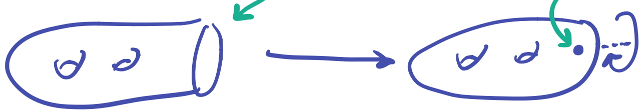
<u>Cor.</u> $\partial S \neq \emptyset$

$\mathrm{Mod}(S)$ is torsion free.

<u>Pf of Cor.</u>

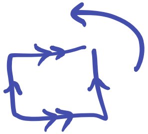
$$\mathbb{Z}^b \longrightarrow \mathrm{Mod}(S_{g,n}^b) \xrightarrow{\text{capping}} \mathrm{Mod}(S_{g,n+b})$$

$\uparrow$ torsion free.

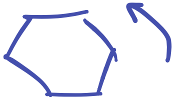


Torus case

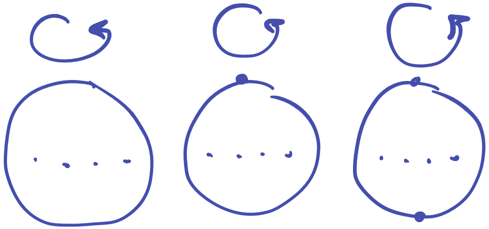
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$



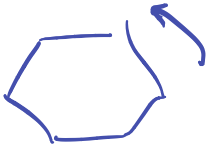
Sphere case



Brouwer: a per. homeo of S^2 is conj to Eucl. rot.

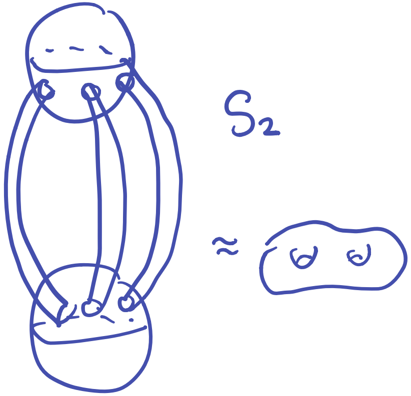
In higher genus, it is complicated to list all periodic elts (number thy).

examples



$4g+2$

not realizable by rotation in $\mathbb{R}^3$ (1 fixed pt)



Torelli $\ker \mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z}$

Thm $\mathcal{I}(S_g)$ is torsion free.

Pf. Say $f \in \mathcal{I}(S_g)$ WLOG $g \geq 2$.

$1 < |f| < \infty$.

$\rightsquigarrow$ representative φ $\rightsquigarrow$ at each fixed pt, rotation.

Apply Lefschetz f.p.t.

$$L(\varphi) = \sum_{i=0}^{2} (-1)^i \, tr\left(\varphi_* : H_i(S_g) \to H_i(S_g)\right)$$

$$1 - 2g + 1$$

$$= 2 - 2g < 0 \qquad \blacksquare$$

$L(\varphi)$ $\|$ #fixed pts > 0

φ = homeo of space X with isolated fixed pts

$L(\varphi)$ = sum of degrees of fixed pts.

degree: deg of induced map on $S^1 \cong UT_pX$ at p

If φ is a rotation at p then degree of φ at p is... $+1$

84(g-1) Thm

Thm. $g \geq 2$, $G \leq \mathrm{Mod}(S_g)$
$|G| < \infty$

$$\Longrightarrow |G| \leq 84(g-1).$$

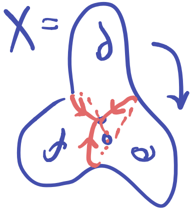
- For G abelian answer: $4g+4$
- Bound is (not) realized for ∞ many g.
- Realized for $g=3$
- Larsen. $\{g : \text{bound is realized}\}$ has same frequency in $\mathbb{N}$ as cubes.

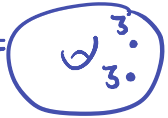
Proof uses hyp orbifolds

X = hyp. surface

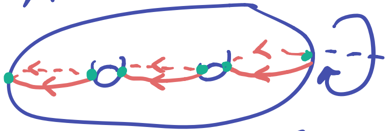
$G \leq \mathrm{Isom}^+(X)$ finite.

$\rightsquigarrow Y = X/G$ orbifold.

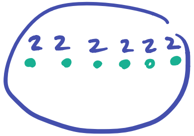
examples $X =$  $G \cong \mathbb{Z}/3$

$Y =$ 

$X =$



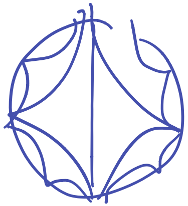
$G \cong \mathbb{Z}/2$

$Y =$ 

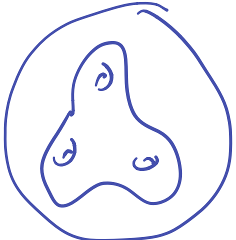
Riemann-Hurwitz formula

In Y, images of fixed pts are marked and label of a marked pt is $|G|/\#\text{ preimages}$

$$\chi(Y) = (2-2g(Y)) - \underbrace{m}_{\#\text{ marked pts}} + \sum_{i=1}^{m} \frac{1}{\underbrace{p_i}_{\text{labels}}}$$



Fact. $\chi(X) = |G|\,\chi(Y)$.

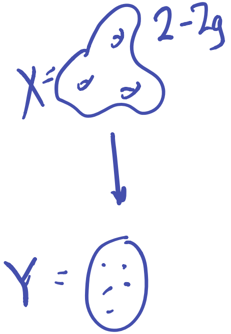


Pf of $84(g-1)$

Want to show for any $Y = X/G$, $\chi(Y)<0$

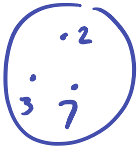
$$\chi(Y) \le -\tfrac{1}{42}.$$

$$= \frac{2-2g}{84(g-1)}$$



Pf. Just check

Only possibility is $Y =$



$$\chi(Y) = -\tfrac{1}{42}$$

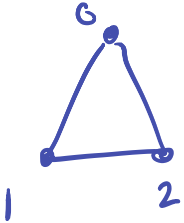
∎

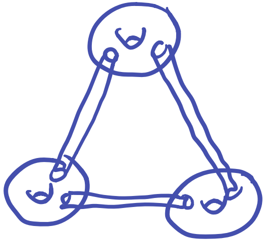
Realizing Finite Groups

<u>Thm.</u> G = finite gp

$\exists\, g$ s.t. $G \leq \mathrm{Mod}(S_g)$

$\mathbb{Z}/3$





<u>Pf #1</u> Build S_g from Cayley graph for G.

vertices : G

edges : differ by generator

vertices $\longrightarrow$ tori

edges $\longrightarrow$ annuli

$G \circlearrowright$ Cayley graph by left mult.

Can replace "$\exists\, g$" with "$g \gg 0$" ?

Yes for cyclic groups.

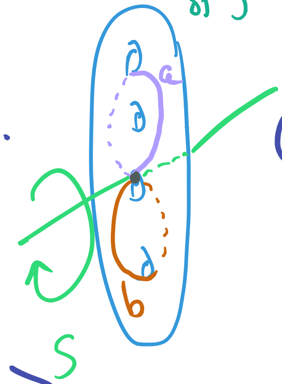
Generating MCG with torsion

Thm $\mathrm{Mod}(S_g)$ is generated by elts of order 2.

Brendle-Farb: only 6 such elts are needed, indep of g.

Pf. $\mathrm{Mod}(S_g)$ is perfect.

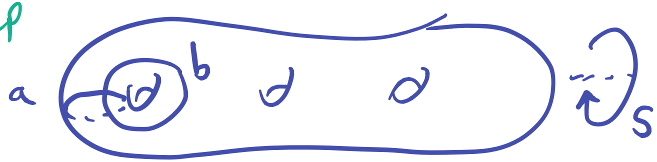
$$\mathrm{Mod}(S_g) = [\mathrm{Mod}(S_g), \mathrm{Mod}(S_g)] = \langle [T_a, T_b] : i(a,b) = 1 \rangle$$



Suffices to write

$$[T_a, T_b] = \prod \text{elts of order } 2$$

Change of coords:



Choose involution s, $s(a) = b$.

$$[T_a, s] = T_a (s T_a^{-1} s^{-1})$$
$$= T_a T_b^{-1}$$

product of 2 elts of order 2: $T_a s T_a^{-1}$, s

Similarly $T_a^{-1} T_b$ is a product of 2 elts of order 2

$\Longrightarrow [T_a, T_b]$ = prod of 4 elts of ord 2

Chap 8 DNB Thm.

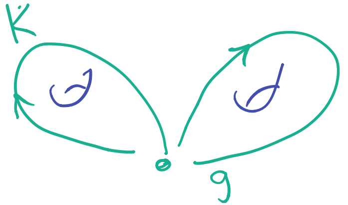
G = group

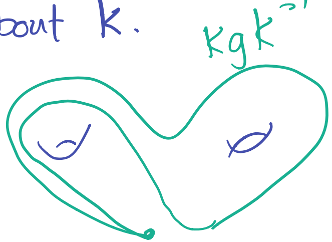
Inner autos:

$$\Phi_k : G \longrightarrow G$$
$$g \longmapsto kgk^{-1}$$

Example: $G = \pi_1(S_g)$

Φ_k = push about k^{-1}.





$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

Example $\Longrightarrow$

$$\sigma : \mathrm{Mod}^{\pm}(S_g) \longrightarrow \mathrm{Out}\,\pi_1(S_g)$$

topology   algebra

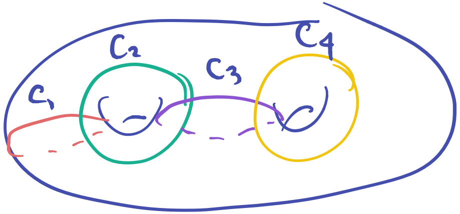
Thm. σ is $\cong$

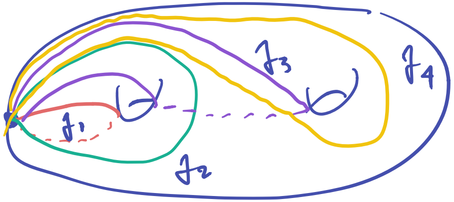
Injectivity: $K(G,1)$ theory

Surjectivity: $K(G,1)$ theory:

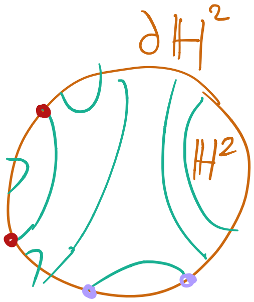
outer auto of $\pi_1 \longrightarrow$ homot. equiv.

Strategy





$c_i = [\gamma_i]$ ← conj class



Let $[\Phi] \in \mathrm{Out}\,\pi_1(S_g)$

1. $\Phi(c_i)$ simple $\forall i$. $\Longleftrightarrow$ all pairs of lifts unlinked at $\partial \mathbb{H}^2$
2. $i(\Phi(c_i), \Phi(c_{i+1})) = 1$ $\Longleftrightarrow$ a little more complicated.
3. $i(\Phi(c_i), \Phi(c_j)) = 0$ $\Longleftrightarrow$ all pairs of lifts unlinked at $\partial \mathbb{H}^2$

   $|i-j| > 1$.

To show:

Φ preserves linking at $\partial \mathbb{H}^2$

Then Alex. method, change of coords...

Cayley graph

$G = \langle S \mid R \rangle$

gen set

vertices: G

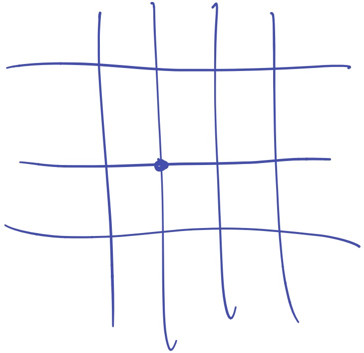
edges:  g gs $s \in S$

Note $G \circlearrowright$ Cayley graph on left

path metric

$\rightsquigarrow$ metric on G.

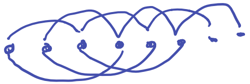
Example: $\mathbb{Z}^2 = \langle a, b \mid [a,b] = 1 \rangle$



metric: taxicab.

$$d(a^m b^n, id) = |m| + |n|$$

Example $\mathbb{Z} = \langle 1 \mid \rangle$ 

$\mathbb{Z} = \langle 2, 3 \mid \cdots \rangle$ 

Quasi-isometries

X, Y metric spaces

$f : X \to Y$

Isometry: $d(f(x), f(y)) = d(x,y)$

Quasi-isometry: $\exists\, K, C, D$ s.t.

① $\frac{1}{K} d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq K\, d_X(x,y) + C$

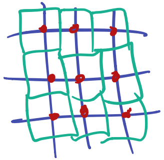
② D-nbd of $f(X)$ is Y

example $\mathbb{Z}^n \hookrightarrow \mathbb{E}^n$

$K = \sqrt{n}$

$C = 1$ (or 0).

$D = 1$



example $\mathbb{E}^n \to \mathbb{Z}^n$ "nearest pt"

Next $\pi_1(S_g) \to \mathbb{H}^2$

example $f : \mathbb{R} \to \mathbb{R}$

$f(x) = Kx$

or $f(x) = \begin{cases} Kx & x \text{ irrational} \\ Kx+1 & x \text{ rational}. \end{cases}$

Milnor–Švarc Lemma

X = proper, geod. metric space

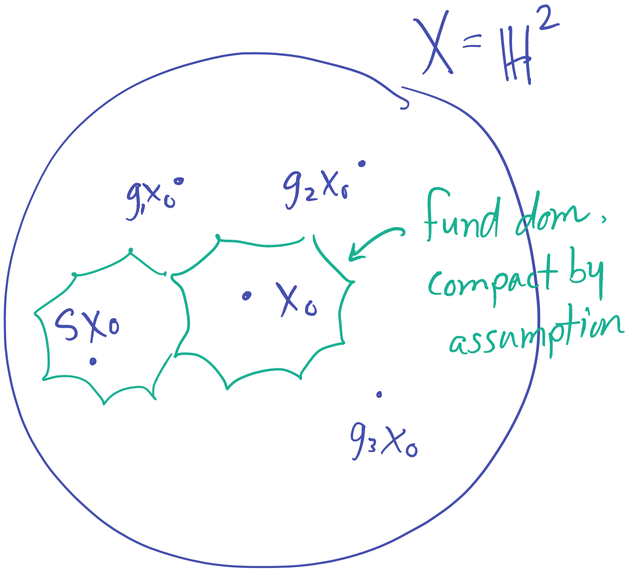
$G \curvearrowright X$ prop. disc., by isometries.

X/G compact

Then ① G is finitely generated

② G quasi-isom to X via any orbit map

$$g \longmapsto g \cdot x_0$$



$G = \pi_1(S_g)$

Gen set for $\pi_1(S_g)$:

elts that take fund dom to an adjacent one.

From Autos to QIs

G = gp $\quad G = \langle S \rangle \quad |S| = \infty$

$\Phi \in \mathrm{Aut}(G)$

$\rightsquigarrow$ quasi-isom of G.

$K = \max \{ \| \Phi^{\pm 1}(s) \| : s \in S \}$

$C = 0$

$D = 0$.

So:

$\Phi \in \mathrm{Aut}\, \pi_1(S_g)$

$\downarrow$ ✓

quasi-isom of $\pi_1(S_g)$

$\downarrow$ ✓

quasi-isom of $\mathbb{H}^2$

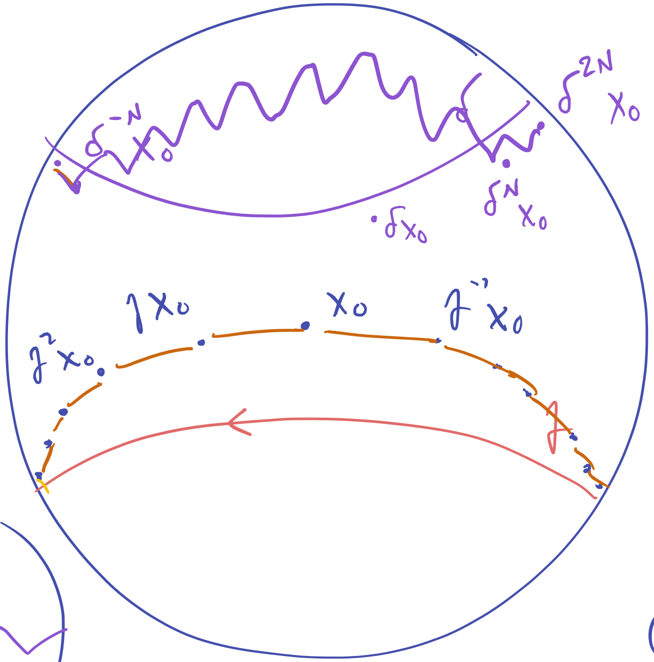
$\downarrow$ or

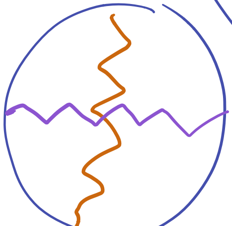
homeo of $\partial \mathbb{H}^2$

next

hence, linking preserved

Quasi-isometries of $\pi_1(S_g) = \mathbb{H}^2$ preserve linking.





