Foliations, Fibrations, and Sutured Manifolds

Our motivating problem: given
- 3-mfd Y
- link $L \subset Y$
- connected surface $\Sigma \subset Y$ st $\partial\Sigma = L$

is there a fibration

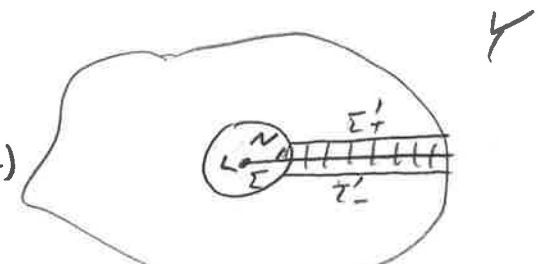
$$\pi : (Y - L) \to S^1$$

st. Σ is a fiber?

(discussion of L a knot, but all works

let $Y(L) = Y - \underbrace{\text{nbhd}(L)}$

call $N = S^1 \times D^2$
(or many copies)



let $N' =$ nbhd of $(\Sigma \cap Y(L))$ |

clearly $N' = \Sigma \times [-1,1]$ (at least can be chosen)

with $\partial\Sigma \times [-1,1]$ an annulus A' on $\partial Y(L) = \partial N$

and $\partial N' = \Sigma'_+ \cup A' \cup \Sigma'_-$

let $C = Y(L) - N'$

note: $\partial C = \Sigma_+ \cup A \cup \Sigma_-$

$$\begin{matrix} & \Sigma_+ & \cup & A & \cup & \Sigma_- \\ & \shortparallel & & \shortparallel & & \shortparallel \\ & \Sigma'_- & & \partial N\text{-}A' & & \Sigma'_+ \end{matrix}$$

$$Y(L) = \underset{\substack{\downarrow \pi \\ [-1,1]}}{N'} \cup C$$

so if C is a product $\Sigma \times [1,2]$

st. $\partial\Sigma \times [1,2] = A$

then π extends to fibration of $Y(L)$)

so to answer main problem we study when C is such a product.

for this.

def^n:
a sutured manifold is

1) a compact oriented 3-mfd M and

2) a set $\gamma \subset \partial M$ of disjoint

   a) annuli $A(\gamma)$ and

   b) tori $T(\gamma)$

3) a choice of oriented core for each annulus in $A(\gamma)$ called a suture

(set of sutures written $s(\gamma)$)

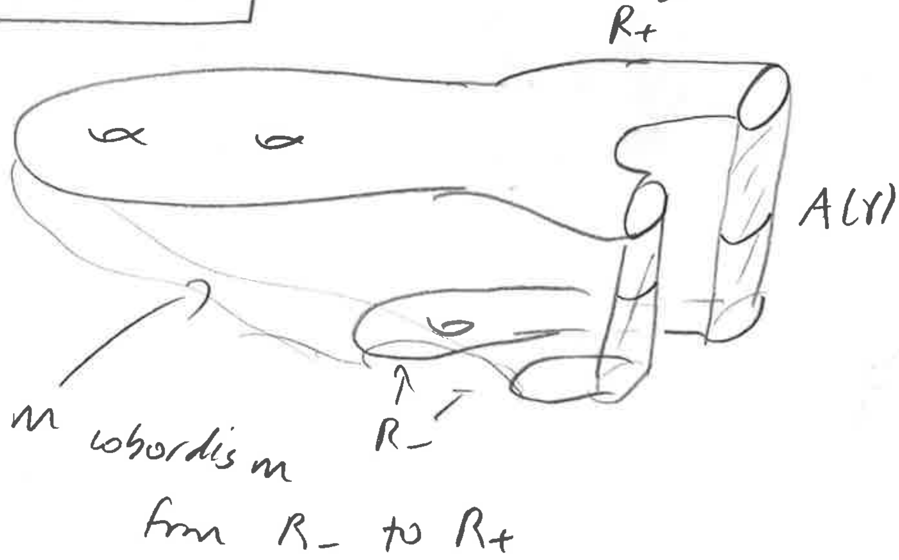
4) a choice of or^n on each cmpnt of $R(\gamma) = \partial M - \mathring{\gamma}$

(let $R_\pm(\gamma)$ = cpts of $R(\gamma)$ where chosen o^n agrees/disagrees w/ or^n on ∂M)

s.t. $\partial R_\pm(\gamma) \simeq s(\gamma)$ as oriented curves in $A(\gamma)$

$\boxed{\text{For this talk } T(\gamma) = \emptyset}$ (annular sutured mfds)

idea:



example: 1) $M = \Sigma \times [-1,1]$ $\qquad R_\pm = \Sigma \times \{\pm 1\}$

$A(\gamma) = (\partial \Sigma) \times [-1,1]$

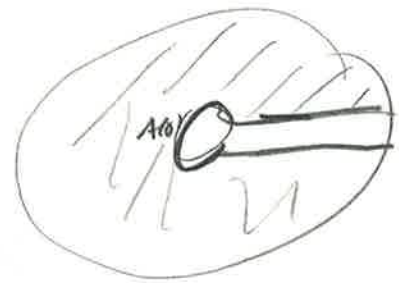
called product sutured mfd

Y 3-mfd $\quad L \subset Y$ link $\quad \Sigma$ sfc $\quad \partial\Sigma = L$

③

2) $\quad N' = \Sigma \times [-1,1] \subset Y(L)$
$\uparrow$ sutured mfd.

