Our motivating problem: given

- 3-manifold $Y$
- link $L \subset Y$
- oriented surface $\Sigma \subset Y$ s.t. $\partial \Sigma = L$

Is there a fibration

$\Pi : (Y - L) \to \mathbb{S}^1$

such that $\Sigma$ is a leaf?

(discount if $L$ a knot, but all works)

Let $Y(L) = Y - \text{Int}(L)$

call $N = \mathbb{S}^1 \times D^2$
(or many copies)

let $N' = \text{Int}(\Sigma \times [1, 2])$

Clearly $N' = \Sigma \times [-1, 1]$ (at least can be chosen)

with $\partial (\Sigma' \times [-1, 1])$ an annulus on $\partial Y(L)$ = $\partial N$

and $\partial N' = \Sigma_+ \cup A' \cup \Sigma_-$

Let $C = Y(L) - N'$

Note: $\partial C = \Sigma_+ \cup A \cup \Sigma_-$

\[\Sigma_- \quad \partial N' - A' \quad \Sigma_+\]

$Y(L) = N' \cup C$

so if $C$ is a product $\Sigma \times [1/2, 1/2]$ it extends to fibration of $Y(L)$

so to answer main problem we study when $C$

is such a product.
for this.

**def:**

a sutured manifold is

1) a compact oriented 3-manifold $M$ and
2) a set $\mathcal{S}$ of disjoint
   a) annuli $A(\mathcal{S})$ and
   b) tori $T(\mathcal{S})$
3) a choice of oriented core for each annulus in $A(\mathcal{S})$ called a *suture* (set of sutures written $\mathcal{S}(\mathcal{S})$)
4) a choice of or a on each unknot of $R(\mathcal{S})$ (set $R(\mathcal{S})$) = pts of $R(\mathcal{S})$ when chosen $\omega$

For this talk $T(\mathcal{S}) = \emptyset$ (annulus sutured only)

**idea:**

\[ \text{m cobordism from } R_- \text{ to } R_+ \]

de

\[ \text{Example 1: } M = \Sigma \times E_0, 1 \]

\[ R_\pm = \Sigma \times \mathbb{R} \]

\[ A(\mathcal{S}) = (\partial \Sigma) \times E_{1,1} \]

called product sutured 3-manifold
2) \( N' = \Sigma (C N) \subseteq Y(L) \)

\[
L \subseteq Y \subseteq L + l + h \quad \Sigma \subseteq Y \subseteq L \quad L = \Sigma
\]

\[C = (C(L)) - N' \quad \text{sutured with } \quad R_+ = C(L+1)
\]

\[A(L) = 2(C(L)) - A'
\]

Ann above \( L \) boxed \( \Theta \text{ product sutural } \)

**Def:** \( D \subseteq C(M, \delta) \) is a **product disk** if

\( \forall D \cap \delta = 2 \text{ pts} \)

**Note:** given such a disk get a new sutural manifold \((M', \delta')\) as follows

**Lemma:** If \((M, Y) \to (M', Y')\) and \( D \) a product disk then \((M, Y) \) product sutural iff \((M', Y')\)
if 

and 

then 

and 

as a product

Proof:

(0)

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)
\[(M, Y) \text{ irreducible and} \]
\[(M, \delta) \text{ a proper subset}\]
\[\Rightarrow\]
\[\exists \text{ a disk } D^n \ni (M, Y) \rightarrow (M, Y_1) \rightarrow D^n \rightarrow \partial D^n \ni (M_{N_k}, \delta_n)\]

where \(N_k = n \times S^2\) and \(\delta_n\) are annuli at each \(x \times S^2\).

Proper for lemons are you know

\[M \text{ irreducible } \iff M/\partial D^2 \text{ irreducible}\]

(Exercise)

to determine if link (say in \(S^3\)) is bend need

implant \# \# to be product so need

to see product dishes in implant

\[
\begin{align*}
I &\text{ in subdisk of } \\
I &\text{ in implant}
\end{align*}
\]
Examples:

1) so fibered

2) cut
Stabilization preserves fibered:

Suppose $\Sigma = K$ fibered
$\alpha \text{ arc in } \Sigma$

if $\Sigma$ fibered so is $\Sigma'$

Now use disks from product disk

Diagram showing fibered structure with cut and product disk indicated.
Homogeneous braids are fibred.

Braid homogeneity if each generator only occurs on $o_1$ or $o_1^{-1}$

$= o_2 o_1^{-1} o_2 o_1^{-1}$ (fig 8)

Stallings: homogeneous braids fibred

of self surface

in each $o_1$ or $o_1^{-1}$, odd band

ribbon of disks is

for each bond, odd
Look at n & n-1 parts

Cut along all of them to get

And weld of n-1 bond
last time we saw how to put a fibration on the complement of some knots
(can't always do this but can tolerate)

recall: 1) F a foliation on M if every point in M has a nbhd diffeo to $D^2 \times I$ with boundary $N \sim F$ mapping to $S^2 \times pt$

2) intuitively fill M with surfaces called (leaves)

3) foliation can also be thought of as a plane field $\xi$ s.t. $\xi = ker \nu$ and $d\nu|_\xi = 0$.

example: 2 diml

(2 diml foliation of non-sing r.f.)