Suppose $\gamma, \delta \in \pi_1(S_g)$ unlinked.

① Choose $N \gg 0$ large compared to QI consts.

$\rightsquigarrow$ orbit pts

$\gamma^i x_0$

$\delta^{Ni} x_0$

far

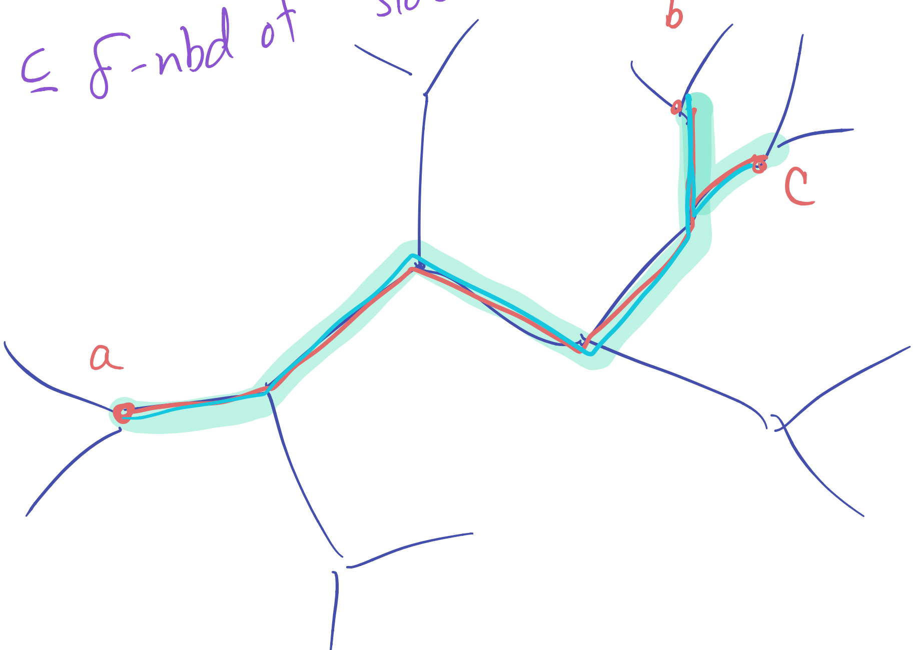
② Connect orbit pts by far paths P_γ, P_δ in $\pi_1(S_g)$

③ If $\Phi(\gamma), \Phi(\delta)$ linked, $\Phi(P_\gamma), \Phi(P_\delta)$ cross $\Rightarrow$ CONT.

Gromov hyperbolic:

$\exists\, \delta$ s.t. for any triangle,

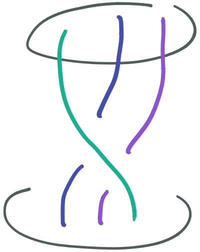
side 3 $\subseteq$ δ-nbd of side 1 $\cup$ side 2



Chap 9 Braid groups

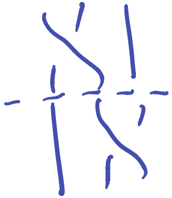
B_n = braid gp on n strands.

Def #1



n strands in $\mathbb{R}^2 \times [0,1]$
monotonic in $[0,1]$ dir.
considered up to isotopy in $\mathbb{R}^3$

Multiplication: stack (& scale vertical)



a
b
ab

id: 

Generators: 

σ_i:

Inverses

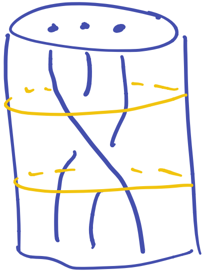
Braid closure:
braids $\longrightarrow$ knots

Defn #2

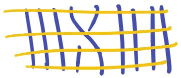
$\mathrm{Conf}_n(\mathbb{R}^2)$ = space of n un-labeled pts in $\mathbb{R}^2$

$$B_n \cong \underbrace{\pi_1 \mathrm{Conf}_n \mathbb{R}^2}_{\text{"dance"}}$$

basept: • • • • •



time

In this defn: σ_i is 

$\mathrm{PConf}_n\ \mathbb{R}^2 = (\mathbb{R}^2)^n \,/\,$ big diagonal.

$\mathrm{Conf}_n\ \mathbb{R}^2 = \mathrm{PConf}_n \mathbb{R}^2 / \Sigma_n$

<u>Fact</u>. $\mathrm{Conf}_n \mathbb{R}^2$ is a $K(G,1)$

$\Longrightarrow$ B_n is torsion free.

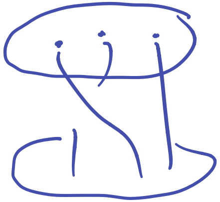
(torsion $\Rightarrow$ ∞-dim $K(G,1)$).

Defn #3

$B_n \cong \mathrm{Mod}(D_n)$ disk with n marked pts in interior.

$\mathrm{Mod}(D_n) \longrightarrow B_n$

Given $[\varphi] \in \mathrm{Mod}(D_n)$ any homotopy φ to id (ignoring marked pts) restricts to a loop in $\pi_1 \mathrm{Conf}_n \mathbb{R}^2$.

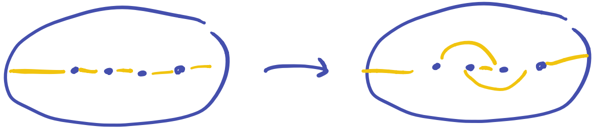


Pf of $\cong$ is BES, forgetting n pts instead of 1.

$$\begin{array}{ccc} \mathrm{Homeo}^+(D^2, \{n \text{ pts}\}) & \longrightarrow & \mathrm{Homeo}^+(D^2) \\ & & \downarrow \\ & & \mathrm{Conf}_n D^2 \simeq \mathrm{Conf}_n \mathbb{R}^2 \end{array}$$

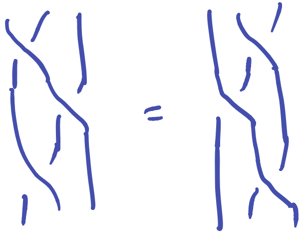
fiber bundle, $\rightsquigarrow$ LES

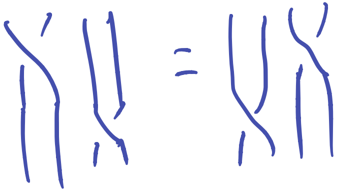
σ_i :



Alg. Structure

- $B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$





braid rel : R3 moves

$\sigma_i \sigma_i^{-1} = id$: R2 moves

- $B_n^{ab} = H_1(B_n; \mathbb{Z}) \cong \mathbb{Z}$

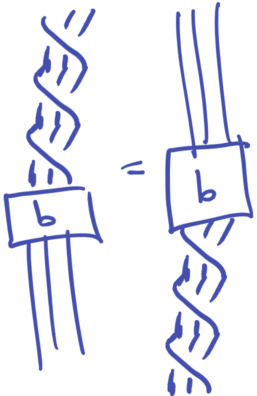
$L : B_n \longrightarrow \mathbb{Z}$

$\sigma_i \longmapsto 1$

"length homom"

- $Z(B_n) = \langle T_\partial \rangle$

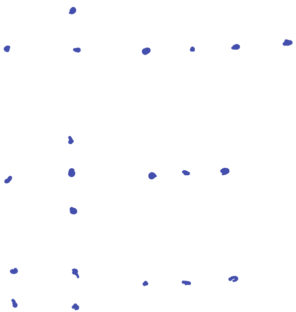
$T_\partial = (\sigma_1 \cdots \sigma_{n-1})^n$



codim 1

codim 2

codim 2.

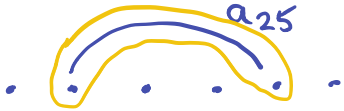


Travaux de Thurston. Exposé 3.

Pure braid gps PB_n

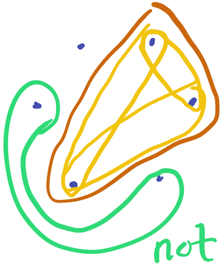
$$1 \to PB_n \to B_n \to \Sigma_n \to 1$$

• PB_n gen by a_{ij} $\binom{n}{2}$



Dehn twist

• Presentation (McCammond-M)



not convex

$PB_n = \langle$ convex Dehn twists | disjointness, lantern $\rangle$

cf. Birman-Ko-Lee

• $Z(PB_n) = Z(B_n) = \langle T_\partial \rangle$

$(a_{12} \cdots a_{1n})(a_{23} \cdots a_{2n}) \cdots (a_{n-1,n})$

• $PB_n \cong PB_n / Z(PB_n) \times \mathbb{Z}$

$$1 \to \mathbb{Z} \rightleftarrows PB_n \xrightarrow{\text{Cap}} PB_n / Z(PB_n) \to 1$$

$1 \mapsfrom a_{12}$

$0 \mapsfrom a_{ij}$

splitting

More on PB_n

- Combing decomp:

$$PB_n \cong F_{n-1} \rtimes PB_{n-1}$$

Iterating:

$$PB_n \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes \overbrace{\mathbb{Z}}^{PB_2}$$

- Abelianization:

$$H_1(PB_n;\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}$$

Need $\binom{n}{2}$ maps $PB_n \to \mathbb{Z}$

$\binom{n}{2}$ forget maps $PB_n \to PB_2 \cong \mathbb{Z}$

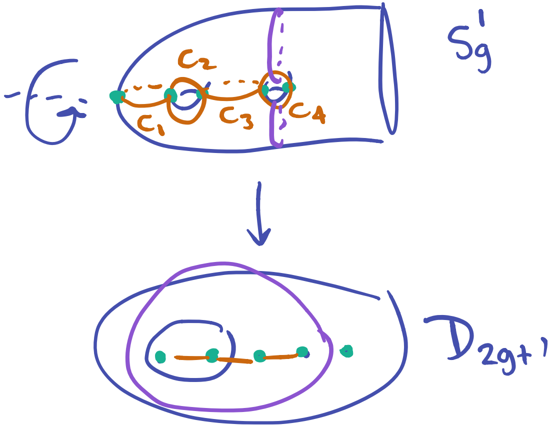
Church–Farb. $H_1(PB_n;\mathbb{Z})$ is rep. stable: As Σ_n reps

$$H_1(PB_n;\mathbb{Z}) = \underbrace{\underbrace{\bigcirc}_{\text{trivial rep.}} \oplus \underbrace{\square}_{\text{std irrep}}}_{\text{std rep.}} \oplus \square\square$$

cf. Farb survey
ICM

Birman-Hilden theory

Survey with Winarski

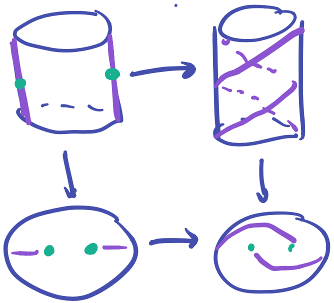


$$B_{2g+1} \longrightarrow \mathrm{Mod}(S_g^1)$$

$$\varphi \xrightarrow{\text{lift}} \tilde{\varphi}$$

BH thm. Injective.

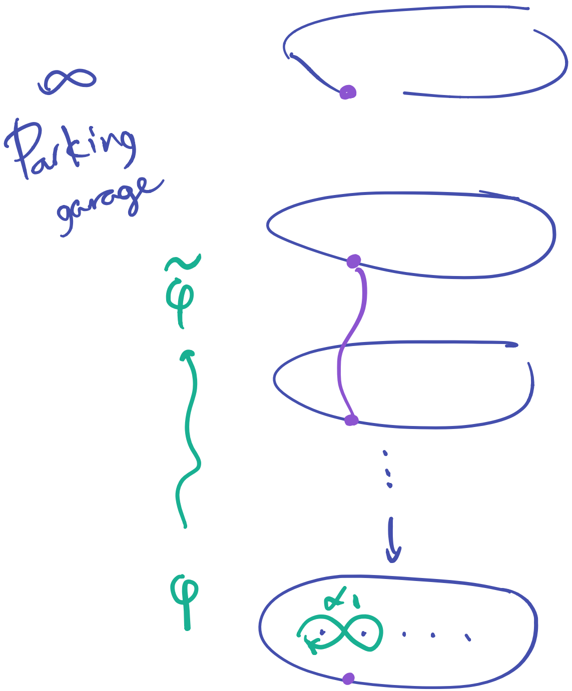
$\sigma_i \longmapsto T_{c_i}$



Braid relns & chain relns come directly from

- braid reln in B_n
- writing center of B_n in terms of σ_i

Burau rep



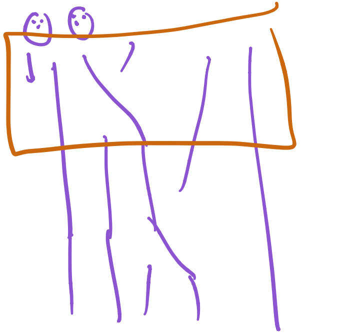
$H_1(X) = \langle t^i \alpha_1, \ldots, t^i \alpha_{n-1} : i \in \mathbb{Z} \rangle$

f.g. as a $\langle t \rangle$-module.

action on $H_1(X)$

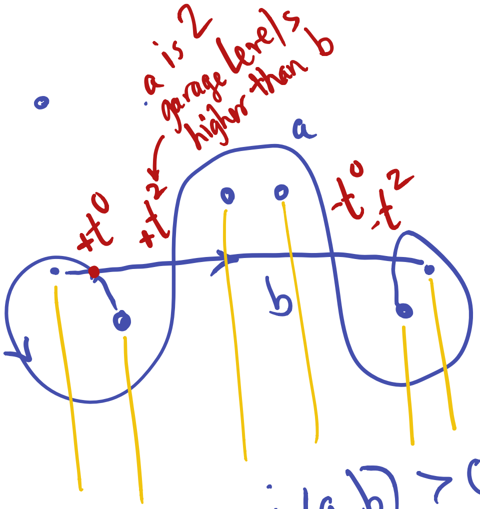
$\tilde{\varphi} \rightsquigarrow$ $n-1 \times n-1$ matrix entries in $\mathbb{Z}[t]$

$$\begin{pmatrix} I & & \\ & \begin{matrix} 1-t & 1 \\ t & 0 \end{matrix} & \\ & & I \end{pmatrix}$$



$F_n \to \mathbb{Z}$

$x_i \mapsto 1$



$i(a,b) > 0$

"$\hat{i}$" $(a,b) = 0.$

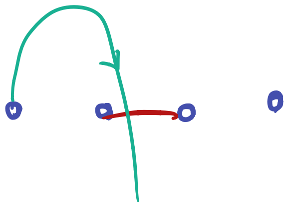
$i(a,b) > 0$

$\hat{i}(a,b) = 0$

$\Rightarrow [T_a, T_b] \in \mathcal{I}(S_g)$

$\neq$

id



Parts II & III

$\mathrm{Mod}(S) \circlearrowright \mathrm{Teich}(S)$

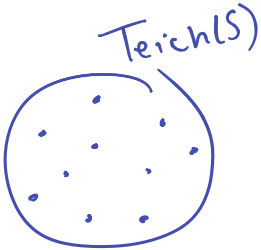
$\quad$ = space of hyp metrics on S / isotopy

This action tells us about <u>both</u> $\mathrm{Mod}(S)$ & $\mathrm{Teich}(S)$

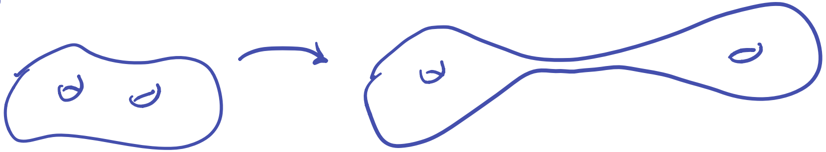
For example:

- $\mathrm{Isom}\,\mathrm{Teich}(S) \cong \mathrm{Mod}^{\pm}(S)$
- Nielsen-Thurston classification for elements of $\mathrm{Mod}(S)$

This <u>is</u> geometric gp thy.



Teich metric.