$C = Y(L) - N'$ sutured mfd w/ $R_\pm = \Sigma \times \{\pm 1\}$
$A(\gamma) = \partial(N(L)) - A'$

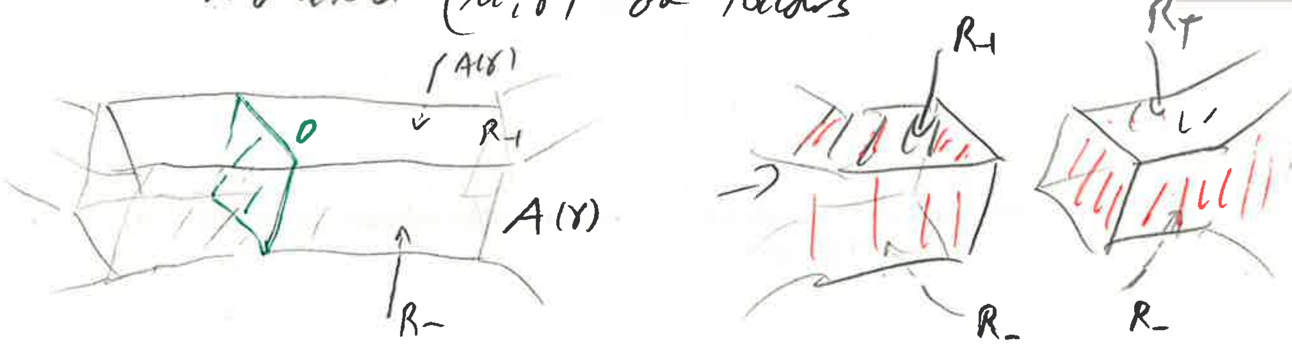


Thm above $\quad L$ fibered $\Leftrightarrow$ C product sutured mfd!

<u>def:</u> $D^2 \overset{\text{proper}}{\subset} (M,\gamma)$ is a <u>product disk</u> if

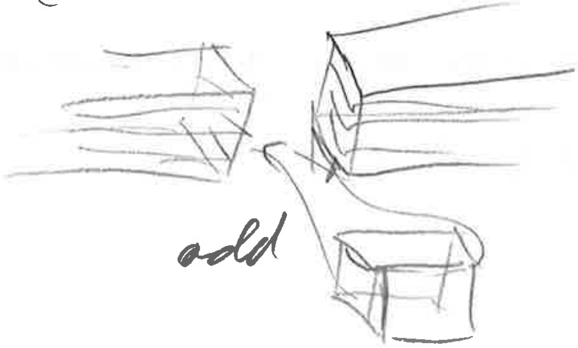
$$\partial D \cap s(\gamma) = 2 \text{ pts}$$

<u>note:</u> given such a disk get a new sutured manifold (M', γ') as follows



lemma: If $(M,\gamma) \xrightarrow{D} (M',\gamma')$ and D a product disk then (M,γ) product sutured mfd $\Leftrightarrow$ (M',γ')

Proof (⇐) obvious: (M', γ')



add

$D^2 \times I$

get product on (M, γ)

(⇒) $M = \Sigma \times [-1,1]$
given: D can isotop so $D = c \times [-1,1]$
$c \subset \Sigma$

now cut

(idea arrange D to $\pitchfork$ $\Sigma \times \{t\}$ transversely $\forall t$

eg

S^2

irred ⇒
$\exists$ B^3 st $\partial B^3 = S^2$

use to isotop

)

lemma:

$$\boxed{\begin{array}{l}\text{if } M \text{ irreducible (i.e. any } S^2 \subset M \text{ bounds } B^3 \text{ in } M) \\ \text{and } \partial M = S^2 \text{ and } \gamma = \text{one annulus} \\ \text{then } \quad M = D^2 \times [-1,1] \quad \text{i.e. is a product}\end{array}}$$

Pf: irred ⇒ $M = B^3 = D^2 \times [-1,1]$ isotope so $(\partial D^2) \times (-1,1) = \gamma$ ✓

Thm: (M,γ) irred sutured mfd

(M,γ) a product " "

$\Longleftrightarrow$

$\exists$ seq $(M,\gamma) \xrightarrow{D_1} (M_1,\gamma_1) \xrightarrow{D_2} \cdots \xrightarrow{D_n} (M_n,\gamma_n)$

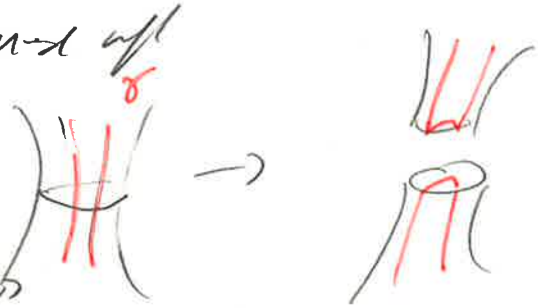
where $\partial M_n = n\ S^2$'s and γ_n = one annulus on each ∂S^2.

Pf: clear from lemma once you know

M irred $\Longleftrightarrow$ $M \setminus D^2$ irred ✓
(exercise)

to determine if link (say in S^3) is fibered need complement of nbhd of lk to be product so need to see product disks in complement