3 diml

$R^3$ glue top to bottom

(eg look at level sets $f$ ($1-x^2+y^4) e^z$ redeft $x, 2+x, 2+y$)
define: Thurston norm

given sfc \( \Sigma \) let

\[ n(\Sigma) = \sum_{\text{link } l \subset \Sigma, \forall l \cap \partial \Sigma} |\chi(l)| \]

\( l_\Sigma \) - total euler characteristic

where closed disks, spheres, tori and annuli (regard disk boundary)

note: in connected sft for \( \Sigma \) with genus \( g \) greater than \( n \) then \( n \leq g \) genus

given a homology class \( \eta \in H_2(M, \partial M) \) (of \( M \approx \mathbb{R}^3 \))

define \( n(\eta) = \min \{ n(\Sigma) \mid \Sigma \text{ proper embedded in } (M, \partial M), \Sigma \cap \eta = \emptyset \} \)

Th\( ^{\text{w}} \) (Thurston):

if \( M \) compact oriented 3-manifold

\( \partial \) a foliated orient.

(2) it to \( \partial \)

(3) no Reeb singular

then any compact leaf of \( \mathcal{F} \) is Thurston norm minimizing

\( \Sigma \) better in unoriented sfc \( \Sigma \approx S^1 \times D^2 \times A \)

\[ \langle \xi(\mathcal{F}), [\Sigma] \rangle \leq -\chi(\Sigma) \]

\( \langle \text{euler class of } \mathcal{F} \rangle \)

(note: for leaf \( \Sigma \)

\[ \langle e(\mathcal{F}), [\Sigma] \rangle = -\chi(\Sigma) = n(\Sigma) \]
Corollary: \( \text{fibers in fibrations are genus minimizing in the homology class} \)

Given a knot \( K \) and slice \( \Sigma \) with \( 2\Sigma = K \),

the question is: Genus(\( \Sigma \)) minimal among all slices with \( \Sigma \)?

This is a difficult and important question.

Let \( \gamma(K) = \Sigma \) normal.

If \( \exists \) foliation \( \gamma(K) \) (not unique) no Reeboris with \( \Sigma \)

or compact leaf then yes.

Recall from last time:

\[ \gamma(K) = N \cup C \]

\[ \Sigma \times \{0\} \cup \gamma(K) - (\Sigma \times \{0\}) \]

\( C \) a sutured manifold \( \Lambda(K) \rightarrow \)

\( R_+ (\gamma(K)) \)

\( \gamma(K) \) has foliation or above \( \iff \exists \) foliation on \( C \)

with \( R_+ (\gamma(K)) \) leaves 2 \( \not\in \Lambda(K) \)

So we try to construct such a foliation.

Given a disk \( D \) in a sutured manifold \( (M, \beta) \)

Let \( M' = M \setminus D^2 \)

\( s(\gamma) \) \( M \not\subset M' \)

\( s(\gamma) \) \( \not\subset \) \( M' \)

\( \not\subset \) \( \not\subset \)

\( \not\subset \) \( \not\subset \)

\( \not\subset \) \( \not\subset \)
we call a \( (M, \phi) \) in \( (Q, Y) \) **taut** if

1. \( \phi \neq Y \)
2. \( \phi \) taut in \( R(Q) \) (trivial in most cases)
3. \( \phi \) here no Reeb components

4. each leaf of \( \phi \) is trivial or a transverse arc or a prop embedded arc.

**Remark**: taut \( \Rightarrow \) no Reeb components.

So \( R(Q) \) is non-minimal!

**Theorem** (Gabai): let \((M, \phi)\) be taut and

\[ \exists (M_t, \phi_t) \rightarrow (M_{\bar{y}}, \phi) \rightarrow (M_{\bar{y}}, \phi) \]

\[ \phi \in (M_{\bar{y}}, \phi) = \bigcup (D^2 \times \{0,1\}, (D^2 \times \{0,1\})) \]

Then \( \exists \) a taut (depth 1) \( \phi' \in (M, \phi) \)
Proof:

Clearly $(x_1, y_1)$ are desired for 1.

We prove it $(x_{n+1}, y_{n+1})$ are for $1^n$ so for $(y_n, x_n)$

then done.

Example: if $D^2$ used is perfect clock can just afford

or lost time

we look at con $D \cap Y_1 = 4$ (other case

simplified)

$M_i$: 

\[ R(\theta) \times [0, e_7] \text{ to } M_i \]

(doent change \( M_i \))

\[ \text{call } M_i' \]

\[ \subseteq R(\theta) \times [0, e_7] \]

\[ R(\theta) \times [0, e_7] \]

follows by \( n \)

\[ 5 \]

\[ 5 \]
on $D^1 \times I$ use

\begin{align*}
\text{vnum of right hand sides of green is } & \text{ glue to } \odot \text{ above} \\
\text{vnum of left hand side of orange is} & \text{ glue to } \odot \text{ above} \\
\text{not: } M_q' = (D^2 \times [0,1]) \cup \text{ maps } - M_0
\end{align*}
example:

\[ \text{Diagram of a knot and its corresponding Seifert surface.} \]

\[ \text{Seifert surface for alternately link carrying from Seifert's algorithm is min genus.} \]