Moduli space

$\chi(S) < 0$

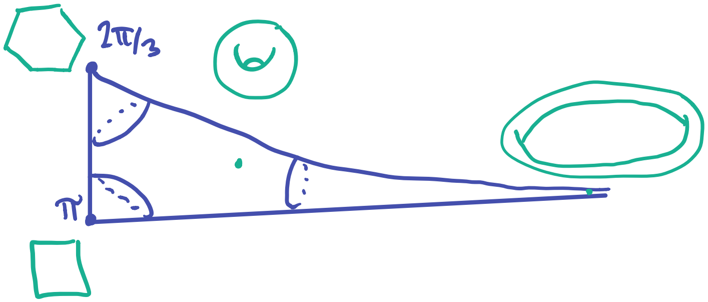
$\mathcal{M}(S) = \{\text{hyp metrics}\}/\text{isometry}.$

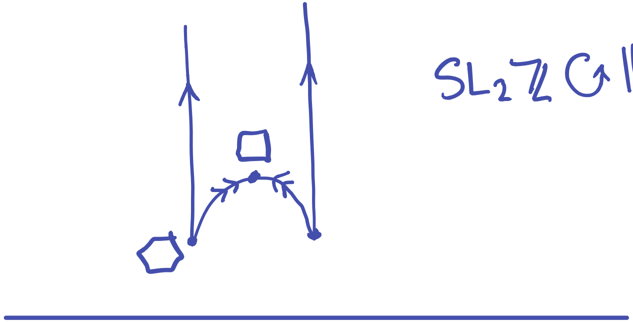
$= \{\text{complex str's}\}/\sim$

$= \{\text{algebraic strs}\}/\sim$

$= \{\text{conformal strs}\}/\sim$

$\mathcal{M}(T^2) = \{\text{unit area Eucl. metrics}\}/_{\text{isom.}}$





$SL_2\mathbb{Z} \curvearrowright \mathbb{H}^2$

Teichmüller space

(orbifold) univ. cover of $\mathcal{M}(S)$.

$$\mathrm{Teich}(S) = \{\text{hyp. metrics}\}/\text{isotopy}$$

$$= \{\text{hyp. metrics}\}/\mathrm{Diff}_0(S)$$

isotopic to id.

(action is pullback)

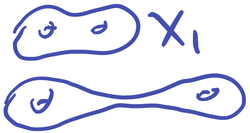
$$= \{(X,\varphi) : X \text{ hyp surf.},\ \varphi : S \to X \text{ diffeo}\}/\sim$$

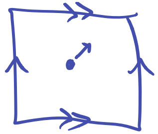
marked surface

S = top surface, fixed forever

$\xrightarrow{\varphi_1}$ X_1

$\xrightarrow{\varphi_2}$ X_2





μ = Eucl. metric

$\varphi \in \mathrm{Diff}_0(T^2)$

$\varphi^*(\mu)$ is a different Eucl. metric on T^2, isometric to μ via φ.

$(X_1,\varphi_1) \sim (X_2,\varphi_2)$ if $\exists$ isometry $I : X_1 \to X_2$

s.t.

$$\begin{tikzcd} & X_1 \arrow[d, "I"] \\ S \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

commute up to isotopy.

The torus

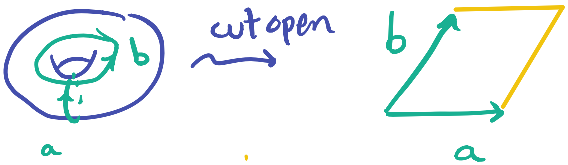
$$\mathrm{Teich}(T^2) = \{\text{Eucl. metrics}\} / \begin{smallmatrix}\text{scale}\\ \text{isometry}\end{smallmatrix}$$

$$= \{(X, \varphi)\}/\sim$$

Prop. $\mathrm{Teich}(T^2) \longleftrightarrow \mathbb{H}^2$

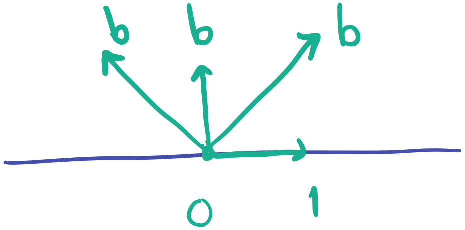
Pf. $\mathrm{Teich}(T^2) \longleftrightarrow$ (marked lattices in $\mathbb{R}^2$)

$\longleftrightarrow$ marked parallelograms / $\begin{smallmatrix}\text{scale}\\ \text{isometry}\end{smallmatrix}$

Why? $X =$ 

Scale so $a = 1 \in \mathbb{C}$

reflect over $\mathbb{R}$ so $\operatorname{im} b > 0$.



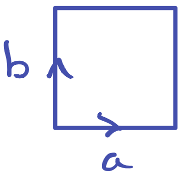
$\square$

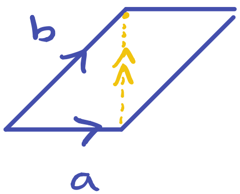
Prop $\rightsquigarrow$ topology on $\mathrm{Teich}(T^2)$.

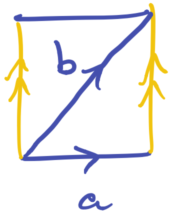
We'll see: Teich metric is hyp metric.

Example tori

① i vs. $i+1$







isometric! via... T_a

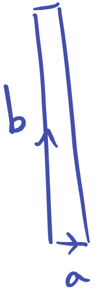
Same pt in $\mathcal{M}(S)$

different in Teich(S)

$\ell_i(b) = 1 \qquad \ell_{i+1}(b) = \sqrt{2}$

② ni vs i/n

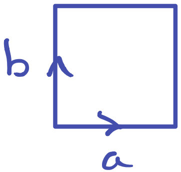
not isometric

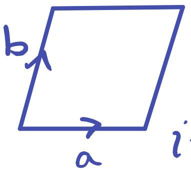


isometric via $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

③ i vs $i+\varepsilon$

not isometric!





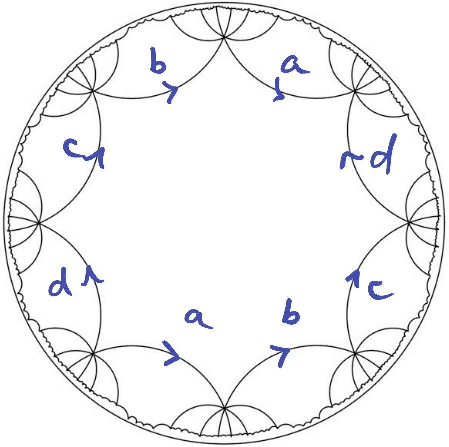
length spectra:

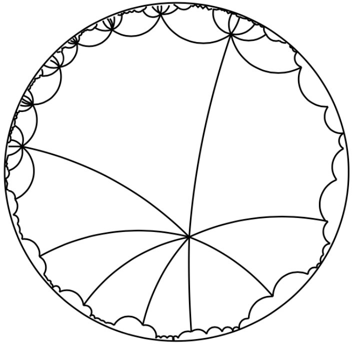
$i: 1, 1, \sqrt{2}, \sqrt{2}, \ldots$

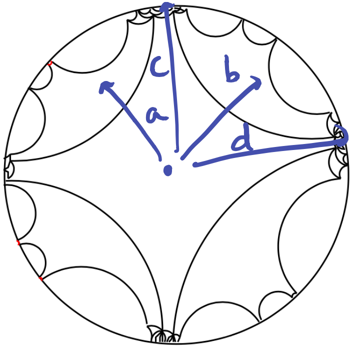
$i+\varepsilon: 1, 1+\varepsilon', \ldots$

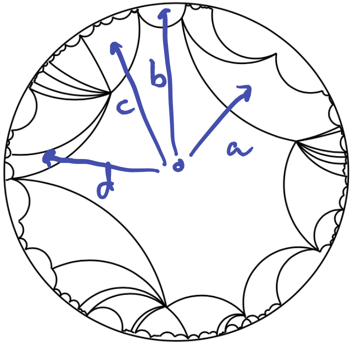
Some points in Teich(S_2)

Marked octagons / isometry of $\mathbb{H}^2$









Length functions

For a curve (isotopy class) in S:

$$\ell_a : \mathrm{Teich}(S) \to \mathbb{R}$$
$$X \longmapsto \ell_X(a)$$

$\ell_X(a)$ = length of a in X-metric.

length of the geodesic

(no such map for $\mathcal{M}(S)$).

Let $\mathcal{S}$ = {curves in S}/isotopy

Will show: $\ell : \mathrm{Teich}(S) \to \mathbb{R}^{\mathcal{S}}$ injective.

(actually: lengths of $6g-5$ curves determine the metric)

The algebraic topology

$DF(\pi_1(S_g), PSL_2\mathbb{R})$

"

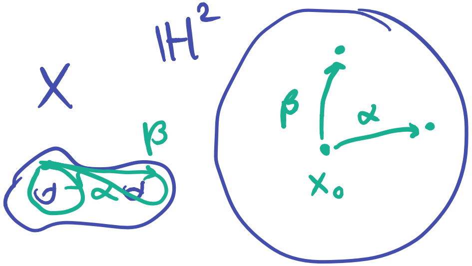
discrete faithful reps

$\pi_1(S_g) \longrightarrow PSL_2\mathbb{R}$

"

cov space actions

$\pi_1(S_g) \longrightarrow \mathrm{Isom}^+\mathbb{H}^2$



Have:

$$\mathrm{Teich}(S_g) \longleftrightarrow DF(\pi_1(S_g), PSL_2\mathbb{R}) \big/ PGL_2\mathbb{R}$$

via deck gp action

conjugation.

Like torus case:

$$\mathrm{Teich}(T^2) \longleftrightarrow DF(\mathbb{Z}^2, \mathrm{Isom}\,\mathbb{E}^2) \big/ \mathrm{Isom}^{\pm}\mathbb{E}^2$$

$DF(\pi_1(S_g), PSL_2\mathbb{R}) \big/ PGL_2\mathbb{R}$ has a natural topology from $(PSL_2\mathbb{R})^{2g}$

10. Teich Space.

$$\mathrm{Teich}(S) = \{\text{hyp metrics}\}/\text{isotopy}$$

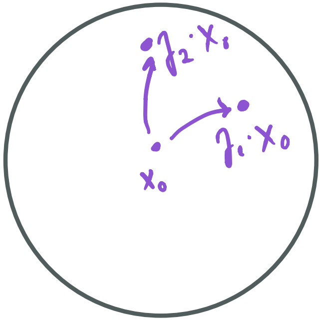
$$= \{(X,\varphi)\}/\sim$$

X: hyp surf

$\varphi: S \to X$

$$= DF(\pi_1(S_g), PSL_2\mathbb{R})/PGL_2\mathbb{R}$$

$\rightsquigarrow$ topology



Note: Teich(S) can intuitively be seen to be a manifold. Which is it?

Dimension count

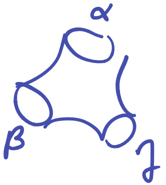
$+6g$: choosing $\rho(\gamma_1), \dots, \rho(\gamma_{2g})$ in $PSL_2\mathbb{R}$

-3 : surface relation.

-3 : conjugation

$6g-6$

Pants

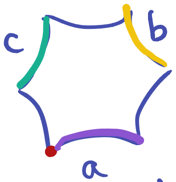
$P =$ 

Thm. The map

$$\mathrm{Teich}(P) \longrightarrow \mathbb{R}^3$$
$$X \longmapsto (\ell_X(\alpha), \ell_X(\beta), \ell_X(\gamma))$$

is a homeo.

Setup: A marked hyp. right-angled hexagon



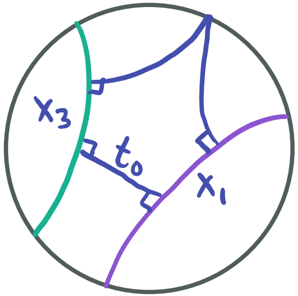
a, b, c counterclockwise

$\mathcal{H}$ = set of these / marked isometry.

Lemma. The map $\mathcal{H} \to \mathbb{R}^3$

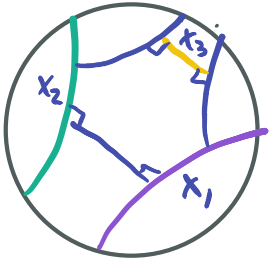
$$H \longmapsto (\ell_H(a), \ell_H(b), \ell_H(c))$$

is a bijection.

Pf. 

Start with (x_1, x_2, x_3) in $\mathbb{R}^3$

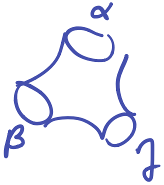
Increase t_0 until get right hexagon



(IVT)

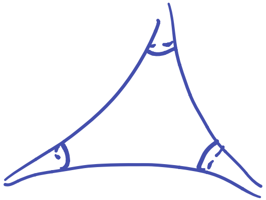
$\square$

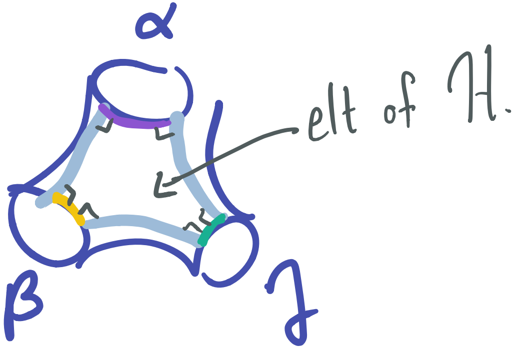
Pants

$P =$ 

Thm. The map
$$\mathrm{Teich}(P) \longrightarrow \mathbb{R}^3$$
$$X \longmapsto (\ell_X(\alpha), \ell_X(\beta), \ell_X(\gamma))$$
is a homeo.

Pf. Draw the geodesics connecting components of P





Also, components of ∂P are cut exactly in half. (by Lemma).

Continuity ✓ $\square$.

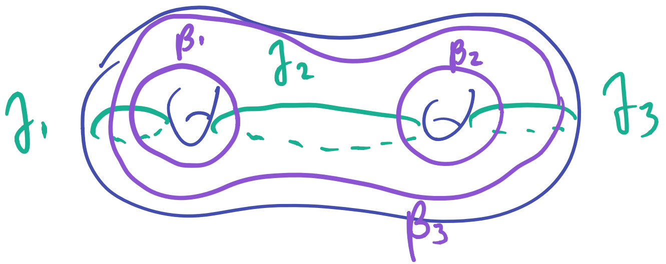
Also: $\mathrm{Teich}(S_{0,3}) = *$

Fenchel-Nielsen Coords

Thm $\mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6}$

$3g-3$ length params

$3g-3$ twist params.



Setup:

$\gamma_1, \dots, \gamma_{3g-3}$ pants decomp

$\beta_1, \dots, \beta_n$ seams:

$(\cup \beta_i) \cap$ one pants

$=$ 3 distinct arcs

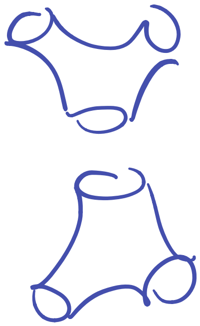
Length params: $\ell_X(\gamma_i)$

these tell us the metric on each pants (by last Thm)

Twist params: harder. how the pants are glued together.

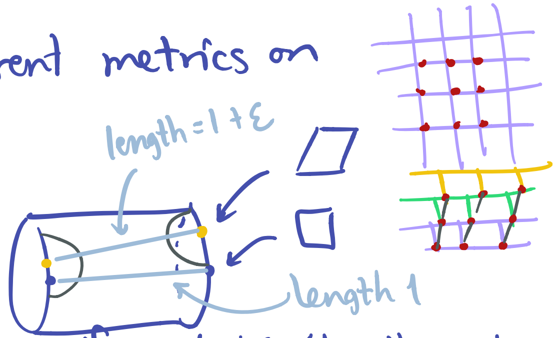
Twist parameters

Given



If you twist before gluing, get different metrics on S_0^4

Similar:



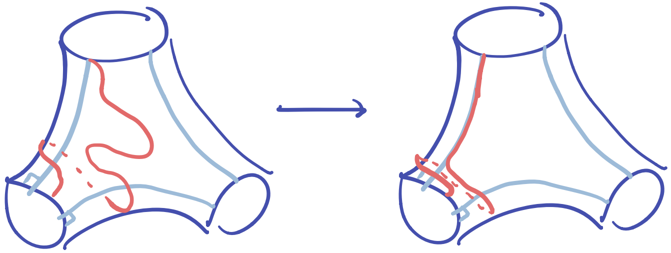
Get different tori if you twist (length spectrum)

For an arc α^* in $X \in \mathrm{Teich}(P)$

$\rightsquigarrow$ twisting about $\partial_i X$

* homotopy class rel ∂X

γ_i



twisting = $2\pi + \varepsilon$

Given $X \in \mathrm{Teich}(S_g)$ & $i \in \{1, \dots, 3g-3\}$

Choose seam β_j crossing γ_i

$\rightsquigarrow$ twisting on left/right of γ_i

$$\Theta_i(X) = 2\pi \frac{t_L - t_R}{\ell(\gamma_i)}$$

Pf of Thm

Given $\ell_1, \dots, \ell_{3g-3}$

$\Theta_1, \dots, \Theta_{3g-3}$.