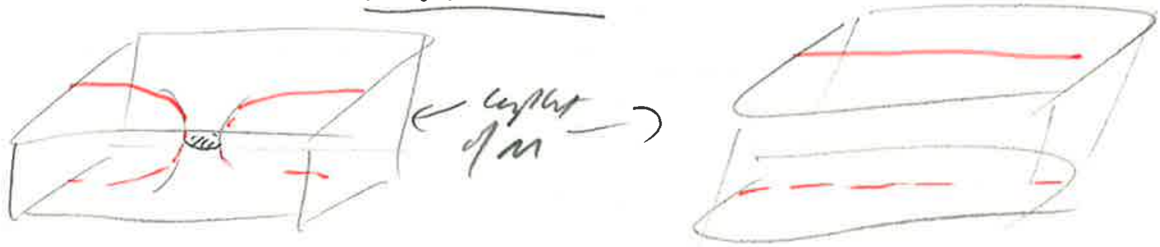
in sutured mfd



is M

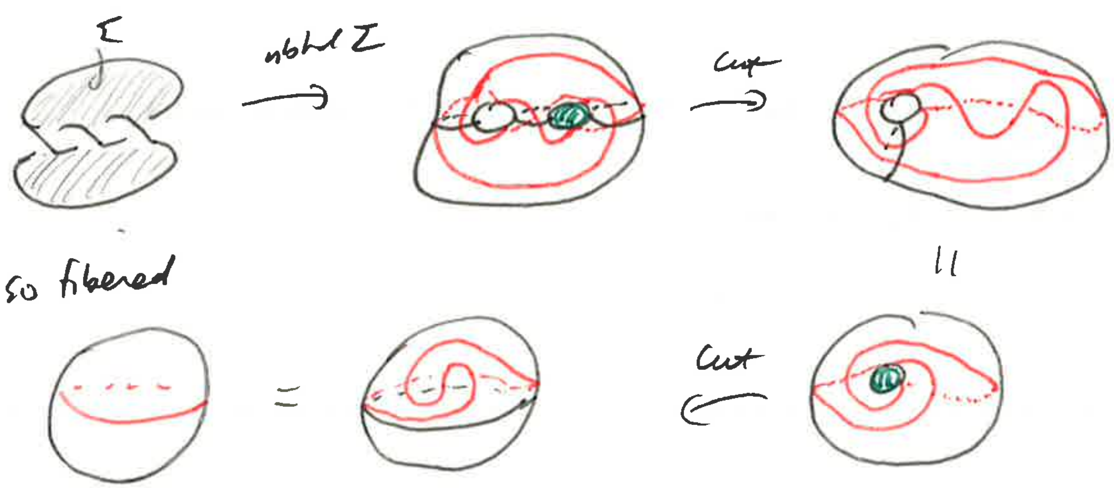
in complement

M on outside

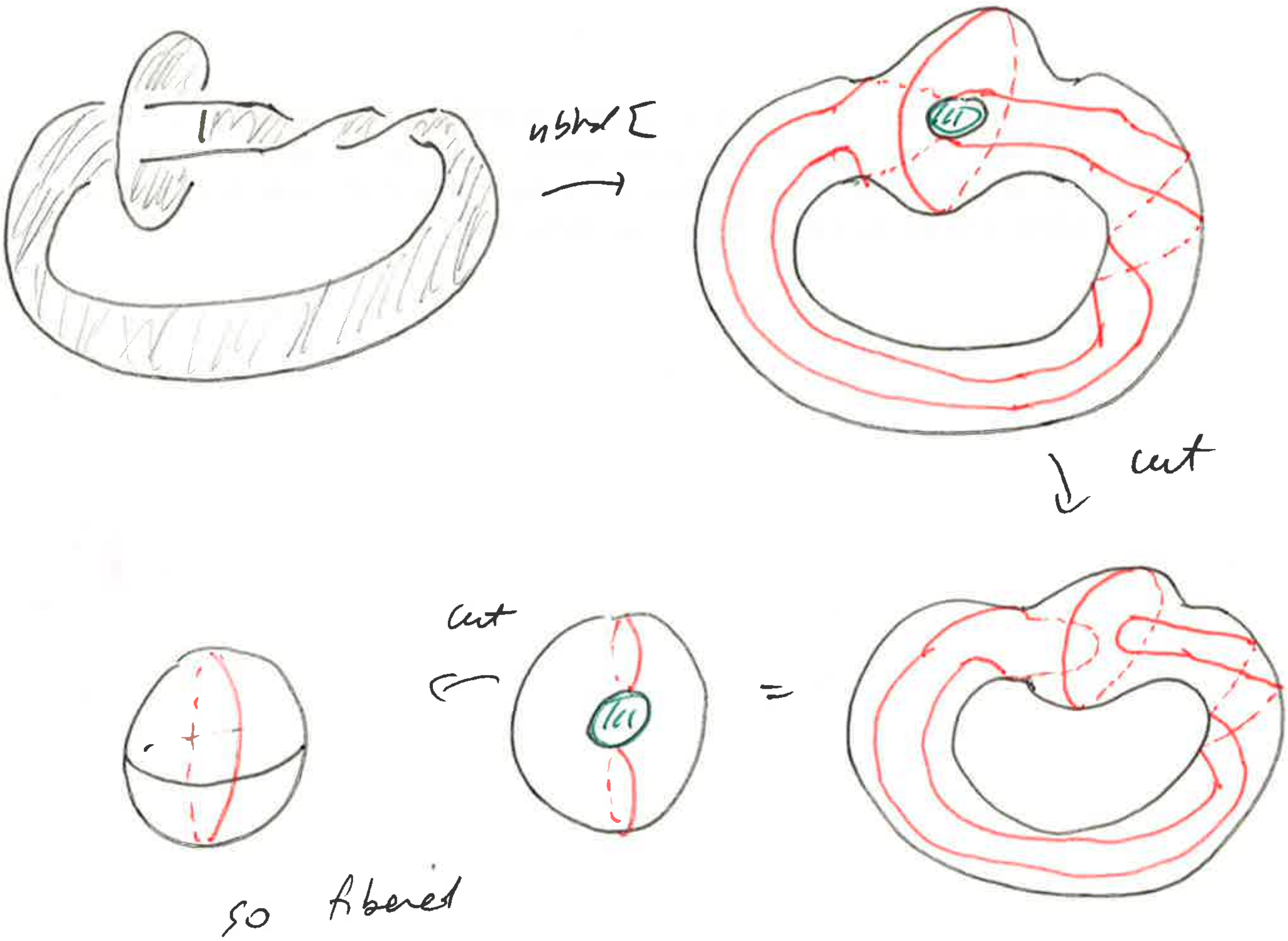


examples:

1)

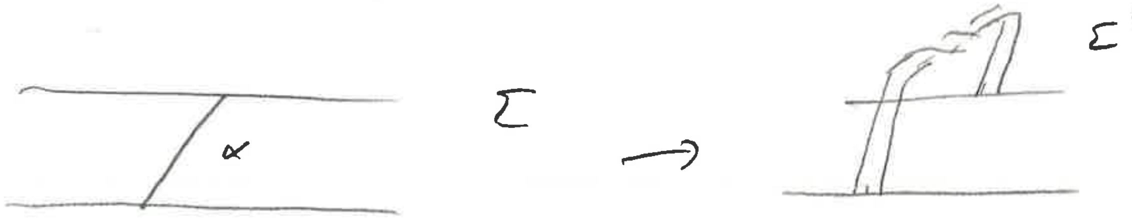


2)