Want to construct unique X with those coords.

Step 1. Make disj union of pairs of pants according to ℓ_i.

Step 2. Draw seams according to Θ_i

Step 3 Glue pants so seams match up. $\rightsquigarrow X$

Step 4. Build marking $\varphi : S \to X$ by change of coords. $\square$

The 9g-9 Thm

Thm $\exists \{\delta_1, \ldots, \delta_{9g-9}\}$

s.t.

$$\mathrm{Teich}(S_g) \longrightarrow \mathbb{R}^{9g-9}$$
$$X \longmapsto (\ell_X(\delta_i))$$

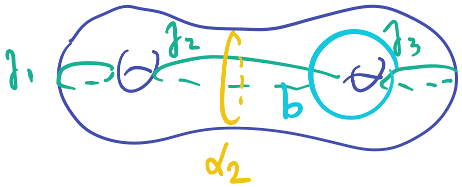
is injective.

Prop. Let X_s be a 1-param family in $\mathrm{Teich}(S_g)$ given by changing i^{th} twist param. & b = curve crossing γ_i

Then the fn $\mathbb{R} \to \mathbb{R}_+$

$$s \longmapsto \ell_{X_s}(b)$$

is strictly convex.



X_s obtained by varying this twist coord

Pf. The 9g-9 curves are:

$\gamma_1, \ldots, \gamma_{3g-3}$

$\alpha_1, \ldots, \alpha_{3g-3}$ any curves with

$$i(\alpha_i, \gamma_j) \neq 0 \iff i = j$$

$\beta_1, \ldots, \beta_{3g-3}$

$$\beta_i = T_{\gamma_i}(\alpha_i)$$

Pf. The $9g-9$ curves are:

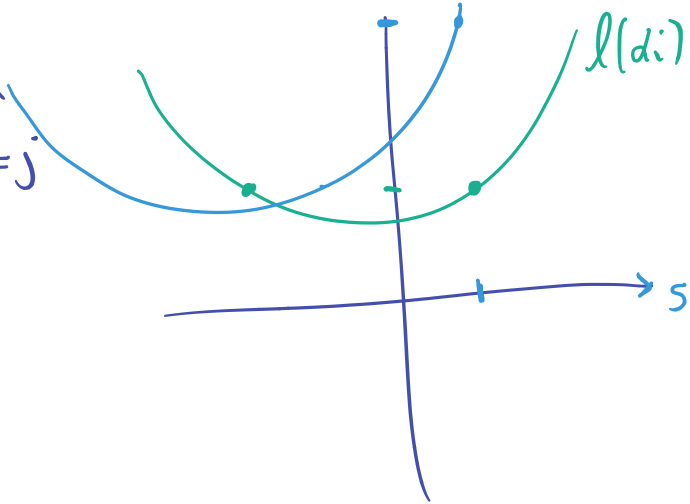
$\gamma_1, \ldots, \gamma_{3g-3}$

$\alpha_1, \ldots, \alpha_{3g-3}$ any curves with

$i(\alpha_i, \gamma_j) \neq 0 \iff i = j$

$\beta_1, \ldots, \beta_{3g-3}$

$\beta_i = T_{\gamma_i}(\alpha_i)$



By design:

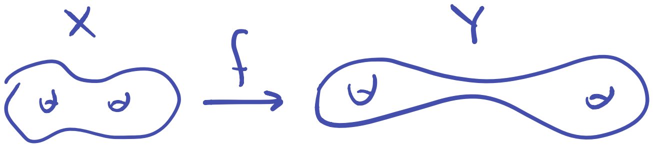
$$\ell_{X_s}(\alpha_i) = \ell_{X_{s+2\pi}}(\beta_i)$$

X_s = family corresponding to γ_i

$\square$

Chapter 11. Teich geom.

Basic question: $X, Y \in \mathrm{Teich}(S)$



What is the best map?

Idea: Measure distortion

"dilatation"

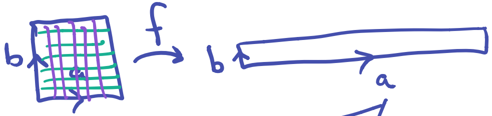
(at a pt)

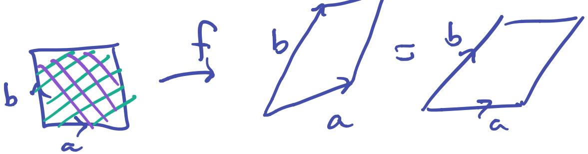
$\rightsquigarrow$ metric on $\mathrm{Teich}(S)$

Take sup of dilatation over X

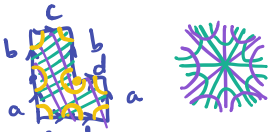
Take inf over f. Take log.

Teichmuller thm: existence & uniqueness of infimal f.





Higher genus:



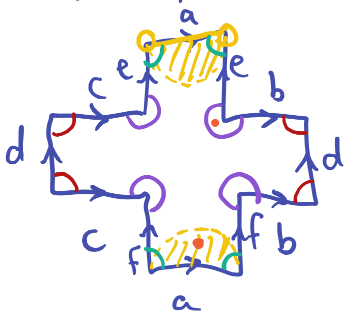
Complex structures

A complex structure on S consists of:

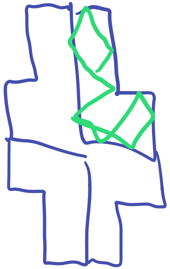
atlas of charts to $\mathbb{C}$ with holomorphic transition maps.

Riemann surface: S with complex structure.

Example of Riemann surface



$S = ✚/\sim$



9 charts: "middle" identity map

6 edge charts: id on half-disk + diameter, translation on other half disk.

2 good corner charts: translation

1 bad corner chart: apply $z^{1/3}$ + translation.

Complex str's vs Hyp strs. $\chi(S) < 0$.

$$\{\text{hyp strs on } S\} \longleftrightarrow \{\text{complex strs on } S\}$$

$\longrightarrow$ isometries of $\mathbb{H}^2$ are holomorphic. (Möbius tr)

+ Cartan-Hadamard: only simply conn. complete surface with $K=-1$ is $\mathbb{H}^2$.

$\longleftarrow$ uniformization thm: only simply conn Riem surf's are $\mathbb{H}^2$, $\mathbb{C}$, $\hat{\mathbb{C}}$.

Linear maps of $\mathbb{R}^2$ via $\mathbb{C}$-analysis

$U, V \subseteq \mathbb{C}$ open

$f : U \to V$ smooth

$Df_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{R}$

Can write as:

$$Df_p(z) = \alpha z + \beta \bar{z}$$

$$\alpha = \frac{(a+ic) - i(b+id)}{2}$$

$$\beta = \frac{(a+ic) + i(b+id)}{2}$$

$1 \in \mathbb{C} \leftrightarrow (1,0) \in \mathbb{R}^2$ $i \in \mathbb{C} \leftrightarrow (0,1) \in \mathbb{R}^2$

Check $Df_p(1) = a + ic$

$Df_p(i) = b + id$

α, β called $f_z, f_{\bar{z}}$

$$Df_p(z) = f_z z + f_{\bar{z}} \bar{z}$$

Complex dilatation:

$$\mu_f = f_{\bar{z}} / f_z$$

$\mu_f = 0 \iff f$ holomorphic.

Dilatation of f

$$K_f(p) = \frac{|f_z(p)| + |f_{\bar z}(p)|}{|f_z(p)| - |f_{\bar z}(p)|} = \frac{1+|\mu_f(p)|}{1-|\mu_f(p)|} = d_{\mathbb{H}^2}(\mu_f(p), 0).$$

$= \text{eccentricity of } Df_p(S^1)$

$$K_f = \sup_p K_f(p)$$

to prove, write S^1 as $e^{i\theta}$, apply Df_p

$$\left| f_z(p)\, e^{i\theta} + f_{\bar z}(p)\, e^{-i\theta} \right|$$

$$1 - |\mu_f(p)| \le \cancel{|e^{i\theta}|}\,\cancel{|f_z(p)|}\,\left| 1 + \mu_f(p)\, e^{-2i\theta} \right| \le 1 + |\mu_f(p)|$$

Quasi-conformal maps

f is q.c. if $K_f < \infty$.

Holomorphic $\Longrightarrow$ 1-q.c.

Note: qc makes sense for Riem. surfaces since transitions maps are holomorphic.

$$\boxed{\text{We only consider maps that are smooth outside a finite set.}}$$

Fact. X, Y Riem surfs.

The set of qc maps $X \to Y$ forms a group

Pf. $K_{f\circ g} \leq K_f K_g$

$K_{f^{-1}} = K_f$ $\square$

Teichmüller's extremal problem

Fix $f: X \to Y$ homeo.

Is this inf realized?

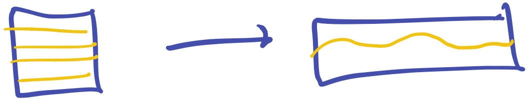
$$\inf\{K_h : h \sim f,\ h \text{ qc}\}$$

If so, what is min. map?

Teichmuller: existence & uniqueness.

$$\rightsquigarrow d_{Teich}(X,Y) = \frac{1}{2}\log K_h$$

Earlier, Grötzch did this for rectangles:

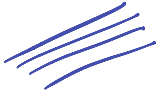


Extemal map is the obvious one & it is unique.

Measured foliations

Sing. foliation on S_g

locally:

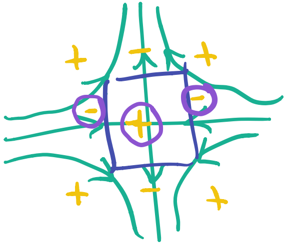


or

$k \geq 3$ prongs.

Prop. (Euler-Poincaré formula)

$$\chi(S) = \sum_{\text{sing}} \left(1 - \frac{k_i}{2}\right)$$



Special case: No singularities

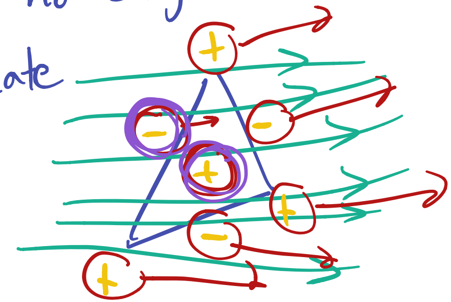
$\iff \chi(S) = 0.$

Pf (assuming foliation is orientable)

(W. Thurston)

Assume no singularities.

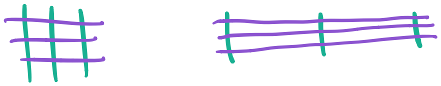
Triangulate



Chap 11. Teich geom.

Teich thms: Given $X, Y \in \mathrm{Teich}(S)$
$\exists$ unique map $h: X \to Y$
homot. to id that minimizes
dilatation K

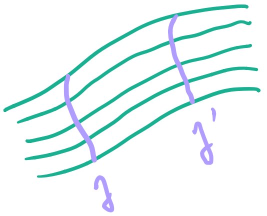
The map is locally:

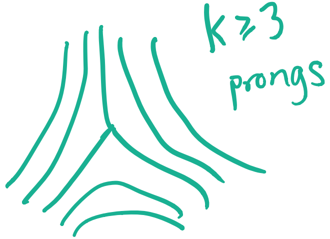


need to make sense of horiz/vert. on S.

$$\begin{pmatrix} \sqrt{K} & 0 \\ 0 & 1/\sqrt{K} \end{pmatrix}$$

Measured foliations



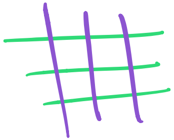


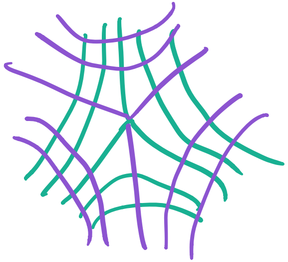
μ = transverse measure ≥ 0.

$$\mu(\gamma) = \mu(\gamma')$$

These allow us to do Teich maps as above.

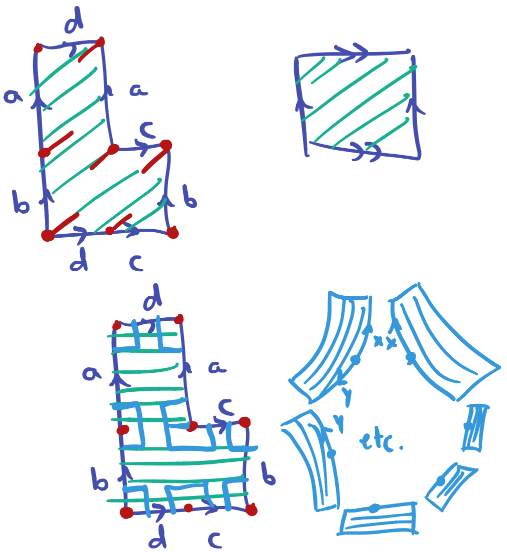
transverse foliations



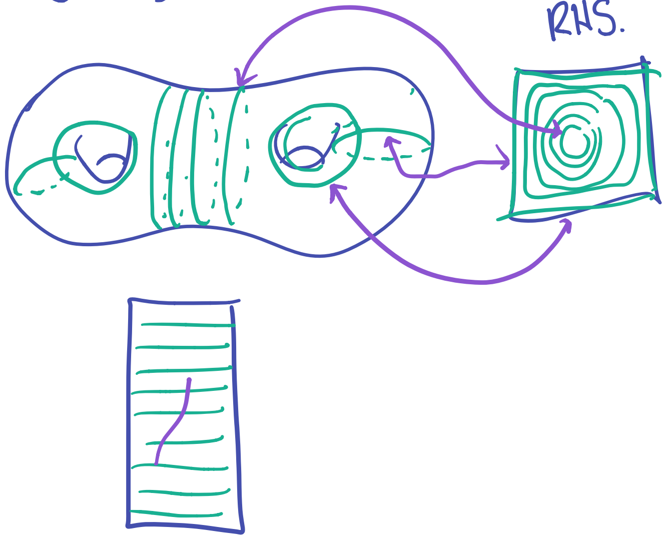


3 constructions

① Polygons

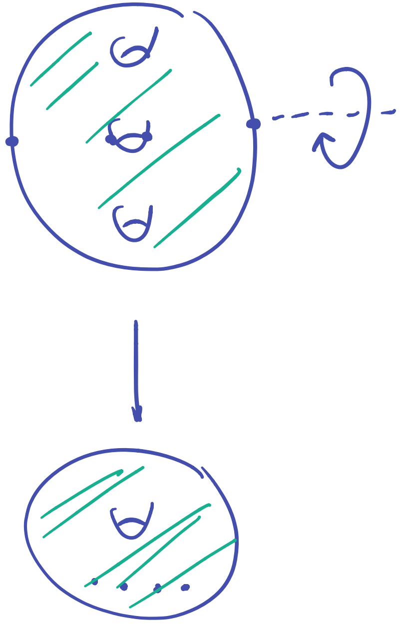


② Curves

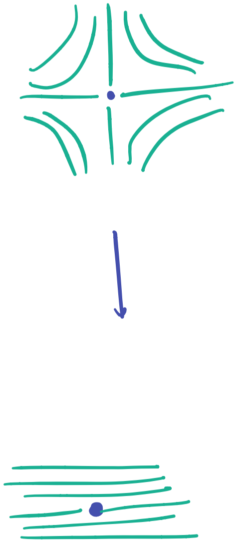


measure: Euclidean, $\perp$ to foliation

③ Branched covers



Lift a foliation from torus



Quadratic differentials

Single, complex analytic object that packages: complex str.

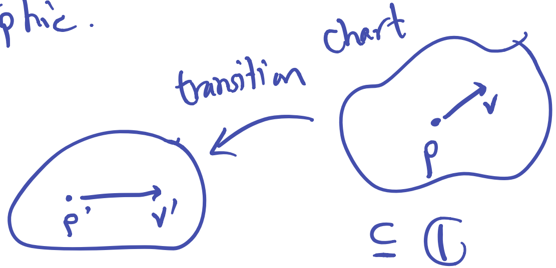
2 transv. foliations with measures.

In a chart:

$$q = \varphi(z)\, dz^2$$

φ holomorphic.

So that...