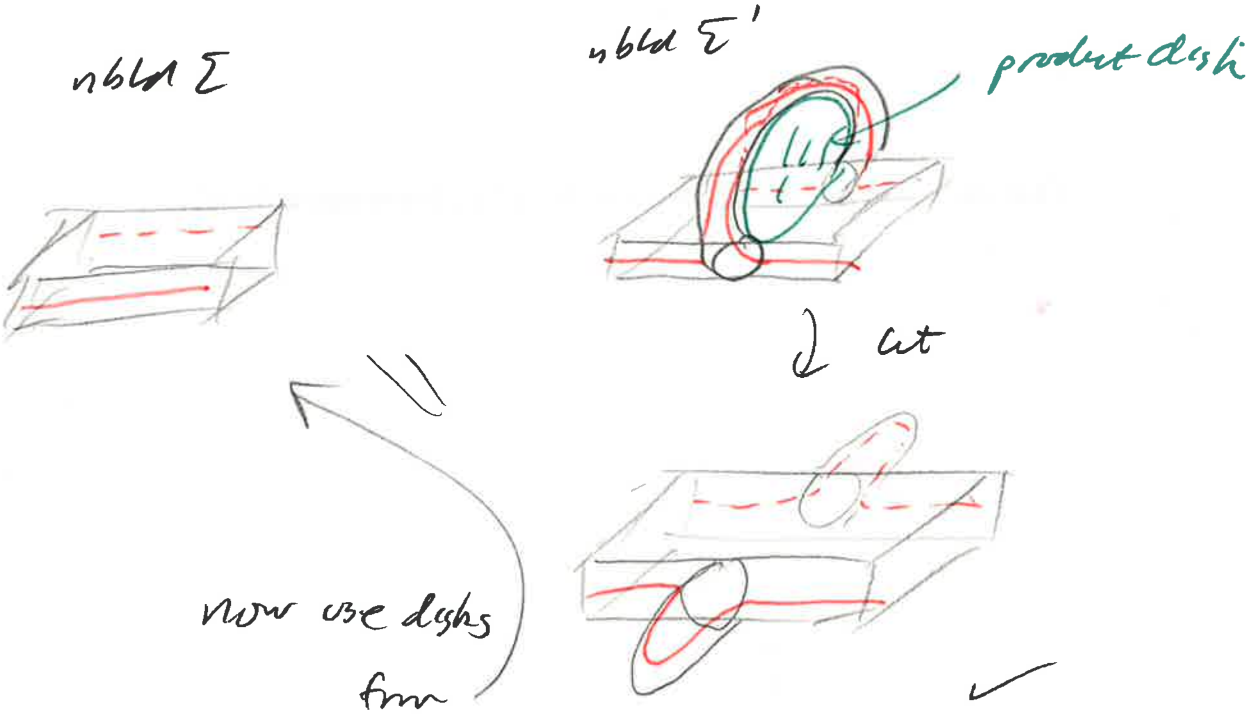


Stabilization preserves fibered:

suppose $\partial\Sigma = K$ fibered
α arc in Σ



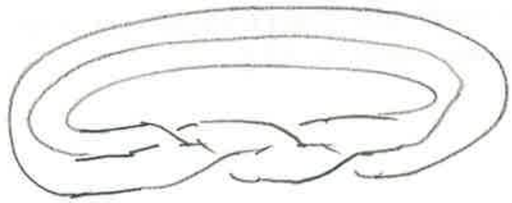
if $\partial\Sigma$ fibered so is $\partial\Sigma'$



closures of Homogeneous braids are fibered

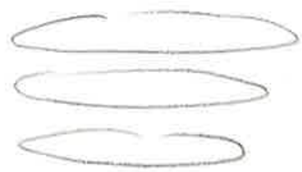
braid homogeneous if each generator only occures as σ_i or σ_i^{-1}

eg $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ (fig 8)



Stallings: $\boxed{\text{homogeneous braids fibered}}$

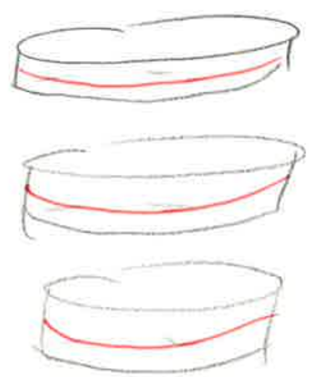
pf: Seif surface



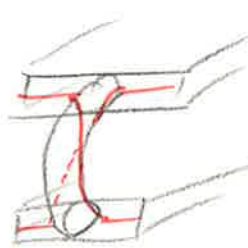
← 1 for each strand of braid

for each σ_i or σ_i^{-1} add band

nbhd of disks is

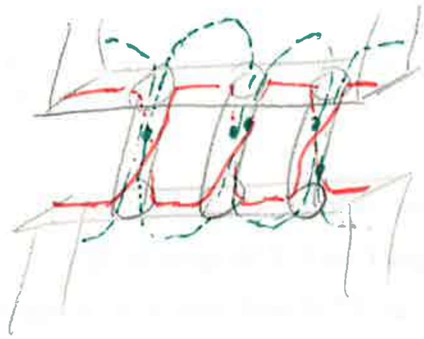


for each band add

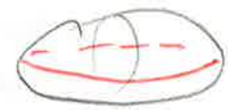


look at n & $n-1$ parts

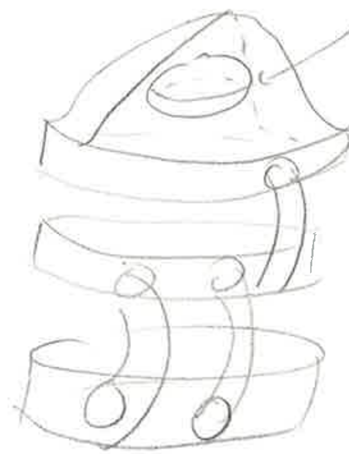
thicken



cut along all of these to get



and nbhd of $n-1$ braid



S^2 inside

last time we saw how to put a fibration on the complement of some knots

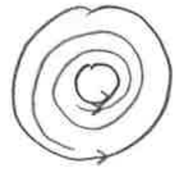
Can't always do this but can foliate

recall: 1) $\mathcal{F}$ a fol^n on M if every point in M has a nbhd N diffeo to $D^2 \times I$ with components of $N \cap \mathcal{F}$ mapping to $D^2 \times pt$