Invariant under transition maps:

$z_\alpha : U_\alpha \to \mathbb{C}$ charts
$z_\beta : U_\beta \to \mathbb{C}$

$q = \varphi_\alpha(z)\, dz_\alpha^2$ or $\varphi_\beta(z)\, dz_\beta^2$ in charts

$$\varphi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^2 = \varphi_\alpha(z_\alpha)$$

q eats tangent vectors, gives complex number.

$$q(v) = \cancel{\varphi(v)^2}$$
$$= \varphi(p)\, v^2$$

From QD's to foliations

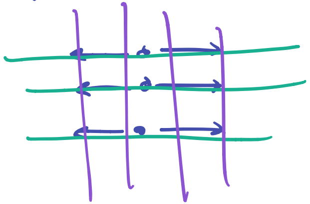
In a chart:

$$q = \varphi(z)\, dz^2$$

Horiz. foliation: $q > 0$

Vert. foliation: $q < 0$.

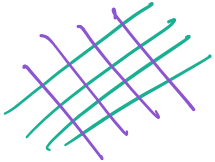
example $q = \varphi(z)\, dz^2 = 1 \cdot dz^2$



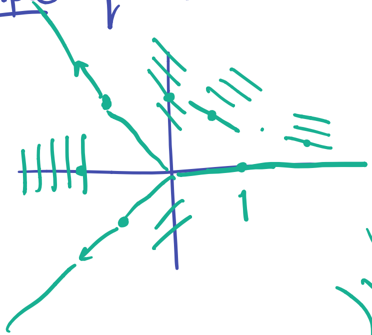
$q > 0$.

$q < 0$

example $q = \alpha\, dz^2 \quad \alpha \in \mathbb{C}$



example $q = z\, dz^2$



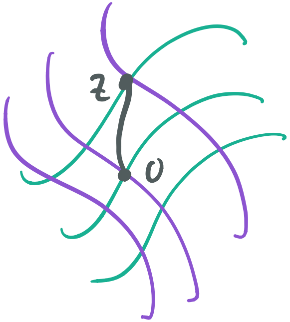
$z^k dz^2$

$k+2$ prongs.

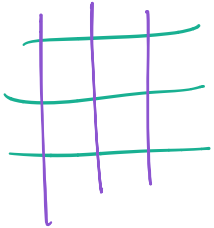
... and the measures

Every q has natural coords where it is $z^k \, dz^2$

So: away from zeros, measure is $|dx|$, $|dy|$



Z



$\eta \circ Z$

Say: $z : U \to \mathbb{C}$ chart

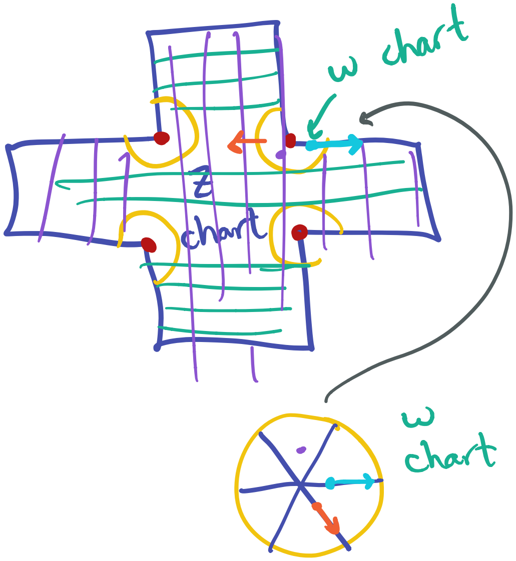
$$\eta(z) = \int_0^z \sqrt{\varphi(w)} \, dw$$

Choose a branch of $\sqrt{\ }$

dummy variable

Check: in these coords, away from zeros of q, $q = 1 \cdot dz^2$.

Example



In z-chart

$$q = 1\, dz^2 \qquad \varphi_z(z) = 1$$

In w-chart

$$q = \varphi_w(z)\, dz^2 = 9z^4 dz^2$$

Change of coords from w to z:

$$z^3 + \text{const.}$$

$$\varphi_z(z) \left(\frac{dz}{dw}\right)^2 = \varphi_w(z)$$

$$1 \cdot (3z^2)^2 = \varphi_w(z)$$

or

$$q = \alpha\, dz^2$$

$$q = \alpha 9 z^4 dz^2$$

$\rightsquigarrow$ foliations rotated by $\arg\alpha$.

Statement of Teich Thms

X, Y Riem surf's

A homeo $f: X \to Y$

is a Teich map if

$\exists$ qd's q_X initial differential

q_Y terminal.

& $K \in (0, \infty)$

s.t.

① $f(\text{zeros of } q_X)$

$= \text{zeros of } q_Y$

② At nonzero pts of q_X:

$$f(x+iy) = \sqrt{K}\, x + \frac{1}{\sqrt{K}}\, y$$

in natural coords

$\rightsquigarrow K_f = \max\{K, 1/K\}$

TET. X, Y Riem surfs

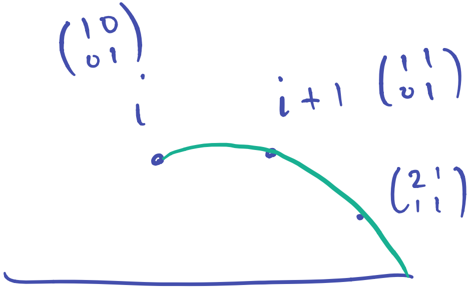
$f: X \to Y$ homeo

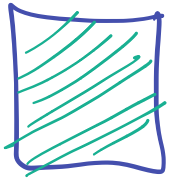
Then $\exists$ Teich map homotopic to f.

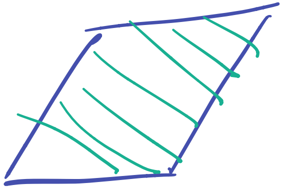
TUT. $h: X \to Y$ Teich map

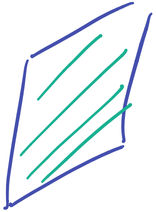
$f \sim h \Rightarrow K_f \geq K_h$

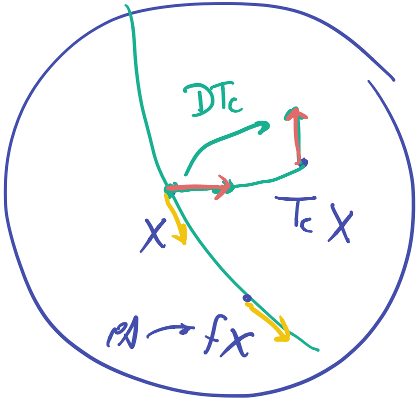
Equality $\Leftrightarrow f \circ h^{-1}$ conformal $\overset{g \geq 2}{\Longleftrightarrow} f = h$


$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
i
$i+1$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$
$\begin{pmatrix}2&1\\1&1\end{pmatrix}$








DT_c
X
$T_c X$

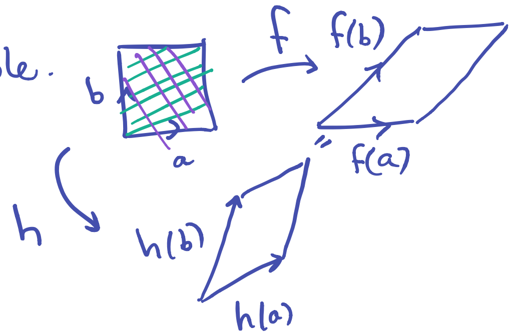
Teich Thms

TET. X, Y Riem surfs

$f : X \to Y$ homeo

$\exists$ Teich map $h \sim f$.

example.



TUT. $h : X \to Y$ Teich map

$f \sim h$

$\Longrightarrow K_f \geq K_h$

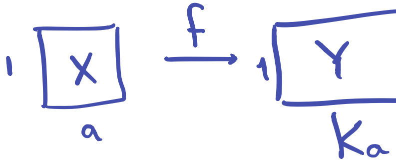
& equality $\Longleftrightarrow f = h \quad (g \geq 2)$

Grötzch's Problem

The rectangle case of TUT

(The 1D version is MVT.
$K = |f'|$)

Thm. Given

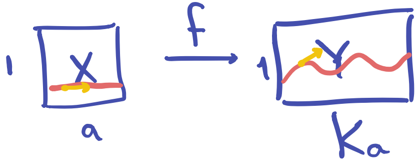


or. pres, side pres, almost smooth (smooth outside finite set)

Then $K_f \geq K$

& equality $\iff$ f is the obvious map.

Thm. Given



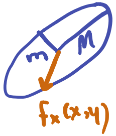
Then $K_f \geq K$

Pf. $K_f(x,y) = $ dil. at (x,y)

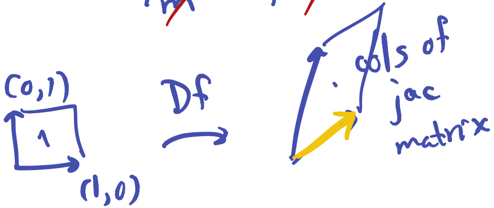
$j_f(x,y) = $ jacob. of f @ (x,y)

Claim 1. $|f_x(x,y)|^2 \leq K_f(x,y)\, j_f(x,y)$

Pf.



$M/\cancel{m} \cdot M\cancel{m}$



Uniqueness

For 1st $\leq$ to be $=$ need hor arcs $\longmapsto$ hor arcs.

Symmetry: vertical arcs $\to$ vertical.

etc.

Claim 2. $\int_X |f_x(x,y)|\, dA \geq K\, \text{Area}(X)$

Pf. Take $\underbrace{\int_0^a |f_x(x,y)|\, dx}_{\text{length}(f(\text{hor arc}))} \geq Ka$

y fixed

& integrate over y.

Now: $(K\, \text{Area}(X))^2 \overset{2}{\leq} \left(\int_X |f_x(x,y)|\, dA\right)^2$

$\overset{1}{\leq} \left(\int_X \sqrt{K_f(x,y)}\sqrt{j_f(x,y)}\, dA\right)^2$

$\overset{C\text{-}S}{\leq} \int_X K_f(x,y)\, dA \int_X j_f(x,y)\, dA$

$\leq K_f\, \text{Area}(X)\, \text{Area}(Y)$

$= K_f\, K\, \text{Area}(X)^2$ $\square$

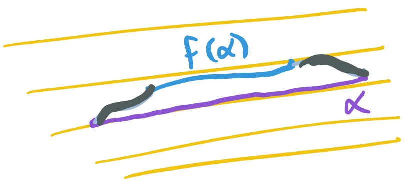
For TUT, need a version of Claim 2. But: leaves might not be closed...

Lemma. $q_Y \in QD(Y)$

$f : Y \to Y$ $f \sim id.$

$\exists M$ s.t. $\forall$ horiz. arcs α geodesic.

$$\ell_{q_Y}(f(\alpha)) \geq \ell_{q_Y}(\alpha) - M$$



Pf. $M = 2 \cdot$ max distance a pt moves under homotopy f to id.

α geodesic

$$\Rightarrow \quad \ell(f(\alpha)) + M \geq \ell(\alpha) \quad \square$$

Next: Analog of Claim 2 using this Lemma.

Prop. $h: X \to Y$ Teich map
init diff q_X term diff q_Y
hor stretch K, $f \sim h$ almost smooth

Then $\int_X |f_x| \, dA \geq K \, \text{Area}(q_X)$

Pf. Define $\delta : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\delta(p, L) = \int_{-L}^{L} |f_x| \, dx$$

$$= \ell_{q_Y} f(\alpha_{p,L})$$

$\alpha_{p,L}$ = hor. arc length $2L$ thru p.

Also: $\ell_{q_Y}(h(\alpha_{p,L})) = 2KL$

Lemma $\Longrightarrow \ell_{q_Y}(f(\alpha_{p,L})) \geq 2KL - M$
some M.

So:

$$\int_X \delta(p,L) \, dA = \int_X \ell_{q_Y}(f(\alpha_{p,L})) \, dA$$

$$\geq \int_X (2KL - M) \, dA$$

$$= (2KL - M) \, \text{Area}(X)$$

Fubini: $\int_X \delta(p,L) \, dA = \int_X \left(\int_{-L}^{L} |f_x| \, dx \right) dA$

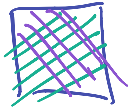
$$= 2L \int_X |f_x| \, dA$$

So: $\int_X |f_x| \, dA \geq \left(K - \frac{M}{2L} \right) \text{Area } X$
$\forall L$. $\square$

Pf of TUT. Repeat Grötzch argument $\square$

Proof of TET

$X \in \text{Teich}(S)$



$QD(X) = \mathbb{C}\text{-vector space}$

$\mathbb{R}\text{-}\dim = 6g-6$ (Riemann-Roch)

Define $\|q\| = \int_X |\varphi| = \text{area}$

$q = \varphi(z)\,dz^2$

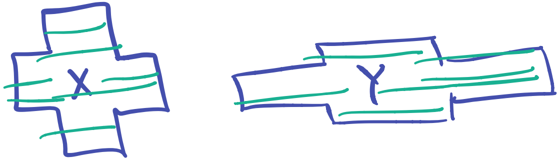
$QD_1(X) = \text{open unit ball}.$

$\rightsquigarrow K = \dfrac{1+\|q\|}{1-\|q\|}$

$\rightsquigarrow Y \in \text{Teich}(S)$

& Teich map $h : X \to Y$.

example.



$q = dz^2$

So $QD(X) \longleftrightarrow T_X\text{Teich}(S)$

line in Teich(S) "exponential map"

$\rightsquigarrow \Omega : QD_1(X) \longrightarrow \text{Teich}(S).$

$\text{TET} \iff \Omega$ surjective.

Prop. Ω continuous

hard part!

Prop. Ω proper.

Also: Ω inj by TUT

& $\dim QD_1 = 6g-6$

Brouwer's Inv. of Domain:

Any proper, inj contin. map

$\mathbb{R}^n \to \mathbb{R}^n$

is a homeo.

Continuity uses Beltrami differentials

PDEs.

Teichmüller metric

$$d_{\text{Teich}}(X,Y) = \frac{1}{2}\log K$$

where K is dilatation of Teich map $h: X \to Y$.

Prop. d_{Teich} is a complete metric.

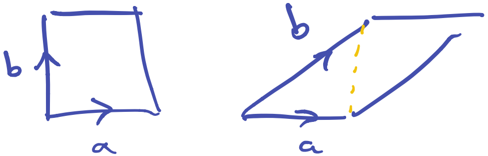
Prop. Teich lines above are geodesics in d_{Teich}. (TUT)

Prop. Teich(S) is a geodesic metric space (TET + prev. prop)

Prop. d_{Teich} for T^2 is hyp metric on $\mathbb{H}^2$ (up to multiple).

Chap 12. Moduli Space

$M(S) = \{$ hyp / complex / alg / conformal structures on $S \} / \sim$



different in Teich

same in M

$F \cdot X = X \quad \forall X$

$\ell_X(F^{-1}(c)) = \ell_{F \cdot X}(c) = \ell_X(c) \quad \forall c$

$\mathrm{Mod}(S) \circlearrowright \mathrm{Teich}(S)$

by pulling back metrics...

In terms of markings:

$$[\psi] \cdot (X, \varphi) = (X, \varphi \circ \psi^{-1})$$

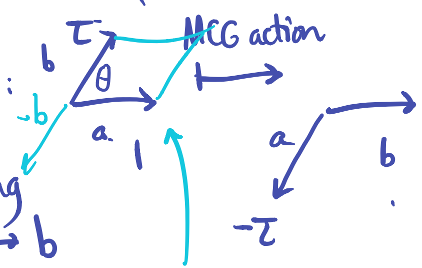
- Action is by isometries.
- $\mathrm{Stab}(X) = \mathrm{Isom}^+(X)$ finite
- Kernel is $\begin{cases} \mathbb{Z}/2 & g = 1,2 \\ 1 & g \geq 3 \end{cases}$ ← hyp. inv.

$$M(S) = \mathrm{Teich}(S) / \mathrm{Mod}(S)$$

The torus

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:



as a mapping class : $a \to b$

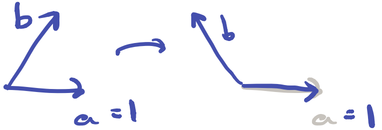
$b \to -a$

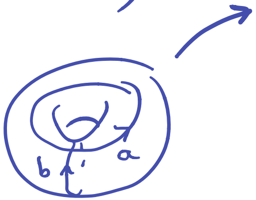
Prop. The action of $\mathrm{Mod}(T^2) = SL_2\mathbb{Z}$ on $\mathrm{Teich}(T^2) = \mathbb{H}^2$ is by Möbius transf. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \dfrac{az-b}{-cz+d}$

Pf. Check on generators.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:





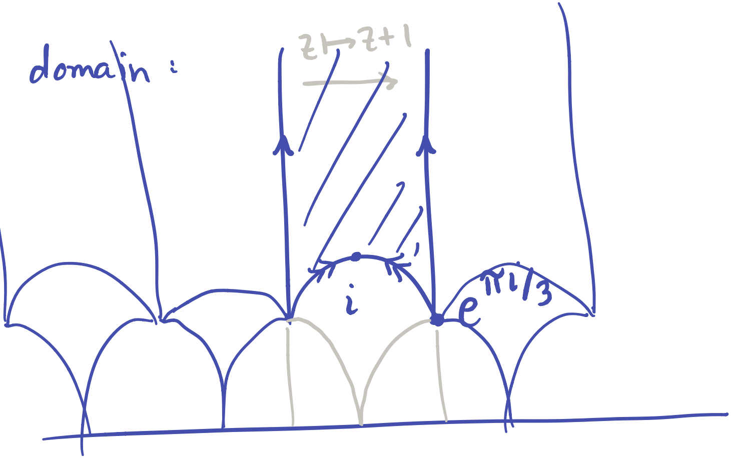
$z \longmapsto z-1$ ✓

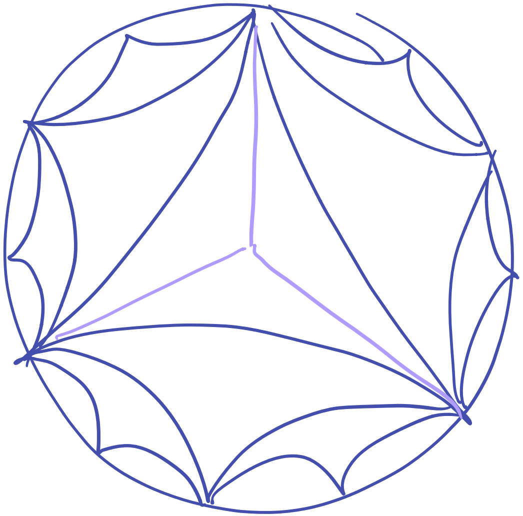
To put latter in std form, divide by $-\tau \rightsquigarrow -\frac{1}{\tau}$ in $\mathbb{H}^2$.

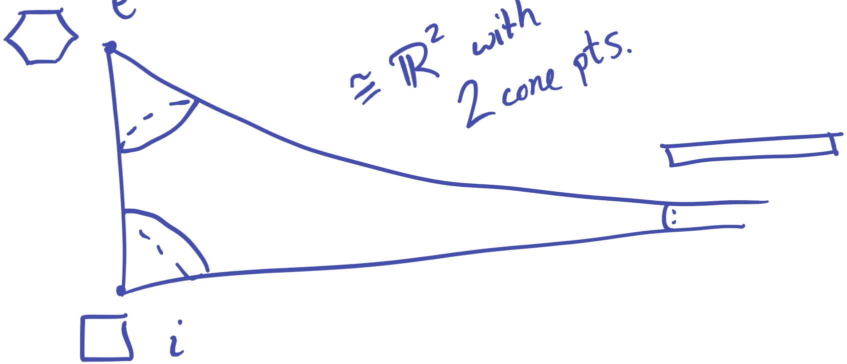
agrees with:

$$\frac{0z-1}{1z+0} = -\frac{1}{z}$$

Fund domain:







$\cong \mathbb{R}^2$ with 2 cone pts.

$$PSL_2\mathbb{Z} \cong \mathbb{Z}/2 * \mathbb{Z}/3$$

Proper Discontinuity

$G \circlearrowright X$ prop disc if

$\forall$ compact $K \subseteq X$

$\#\{g \in G : gK \cap K \neq \emptyset\} < \infty.$

Thm (Fricke) $\mathrm{Mod}(S_g) \circlearrowright \mathrm{Teich}(S_g)$ is prop. disc.

Thm + Teich metric $\Longrightarrow$ metric on $\mathcal{M}(S)$.

(inf of dist. b/w lifts).

Tool: Raw length spectrum.

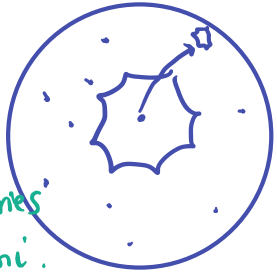
$$rls(X) = \{\ell_X(c)\} \subseteq \mathbb{R}.$$

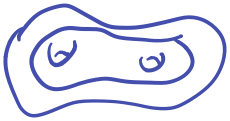
Lemma. $X \in \mathrm{Teich}(S)$. $\forall L$

Then $\#\{c : \ell_X(c) \leq L\} < \infty$

In partic, $rls(X)$ closed, discrete in $\mathbb{R}$.

Pf. Prop. disc. of $\pi_1(S) \circlearrowright \mathbb{H}^2$.





$\square$

Birman-Series

cf. Mirzakhani

Wolpert's Lemma

X_1, X_2 hyp. surfaces

$\varphi : X_1 \to X_2$ quasi-conf homeo $(K < \infty)$.

For all c:

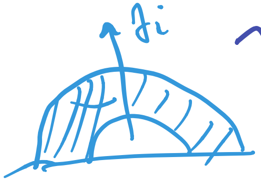
$$\frac{1}{K} \ell_{X_1}(c) \le \ell_{X_2}(\varphi(c)) \le K \ell_{X_1}(c)$$

"curves get stretched by at most K."

Pf. $\gamma_1, \gamma_2 \in \mathrm{Isom}^+(\mathbb{H}^2)$

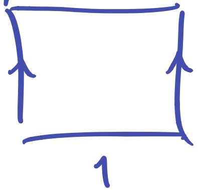
$\updownarrow \qquad \updownarrow$

$c \subseteq X_1 \quad \varphi(c) \subseteq X_2$



$\rightsquigarrow$ hyp annuli $\mathbb{H}^2 / \langle \gamma_i \rangle = A_i$ cover of X_i.

A_i is conformally equiv. to a unique* std annulus



* Grötzch+

φ lifts to $\tilde{\varphi} : A_1 \to A_2$ (lifting criterion)

Same qc const.

Grotzch $\Rightarrow \frac{m_1}{K} \le m_2 \le K m_1$

$\Rightarrow$ lemma $\blacksquare$

Proof of PD

$d(X,Y) = \frac{1}{2}\log K$

$B \subseteq \mathrm{Teich}(S_g)$ compact.

$X \in B$ arbitrary.

$D = \operatorname{diam} B$.

c_1, c_2 curves that fill S_g.

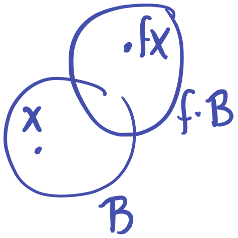
$L = \max\{\ell_X(c_1), \ell_X(c_2)\}$

Say $f \cdot B \cap B \neq \emptyset$.

(WTS finitely many such f)

$f \cdot B \cap B \neq \emptyset$

$\Rightarrow d(X, f\cdot X) \leq 2D$



Wolpert $\Rightarrow \ell_{f\cdot X}(c_i) \leq KL$

where $K = e^{4D}$

$\Rightarrow \ell_X(f^{-1}(c_i)) \leq KL$

Lemma $\Rightarrow$ finitely many choices for $f(c_1)$ & $f(c_2)$.

Alex method $\Rightarrow$ finitely many choices for f. $\square$

Moduli space

$\mathcal{M}(S) = \{\text{hyp. str}\}/\text{isometry}.$

Also: $\mathcal{M}(S) = \text{Teich}(S)/\text{Mod}(S)$

Mod(S) acts by pullback:

$$[\varphi] \cdot X = (\varphi^{-1})^* X$$

Torus case

"A"

Action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} = \text{Mod}(T^2)$

on marked lattice $\Lambda \cong \mathbb{Z}^2 \in \text{Teich}(T^2)$

$\updownarrow$

matrix $M \in M_2\mathbb{R}$

Action is by Möbius trans

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az-b}{-cz+d}$$

The torus

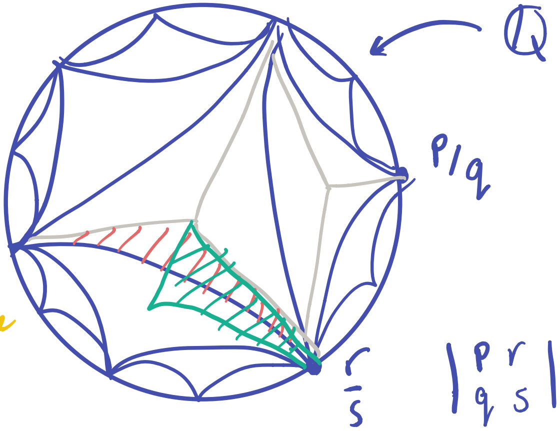
$\mathcal{M}(T^2) =$

Möb action $\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$

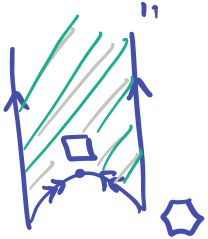
$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$

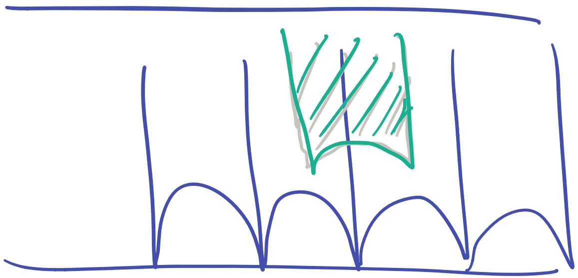
lin alg action.

in upper half-plane



$\begin{vmatrix} p & r \\ q & s \end{vmatrix} = 1$





$\mathbb{R}$ Mobius trans act on this 2-complex.

$SL_2\mathbb{Z}$ acts trans. on Δ's

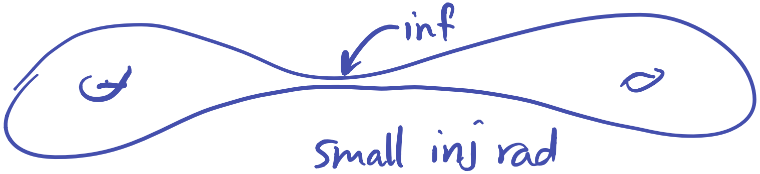
Stab of Δ rotates

Mumford's Compactness Criterion

$\ell : \mathcal{M}(S) \longrightarrow \mathbb{R}_+$

$X \longmapsto$ length of shortest curve in X.

$$\ell(X) = 2\,\underbrace{\text{inj rad}(X)}_{\inf\limits_{x \in X}\{\text{largest embedded disk at } x\}}$$



ℓ is continuous.

$\mathcal{M}(S)$ is not compact because ℓ has no minimum. (keep pinching)

Define $\mathcal{M}_\varepsilon(S) = \{X \in \mathcal{M}(S) : \ell(X) \geq \varepsilon\}$

"ε-thick part"

<u>Thm</u>. $\forall\, \varepsilon$, $\mathcal{M}_\varepsilon(S)$ compact.

So: only way to go to ∞ is to pinch curves.

Torus case: evident from picture

Bers constant

Thm. $\chi(S)<0$ Maybe $\partial S \neq \emptyset$.

① $\exists\, L_0 = L_0(S)$ s.t.

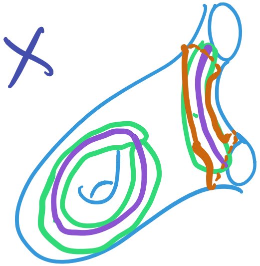
$\forall X \in M(S)$ $\exists$ curve in X

of length $\leq L_0$.

② $\exists\, L = L(S)$ s.t. $\forall X$

$\exists$ pants decomp. of

length $\leq L$.



Pf. ① $\Rightarrow$ ② by induction

on # curves (cut open)

Given X find largest radius disk D

with interior embedded & disjoint

from ∂X.

D is a hyperbolic disk. radius r.

$$\text{Area } D = 2\pi(\cosh(r)-1)$$

$$\leq \text{Area } X = -2\pi\chi(S)$$

If ∂D touches itself $\Rightarrow$ short curve.

If ∂D touches ∂X, it touches in at

least two points $\Rightarrow$ short arc $\Rightarrow$ short curve.

One of these 2 situations must happen $\square$

Define $\mathcal{M}_\varepsilon(S) = \{X \in \mathcal{M}(S) : \ell(X) \geq \varepsilon\}$

"ε-thick part"

<u>Thm</u>. $\forall\, \varepsilon$, $\mathcal{M}_\varepsilon(S)$ compact.

<u>Pf</u>. $\mathcal{M}(S)$ metrizable. $\Rightarrow$ enough to show seq. compact.

$(X_i) \subseteq \mathcal{M}_\varepsilon(S)$

We will find lifts to Teich(S) lying in closed cube in FN coords.

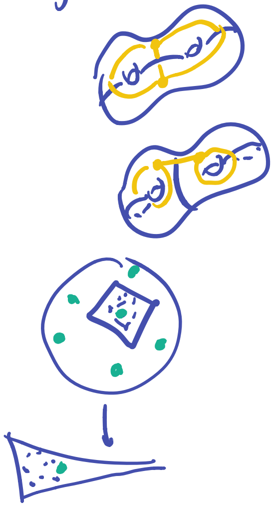
Bers: Each X_i has pants decomp where curves have length in $[\varepsilon, L]$.

Pass to subseq so these pants decomps are topologically equivalent.

Choose lifts to Teich(S) where a <u>specific</u> pants decomp. has length in $[\varepsilon, L]$.

So length params in $[\varepsilon, L]$.

Can modify twist params to be in $[0,1]$ (Dehn twists)



$\square$

The end of moduli space

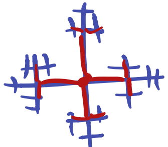
Z = connected, locally compact ~~metric~~ space

Z has one end if $Z \setminus K$ has one unbounded component $\forall$ compact K.

or if $\exists$ exhaustion $K_0 \subseteq K_1 \subseteq \cdots$ by compact sets so $Z \setminus K_i$ connected $i \gg 0$.

one end: $\mathbb{R}^n$ $n \geq 2$.

not one end: $\mathbb{R}^n$ $n \leq 1$.

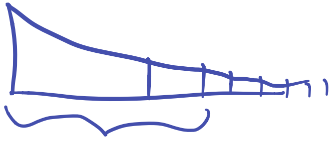


∞ many ends

Cantor set.

Thm. $\mathcal{M}(S)$ has one end.

Pf. $\mathcal{M}_\varepsilon(S)$ form an exhaustion by compact sets



$\mathcal{M}_\varepsilon$

To show $\mathcal{M} \setminus \mathcal{M}_\varepsilon$ connected $\forall \varepsilon$.

Let $X, Y \in \mathcal{M} \setminus \mathcal{M}_\varepsilon$

Lift to $\tilde{X}, \tilde{Y} \in$ Teich.

short curve c   short curve d.

Connect c,d in $\mathcal{C}(S)$....

pinch consec. curves one at a time... $\square$

THEOREM 2. *Modulus space is simply-connected.*

PROOF. It is proved in [4] that each element of finite order in $M(K_g)$ has a fixed point in $T(K_g)$, so that, by Theorem 1, $M(K_g)$ is generated by elements which have fixed points. Also $M(K_g)$ is a properly discontinuous group of homeomorphisms of a space homeomorphic to $\boldsymbol{R}^{6g-6}$. Furthermore, the stabiliser of a point $[\phi]$ of $T(K_g)$ is isomorphic to the group of conformal self-homeomorphisms of the compact Riemann surface $D/\phi(K_g)$ and hence is finite. Thus, applying a result of Armstrong [1] we have that $T(K_g)/M(K_g)$ has trivial fundamental group.

Chap 13. Nielsen-Thurston Classification.

<u>Thm</u> (Thurston) Every $f \in \mathrm{Mod}(S)$ has a representative φ s.t.

① periodic: $\varphi^n = 1$

② reducible: $\varphi(M) = M$ some multicurve M

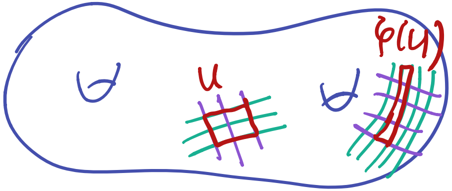
③ pseudo-Anosov:

$\exists$ transv. meas. fol's $\mathcal{F}_u, \mathcal{F}_s$ & $\lambda > 1$ s.t. "stretch factor"

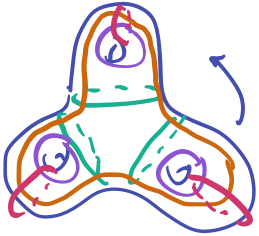
$$\varphi \cdot \mathcal{F}_u = \lambda\, \mathcal{F}_u$$

$$\varphi \cdot \mathcal{F}_s = \frac{1}{\lambda}\, \mathcal{F}_s$$

Moreover ③ is exclusive from ① & ②.