2) intuitively fill M with surfaces (called leaves)

3) foliation can also be thought of as a plane field ξ st $\xi = \ker \alpha$ and $d\alpha|_\xi = 0$.

example: 2 dim'l



(flow lines of non sing v.f.) (2D)

← called Reeb component

3 dim'l

take 2 dim'l & $\times [0,1]$ or S^1

3 dim'l Reeb component

$\mathbb{R}^3$

glue top to bottom

(eg look at level sets of $(1-(x^2+y^2)^2)e^z$ mod out by $z \mapsto z+1$)

define: Thurston norm

given sfc Σ let

$$n(\Sigma) = \sum_{\substack{\text{compnts} \\ \Sigma' \text{ of } \Sigma \\ \text{with } \chi(\Sigma')<0}} |\chi(\Sigma')|$$

(i.e. - total euler characteristic where disregard disks, spheres, tori and annuli (really disks & spheres))

note: for connected surface with given # of ∂ cpts then $n \longleftrightarrow$ genus

given a homology class $h \in H_2(M, \partial M)$ (if $\partial M \neq \emptyset$ ok)

define $n(h) = \min\{n(\Sigma)\ \forall\ \Sigma$ properly embedded in $(M, \partial M)$ st $[\Sigma] = h\}$

Thm (Thurston):

if M compact oriented 3-mfd
$\mathcal{F}$ a fol'n ① $\pitchfork$ oriented
② $\pitchfork$ to ∂
③ w/o Reeb compnts.
then any compact leaf of $\mathcal{F}$ is Thurston norm minimizing.

(better for connected sfc $\Sigma \neq S^2, D^2, A$

$$\langle e(\mathcal{F}), [\Sigma] \rangle \leq -\chi(\Sigma)$$

euler class of $\mathcal{F}$)

(note: for connected leaf Σ

$$\langle e(\mathcal{F}), [\Sigma] \rangle = -\chi(\Sigma) = n(\Sigma))$$

Corollary:

$\boxed{\text{fibers in fibrations are genus minimizing in their homology class}}$

given a knot K (in 3-mfd Y) and sfc Σ with $\partial\Sigma = K$

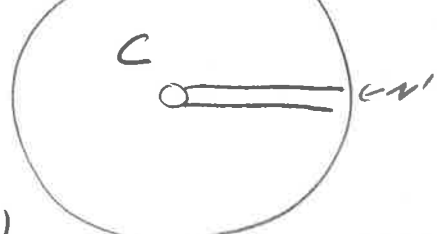
question is genus(Σ) minimal among all sfcs with $\partial = K$? (in homology class $[\Sigma] \in H_2(Y, K)$)

this is difficult & important question

let $Y(K) = \overline{Y - \text{nbhd}\, K}$

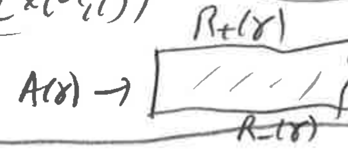
if $\exists$ fol^n on $Y(K)$ ($\pitchfork$ to ∂ w/ no Reeb cpts) with Σ a compact leaf then yes

recall from last time



$$Y(K) = \underset{\Sigma\times[-1,1]}{\overset{\shortparallel}{N'}} \cup \underset{Y(K) - (\Sigma\times(-1,1))}{\overset{\shortparallel}{C}}$$

C a sutured mfd



$Y(K)$ has fol^n as above $\iff$ $\exists$ fol^n $\mathcal{F}$ on C with $R_\pm(\gamma)$ leaves & $\mathcal{F} \pitchfork A(\gamma)$

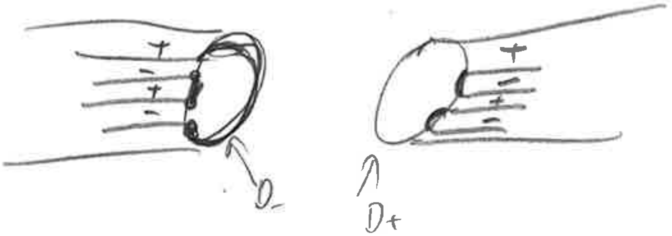
so we try to construct such a fol^n.

given a disk D^2 in a sutured mfd (M, γ)

let $M' = M \setminus D^2$

$s(\gamma')$ in $\partial M'$ is

sutures (def γ')



i.e. on $D_{\mathcal{F}}$ connect end pts of $s(\gamma) \cap D$ across $R_\pm$ region

We call a oriented fol^n $\mathcal{F}$ on (M,γ) <u>taut</u> if

(1) $\mathcal{F} \pitchfork \gamma$

(2) $\mathcal{F}$ tangt to $R(\gamma)$ w/ normal direction agreeing w/ R_+, disagree w/ R_-

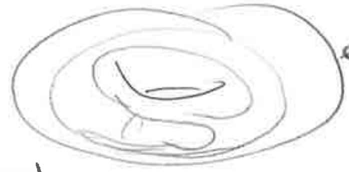
(3) $\mathcal{F}|_\gamma$ has no Reeb components

<u>eg</u>:

but no

(4) each leaf of $\mathcal{F}$ intersects a transverse curve or prop embedded arc.

<u>Remark</u>: taut $\Rightarrow$ no 3D Reeb components.



So $R_+(\gamma)$ & $R_-(\gamma)$ are norm minimizing!

<u>Th^m (Gabai)</u>:

let (M,γ) be sutured mfd

if $\exists$ $(M,\gamma) \xrightarrow{D} (M_1,\gamma_1) \xrightarrow{D} (M_2,\gamma_2) \text{ --- } (M_n,\gamma_n)$

st $(M_n,\gamma_n) = \coprod (D^2\times[0,1], \partial D^2 \times [0,1])$

then $\exists$ a taut (depth 1) fol^n $\mathcal{F}$ on (M,γ)

"Proof":

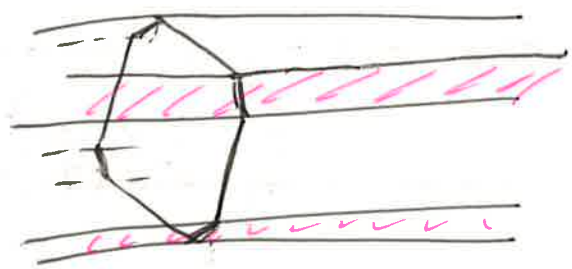
clearly (M_k, γ_k) has desired fol^n

we prove if (M_{i+1}, γ_{i+1}) has fol^n so does (M_i, γ_i) then done

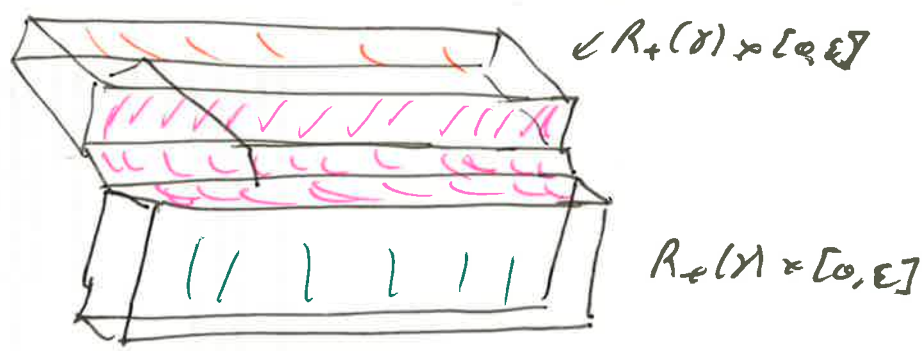
exercise: if D^2 used is product disk can just extend over last time

we look at case $D \cap \gamma_i = 4$ (other cases similar)

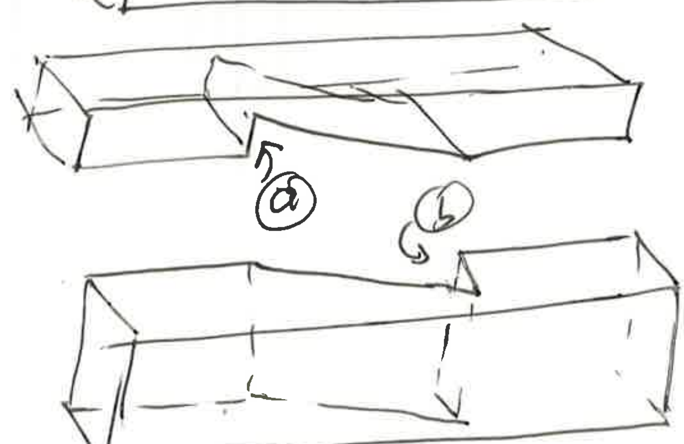
M_i':

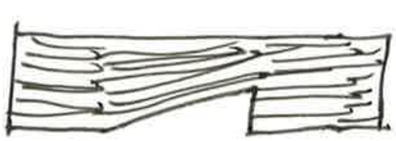


add $R(\gamma) \times [0, \varepsilon]$ to M_i (doesn't change M_i) call M_i'



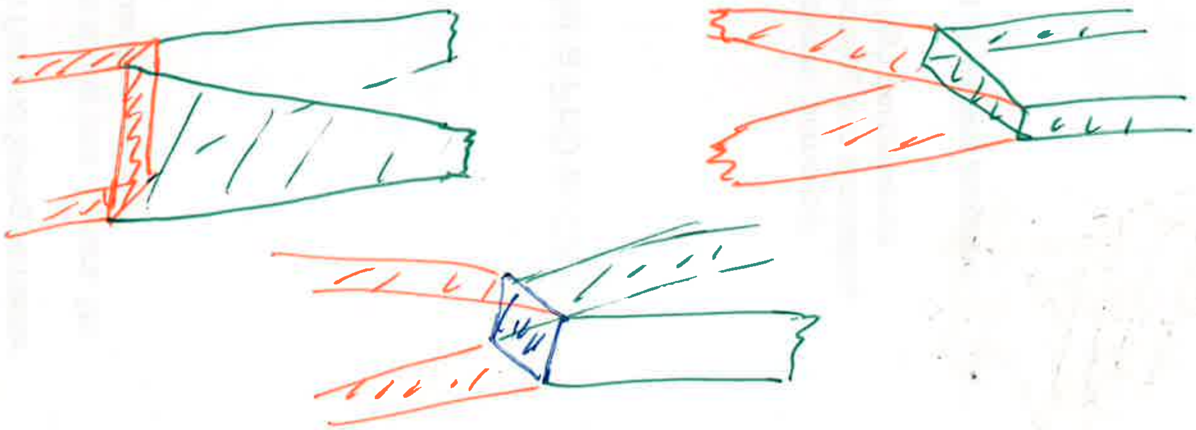
foliate: by





on M_{q+1} see

on $D^2 \times I$ use



union of right hand sides of green is

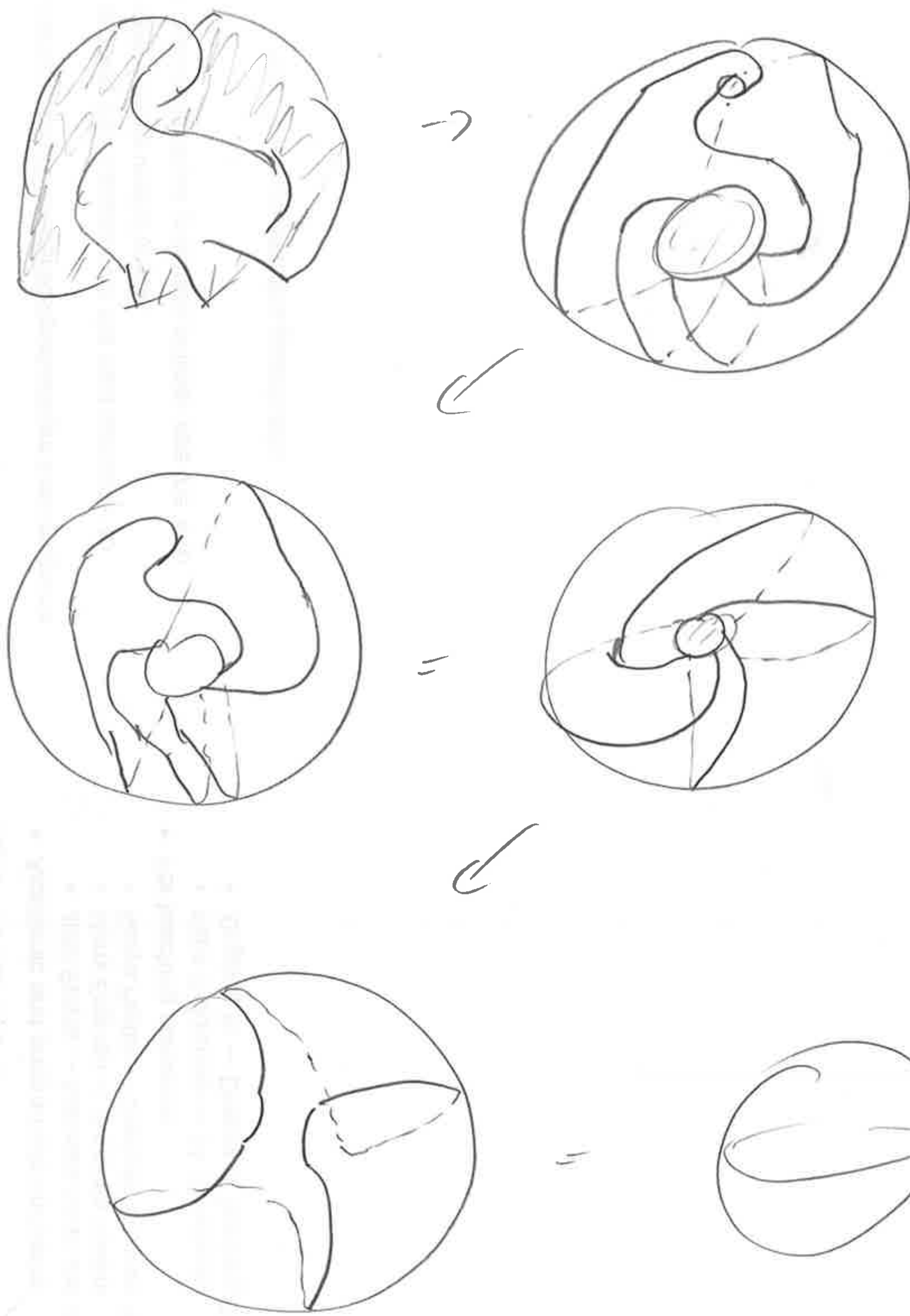
← glue to ⓑ above

union of left hand sides of orange is

glue to ⓐ above

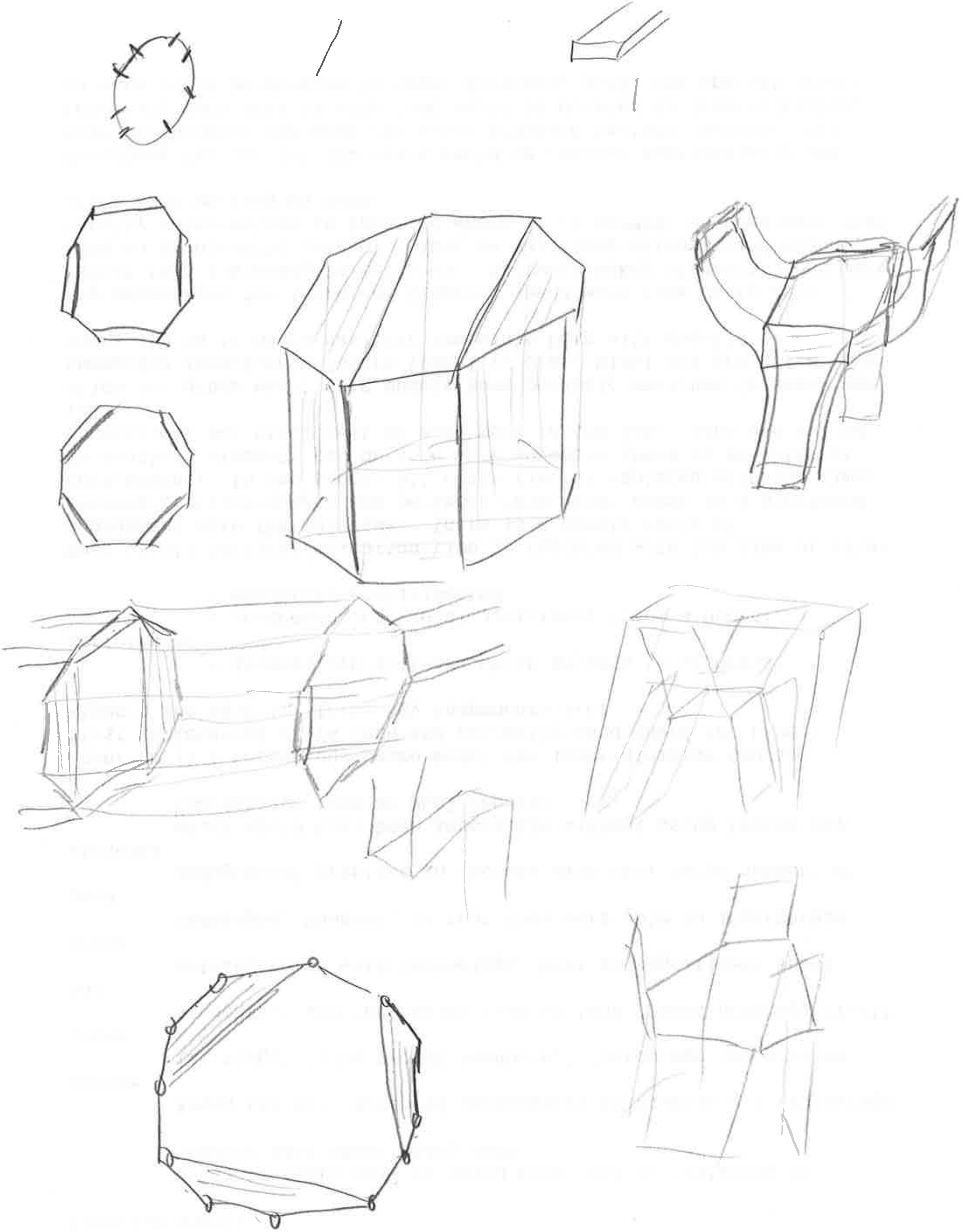
note: $M_q' - \underbrace{\left((D^2 \times I) \cup R(\gamma) \times [0,\varepsilon]\right)}_{} = M_{q+1}$

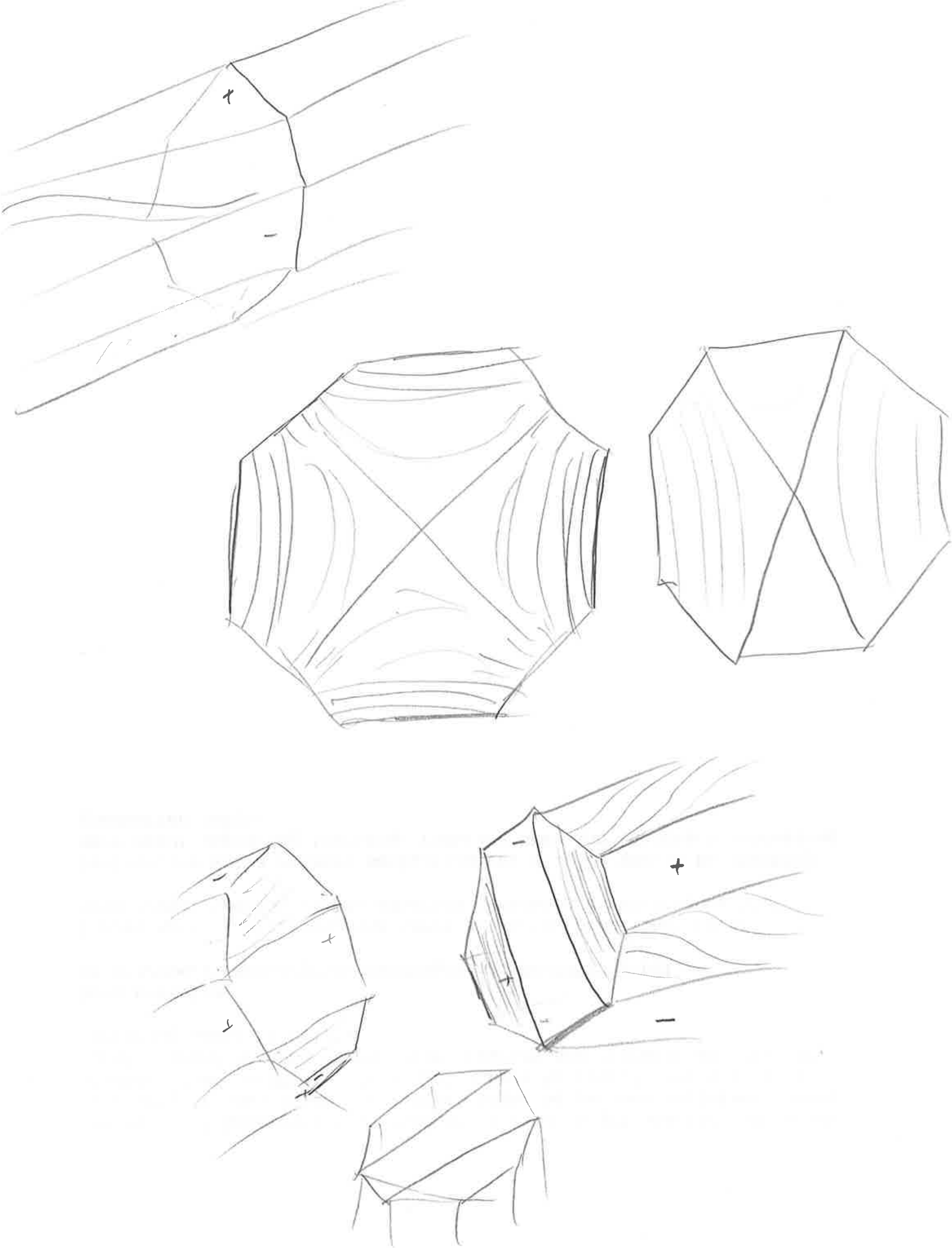
we foliated this with "∂ conditions" γ_{q+1} on example:

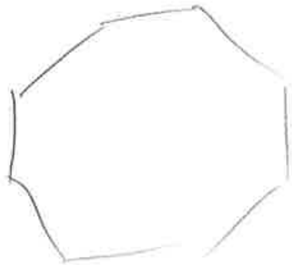


Thm: (Gabai's pf Murasugi-Crowell) So min genus

Seifert surface for alternating link coming from Seifert's algorithm is min genus







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