Examples



per.

& red.

CRS = $\emptyset$

per.

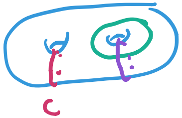
(not. red)

b/c quotient orbifold is 

T_c

red

not per.



CRS = c

 $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

pA

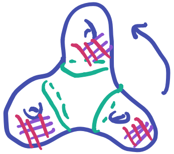
foliations $\longleftrightarrow$ eigenvectors

stretch factor $\longleftrightarrow$ eigenvalues.

Birman - Lubotzky - McCarthy:

Canonical reduction system = intersection of max reduction systems.

Restatement of NTC: Every mapping class reduces to per & pA pieces.



a typical mapping class.

Jordan form

Torus case

$f \in \mathrm{Mod}(T^2) \longleftrightarrow A \in SL_2\mathbb{Z}$

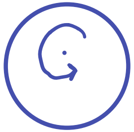
$\longleftrightarrow \tau \in \mathrm{Isom}^+(\mathbb{H}^2)$

Case 1 2 complex eigenvals

per.

$\longleftrightarrow \tau$ rotation.

prop disc. $\Rightarrow |\tau| < \infty$.



Case 2. 1 real eigenval

red

product of eigenvals = det = 1.

$\lambda = \pm 1$

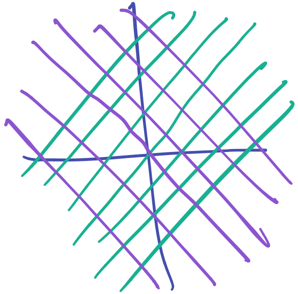
$\Rightarrow$ eigenvector rational.

$\rightsquigarrow$ fixed curve

Case 3 2 real eigenvals. $\lambda, \frac{1}{\lambda}$

pA

$\Longrightarrow f$ (pseudo-) Anosov.



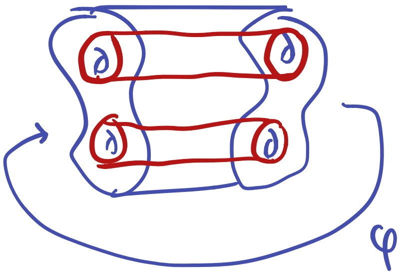
$$X^2 + X + 1 = 0$$

$$(X-1)(X^2+X+1) = 0$$

$$X^3 = 1.$$

3-manifolds

$f \in \mathrm{Mod}(S) \rightsquigarrow M_f$ = mapping torus



$[\varphi] = f$.

Thm (Thurston) $f \in \mathrm{Mod}(S)$ $\chi(S) < 0$.

- f per $\iff$ M_f admits metric locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.
- f red $\iff$ M contains incompressible torus
  $\hookrightarrow \pi_1$-inj.
- f pA $\iff$ M admits hyperbolic metric.
  $\Rightarrow$ hard.

Periodic elements

"Easy Nielsen realization" (Fenchel)

Thm. $f \in \mathrm{Mod}(S)$ periodic $\Rightarrow$ f has a periodic rep:

$$\varphi^n = 1.$$

Pf. To show f has fixed pt in $\mathrm{Teich}(S)$.

Indeed: $f \cdot X = X$

$$\varphi^* X \sim X$$

Can change φ by isotopy so fixes X on nose.

Note $\langle f \rangle \cong \mathbb{Z}/m$

Fact. $\mathbb{Z}/m$ cannot act freely on a fin. dim contractible space.

(otherwise quotient is a fin. dim $K(\mathbb{Z}/m, 1)$ & $H^k(\mathbb{Z}/m) \neq 0$ arb. large k).

So f^j fixes $X \in \mathrm{Teich}(S)$ some j.

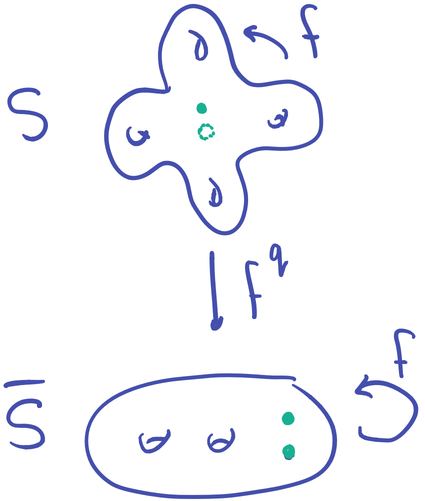
Special case. m prime.

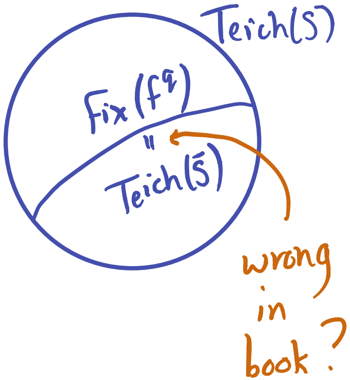
$f = (f^j)^k$ some k. $\Rightarrow f \cdot X = X$.

Assume now $m = pq$, p prime, q prime.

Note f^q has order p.

As above f^q fixes $X \in \mathrm{Teich}(S)$.





The map

$$\mathrm{Teich}(\bar{S}) \to \mathrm{Fix}(f^q)$$

is: lift complex structures.

Injectivity: Teich. U.T.*

$\bar{X} \neq \bar{Y} \in \mathrm{Teich}(\bar{S})$

$\rightsquigarrow$ Teich map of $\bar{S}$

$\rightsquigarrow$ Teich map of S between lifts X, Y.

$\Longrightarrow X \neq Y$.

Surjectivity: Special case.

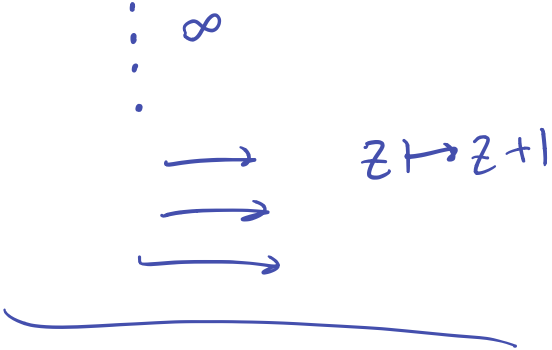
$\square$

Outline of proof of NTC (Bers)

$f \in \mathrm{Mod}(S)$

$$\tau(f) = \inf\{X \in \mathrm{Teich}(S) : d(X, f\cdot X)\}$$

"translation length"



To show:

$\tau(f) = 0$ & realized $\iff$ f periodic

$\tau(f)$ not realized $\iff$ f reducible

$\tau(f) > 0$ & realized $\iff$ f pA.

like torus case.

Nielson-Thurston Classification

Thm. Every $f \in Mod(S)$ has a rep φ s.t.

1. periodic: $\varphi^n = 1$
2. reducible: $\varphi(M) = M$
3. pseudo-Anosov:

$$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u$$
$$\varphi \cdot \mathcal{F}_s = \frac{1}{\lambda} \mathcal{F}_s$$

(3) is exclusive from (1) & (2)

Exclusivity: we show in Chap 14

for any curve γ

$$\ell_X(f^n(\gamma)) \to \infty$$
$$\text{IS}$$
$$\lambda^n \ell(\gamma)$$

Proof. $\tau(f) = \inf_X d_{Teich}(X, f \cdot X)$

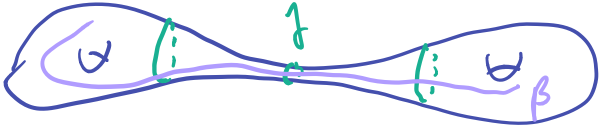
"translation length"

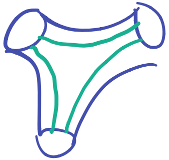
elliptic: $\tau(f) = 0$, realized $\Rightarrow$ f periodic ✓

parabolic: $\tau(f)$ not realized $\Rightarrow$ f reducible

loxodromic: $\tau(F) > 0$, realized $\Rightarrow$ f pA

Collar Lemma





Prop. γ = scc on hyp $X \implies$ r-nbd of γ is an embedded annulus where $r = \sinh^{-1}\left(\dfrac{1}{\sinh \frac{1}{2}\ell(\gamma)}\right)$

Note $r \longrightarrow \infty$ as $\ell(\gamma) \to 0$.

Cor. X = hyp surf

$\exists\, \delta$ s.t.

$\ell(\beta), \ell(\gamma) < \delta$

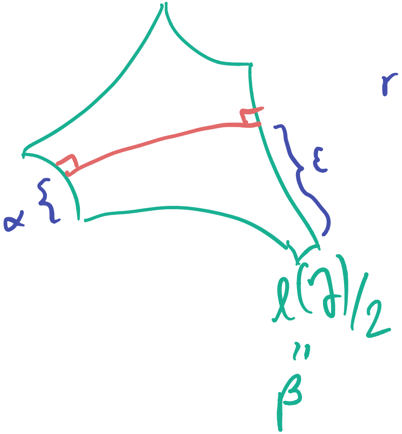
$\implies i(\beta, \gamma) = 0$.

δ = universal $\forall X$ in all Teich(S).

Pf. Choose pants decomp $\{\gamma_1, \dots\}$

$\rightsquigarrow$ hyp pants $\rightsquigarrow$ right angled hexagons

$\rightsquigarrow$ right angled pentagons



rt ang pent formula:

$$\sinh \varepsilon \sinh \beta = \cosh \alpha \geq 1$$

$\square$

Parabolic $\Rightarrow$ Reducible

Assume $\tau(f)$ not realized.

Choose X_i s.t.

$$d(X_i, f\cdot X_i) \longrightarrow \tau(f)$$

Step 1. $\ell(X_i) \longrightarrow 0$.

Pf is essentially prop disc.
(next)

Step 2. Find reduction curves.

Wolpert Lemma: $d(X,Y) \le \tau(f)+1$

$$\Rightarrow \ell_X(c) \le K\,\ell_Y(c)$$

some fixed K.

Choose $X = X_N$ s.t.

① $d(X, f\cdot X) \le \tau(f)+1$

② $\ell(X) < \left(\frac{1}{K}\right)^{3g-3} \delta$ (Step 1)

#curves in a pants decomp.

from Wolpert

collar lemma const.

Choose c s.t. $\ell_X(c) = \ell(X)$

Will show $c, f^{-1}(c), f^{-2}(c), \ldots, f^{-(3g-3)}(c)$ is a reduction system.

Wolpert ②

Have $\ell_X(f^{-i}(c)) = \ell_{f^i X}(c) \le K^i \ell(c) < \delta$

So the $c, f^{-1}(c), f^{-2}(c), \ldots, f^{-(3g-3)}(c)$ are disjoint by collar lemma. Must repeat (only $3g-3$ disjoint curves)! $\square$

Loxodromic $\Rightarrow$ pA

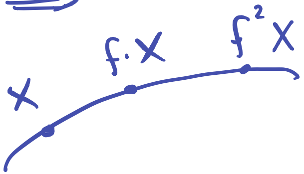
Choose X s.t.

$$d(X, f\cdot X) = \tau(f) > 0.$$

Let γ = Teich geod $X \longrightarrow f(X)$

Claim: $f\cdot\gamma = \gamma$

Claim $\Rightarrow$

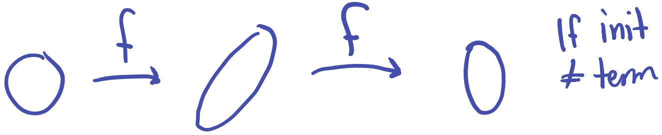


Let $h : X \to X$ Teich map in homotopy class of f.

Then h^2 is a Teich map in homotopy class of f^2.

$$\boxed{\text{We have: Initial \& terminal qd's for } h \text{ are } \underline{\text{equal}}.}$$

If not: $d(X, f^2\cdot X) < 2\, d(X, f\cdot X)$

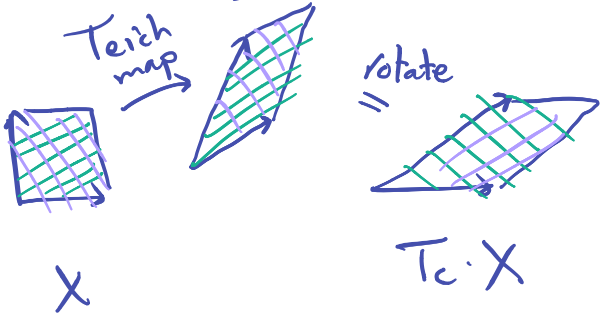


Violates

By the yellow box: h, hence f is pA with foliations from initial qd $\square$

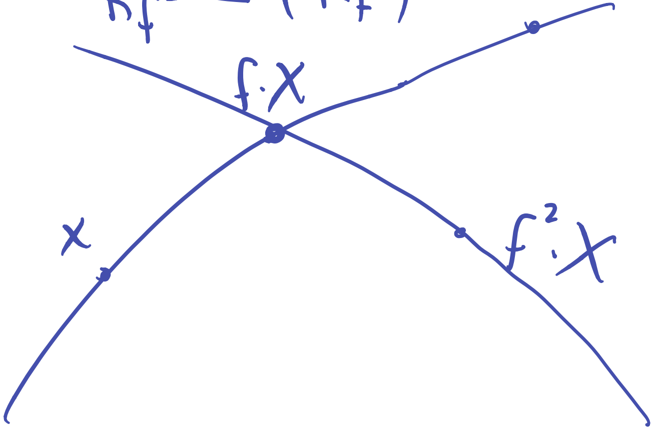
Picture for $T_c \in \mathrm{Mod}(T^2)$

"same" foliations in $\mathbb{R}^2$



Initial $\neq$ Terminal $\Longrightarrow$

$$K_{f^2} < (K_f)^2$$

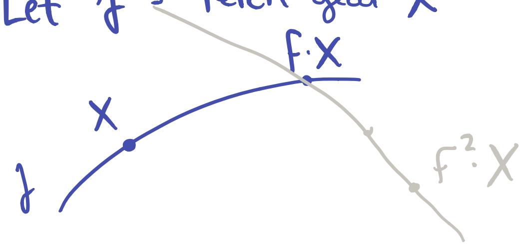


Loxodromic $\Rightarrow$ pA

Choose X s.t.

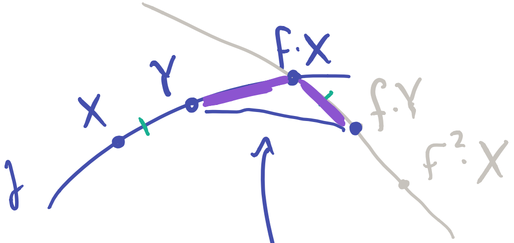
$$d(X, f\cdot X) = \tau(f) > 0.$$

Let γ = Teich geod $X \longrightarrow f(X)$



Claim: $f\cdot\gamma = \gamma$

Pf: Must rule out above picture.



$$d(Y, f\cdot Y) < d(X, f\cdot X)$$

violating $d(X, f\cdot X) = \tau(f)$

Indeed: purple path has length

$$d(X, f\cdot X).$$

Minimality of $X \Rightarrow f\cdot Y$ lies on γ.....

Some things about pA's

f pA with $\mathcal{F}_u, \mathcal{F}_s, \lambda$.

h commutes with f

$\Longrightarrow$ h preserves $\mathcal{F}_u, \mathcal{F}_s$

$\Longrightarrow$ h pA with same foliations $\Longrightarrow$ h is a power of a root of f.

or h periodic (if $\mathcal{F}_u, \mathcal{F}_s$ have symmetries)

$\Longrightarrow$ Centralizer of f is virtually cyclic.

Ian Runnels seminar @ 2

Writing assignment Dec 9.

Chap 14. pA Theory.

pseudo-Anosov

$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u$

$\varphi \cdot \mathcal{F}_s = \lambda^{-1} \mathcal{F}_s$

NTC. $f \in \mathrm{Mod}(S)$

① periodic
② reducible
③ pA

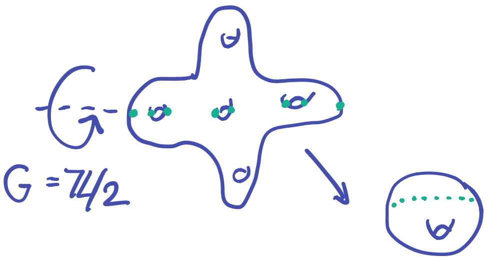
Today: constructions.

Construction #1 Branched covers.

$p: M \to N$ is a branched cover if it is a cover over $N \setminus B$, B small.

For surfaces: B = finite set.

Example. $G \circlearrowright S_g$ $|G| < \infty$.



$p: S_g \to X$ branched cover.

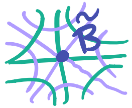
Assume $X \approx (T^2, B)$.

Take $\varphi : T^2 \to T^2$ Anosov.

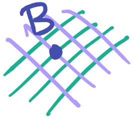
Up to power, isotopy φ fixes B. (periodic pts dense)

Further power: φ lifts to S_g. (lifting criterion)

The lift is pA. with $\mathcal{F}_s, \mathcal{F}_u$ lifts of foliations in T^2.



$\longrightarrow$

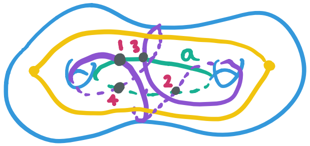


Note: All resulting stretch factors are quadratic integers.

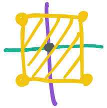
Construction #2 Thurston's construction

Thm $a, b \subseteq S_g$ filling.

$\exists$ sing Eucl. structure and



$$\rho : \langle T_a, T_b \rangle \longrightarrow PSL_2\mathbb{R}$$



$$f \longmapsto Df$$

$$T_a \longmapsto \begin{pmatrix} 1 & -i(a,b) \\ 0 & 1 \end{pmatrix}$$

$$T_b \longmapsto \begin{pmatrix} 1 & 0 \\ i(a,b) & 1 \end{pmatrix}$$

With:

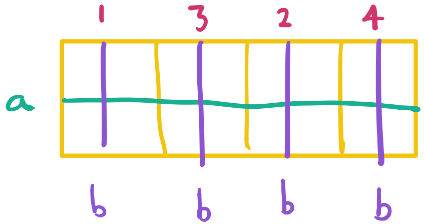
$\rho(f)$ elliptic $\iff$ f periodic

$\rho(f)$ parabolic $\iff$ f reducible

$\rho(f)$ hyperbolic $\iff$ f pA e.g. $T_a T_b^{-1}$

Cor. $\exists$ pA's in $\mathcal{I}(S_g)$. (take a,b sep)

Pf. From $a,b \rightsquigarrow X$ = dual square complex.



T_a acts on Eucl. structure.

by $\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$

Similar for T_b.

If $\rho(f)$ hyperbolic $\rightsquigarrow$ eigenvals λ, λ^{-1}

2 foliations

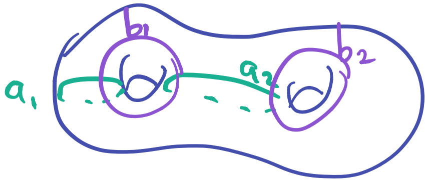
Those are stretch factor, foliations for f.

- There is a version with multicurves A, B.
- All resulting stretch factors totally real.

Penner's construction

Thm. $A = \{a_1, \dots, a_m\}$
$B = \{b_1, \dots, b_n\}$
filling multicurves.

Any
f = product of pos. powers of T_{a_i} & neg powers of T_{b_i} s.t. each a_i, b_i appears at least once.
is pA.



$$T_{a_2}^{15} T_{b_2}^{-7} T_{b_1}^{-1} T_{a_1}^{100}$$

Penner: Do all pA have a power coming from this construction?

Shin-Strenner: No. The Galois conjugates of Penner stretch factors all on S^1.

Construction # 3 Homological criterion.

$A \in Sp_{2g}\mathbb{Z} \Rightarrow$ char poly is monic & palindromic

Why? roots come in pairs λ, λ^{-1}

So do sub: $x^g P(\frac{1}{x})$

Why? $A^T J A = J \Rightarrow A^T \sim A^{-1}$

Thm (Casson-Bleiler, M-Spallone w/ Bestvina)

If char. poly of $\Psi(f)$ satisfies:

1. symplectically irred
2. not cyclotomic
3. not poly in t^k, $k>1$.

Then f is pA.

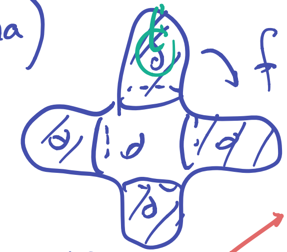
Pf. Suppose f not pA.

f periodic $\Rightarrow \Psi(f)$ has root of 1 as eigenval.

$\Rightarrow$ cyclotomic factor, violates 1 or 2.

f reducible, a power fixing nonsep $\Rightarrow$ as above.

f reducible, a power fixes a sep curve



$$\Psi(f) = \left(\begin{array}{ccc|c} I & & & \\ & I & & 0 \\ & & I & \\ \hline & 0 & & C \end{array}\right)$$

action on H_1(middle)

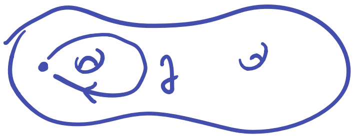
not 0×0.

If C there, violate 1

If the I's are +, violate 3

Construction #4 Kra's construction.

$\text{Push}: \pi_1(S_g) \to \text{Mod}(S_{g,1})$

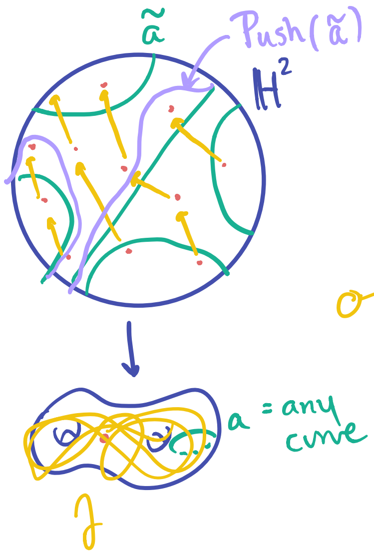


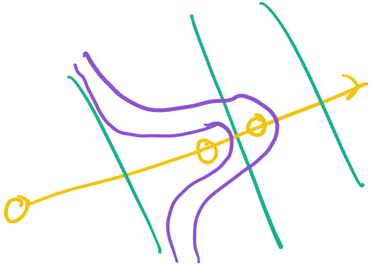
Thm. $\text{Push}(\gamma)$ is pA

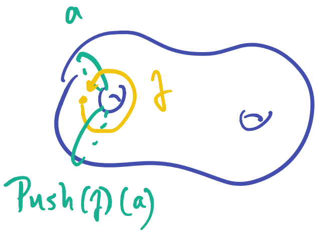
$\iff \gamma$ filling

Pf. Enough to show:

γ filling $\Rightarrow \text{Push}(\gamma)$ does not fix any curve.





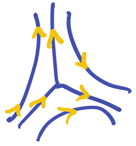


Suppose $\text{Push}(a) \simeq a$ (homotopic.)

Then could lift homotopy

cf. Dowdall's thesis.

Pseudo-Anosov theory



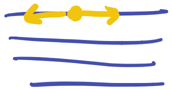
Part I. Stretch factors

Thm. $g \geq 2$, $f \in \mathrm{Mod}(S_g)$ pA.
$\lambda(f)$ is alg int of deg $\leq 6g-6$.

Pf. Show $\lambda(f)$ is eigenval. of $\mathbb{Z}$ matrix of size $\leq 6g-6$.

Matrix comes from action on $H_1(S_g;\mathbb{Z})$ or subspace of $H_1(\tilde{S}_g;\mathbb{Z})$ $\quad \tilde{S}_g$ = branched double cover.

Pf. If $\mathcal{F}_u$ orientable then $(\mathcal{F}_u, \mu)$ is a 1-form ω on S_g.



$\cancel{|dy|}$ $\quad dy$

$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u \Rightarrow \omega$ is an eigenv. for $\psi(f)$.

If $\mathcal{F}_u$ not orientable, pass to orient. double cover. $\tilde{S}_g$

$\tilde{S}_g = \{(p,v) : p \in S_g,\ v \text{ points along } \mathcal{F}_u\}$

2-fold cover, branched over odd sing.

$\tilde{S}_g$ has bounded genus, lift & apply prev case.

Q. Which alg. degrees occur for given S_g?

Strenner: exactly

$2, 4, 6, \ldots, 6g-6$

$3, 5, 7, \ldots, 3g-4$ or $3g-3$.

Q. What if you fix a subgp such as $\mathcal{I}(S_g)$.

Fried's Conjecture. $\lambda \in \mathbb{R}$ is a stretch factor $\Leftrightarrow$ all alg. conj's have abs val in $(1/\lambda, \lambda)$ except $\lambda, 1/\lambda$. cf. (Pankau, Kenyon)

Spectrum of $\mathcal{M}(S)$ $\{\log \lambda(f) : f \in \mathrm{Mod}(S)$ pA$\}$.

Thm. This is a closed, discrete subset of $\mathbb{R}$.

Pf. Set of alg. ints of deg $\leq N$ is discrete.

In particular, there is a smallest one.

Q. What is it? Only known $g=1,2$.

Penner. Smallest $\log \lambda(f)$ in $\mathrm{Mod}(S_g)$ $\asymp 1/g$.

Farb-Leininger-M Smallest $\log \lambda(f)$ in $\mathcal{I}(S_g) \asymp 1$

Lanier-M Any proper normal subgp

Thm. ρ = any Riem. metric on S.
α = any closed curve.

$$\lim_{n\to\infty} \sqrt[n]{\ell_\rho(f^n(\alpha))} = \lambda$$

i.e. $\ell_\rho(f^n(\alpha)) \sim \lambda^n$ geometry

Thm. a, b any s.c.c. in S

$$\lim_{n\to\infty} \sqrt[n]{i(f^n(a), b)} = \lambda$$

i.e. $i(f^n(a), b) \sim \lambda^n$ topology

Thm. $\alpha \in \pi_1(S)$

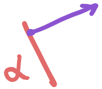
$$\lim_{n\to\infty} \sqrt[n]{|f^n(\alpha)|} = \lambda$$

i.e. $|f^n(\alpha)| \sim \lambda^n$ group theory.

word length

Thm. $\log \lambda$ = top. entropy of f. dynamics

Part II. Foliations



<u>Poincaré recurrence for foliations</u>

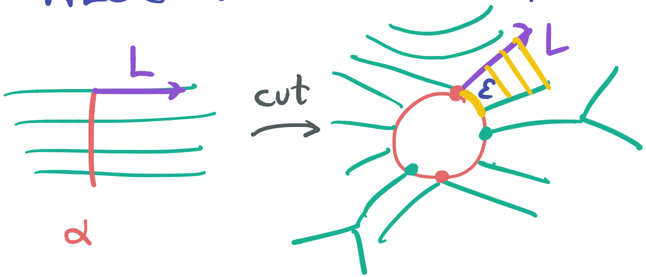
$(\mathcal{F}, \mu)$ meas. fol.

$L = \infty$ half leaf.

α = arc transverse to $\mathcal{F}$

$\alpha \cap L \neq \emptyset \Rightarrow |\alpha \cap L| = \infty$.

<u>Pf.</u> WLOG L & α share endpt.



Choose small arc ε along new ∂.

Push along foliation.

$\rightsquigarrow$ sweep out rectangle.

Can choose ε small enough

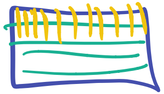
so this rectangle never hits

a singularity.

$\Longrightarrow$ If L never hits ∂ again.

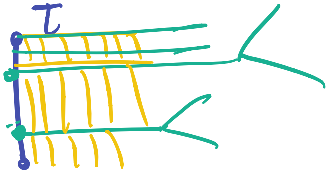
can push forever. CONTRAD.

Really using: Can cover S by finitely

many charts like  

<u>Cor.</u> f pA $\Longrightarrow$ every leaf of $\mathcal{F}_u$ is dense.

<u>Pf.</u> τ = small arc transverse to $\mathcal{F}_u$



No closed leaves ⇒ these swept out rectangles eventually return by Poinc. rec.

The union of these rectangles is the whole surface (otherwise the ∂ is a reducing curve).

<u>Thm.</u> $\mathcal{F}_u$ is uniquely ergodic i.e. μ is unique up to scale.

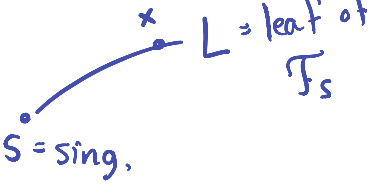
Part III. Dynamics.

Thm. φ pA $\Longrightarrow$ φ has dense orbit.

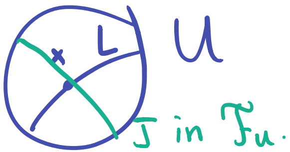
Pf. **Claim.** $U \neq \emptyset$, open, φ-invt $\Longrightarrow$ U dense.

Assume WLOG φ fixes sing's...

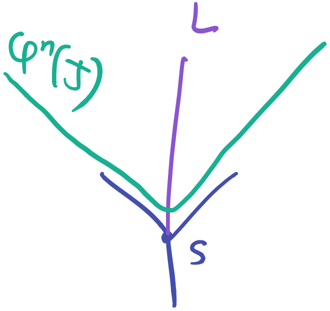
Choose:



L dense $\Longrightarrow$



Apply powers of φ.



$x \longrightarrow s$

J gets longer

$\Longrightarrow \bigcup \varphi^n(J)$ dense.

$\Longrightarrow \underbrace{\bigcup \varphi^n(U)}_{\overset{\shortparallel}{U}.}$ dense

Now: Take $\{U_i\}$ countable basis for S.

Let $V_i = \bigcup_{n \in \mathbb{Z}} \varphi^n(U_i)$ satisfies claim hence dense $\forall i$.

Baire category thm $\Longrightarrow \bigcap V_i$ dense

$\Longrightarrow \bigcap V_i \neq \emptyset$. Say $x \in \bigcap$

$\Longrightarrow \{f^i(x)\}$ intersects every U_i $\square$

Thm. φ pA $\Rightarrow$ periodic pts dense.

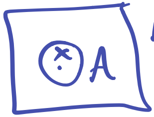
Poincaré Recurrence. M = finite meas. sp.

$T : M \to M$ meas. pres.

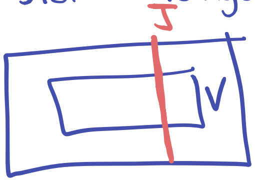
$A \subseteq M$ pos. meas.

Then for a.e. $x \in A$ $\exists$ inc. seq n_i

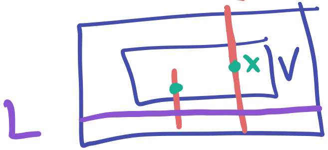
s.t. $T^{n_i}(x) \in A$.



Pf. Choose std rectangles



P.R. $\Rightarrow$ $\varphi^{n_i}(V) \cap V \neq \emptyset$.



continuous $\rightsquigarrow$ $J \to J$

(push right)

1D Brower $\Rightarrow$ fixed pt.

i.e. horiz leaf L mapping to itself.

Fund. thm of 1D dynamics:

Any map $f : [0,1] \to \mathbb{R}$

with im $f \supseteq [0,1]$

has a fixed pt.

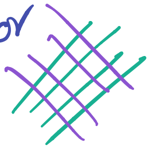
Apply to $L \cap U$. $\square$

Chap 15. Thurston's Proof

Reference. Thurston's Work on Surfaces. Fathi, Laudenbach, Poénaru

<u>NTC</u>. $f \in \mathrm{Mod}(S)$ is

① periodic
② reducible
③ pseudo-Anosov



<u>Setup</u>. $\mathcal{S} = \{\text{s.c. curves in } S\}/\text{isotopy}$

$$\mathrm{Teich}(S) \hookrightarrow P\underbrace{\mathbb{R}^{\mathcal{S}}}_{\text{fns } \mathcal{S} \to \mathbb{R}}$$

$$PMF(S) \hookrightarrow P\mathbb{R}^{\mathcal{S}}$$

<u>Thm</u>. $PMF(S) \cong S^{\dim \mathrm{Teich}(S) - 1}$

$\mathrm{Teich}(S) \cup PMF(S)$ is a closed ball, on which $\mathrm{Mod}(S)$ acts continuously.

Thm. $PMF(S) \cong S^{\dim \mathrm{Teich}(S) - 1}$

$\mathrm{Teich}(S) \cup PMF(S)$ is a closed ball, on which $\mathrm{Mod}(S)$ acts continuously.

Pf of NTC. Brouwer $\Rightarrow$ f fixes some X in the ball.

$X \in \mathrm{Teich}(S) \Rightarrow f$ periodic.

$X \in PMF$ & X has closed leaf $\Rightarrow$ reducible.

& X has no closed leaf

& $\lambda = 1 \Rightarrow$ periodic

& $\lambda > 1 \Rightarrow$ pA $\square$

Torus case: