Permutations & polynomials in algebra and topology

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(Joint w/ T. Church, J. Ellenberg)

Example: Configuration spaces:

M is connected, oriented mfld.

$$\mathrm{Conf}_n(M) = \{(z_1, \dots, z_n) \mid z_i \in M,\ z_i \neq z_j\}$$

Classification Problem:

Understand topology of $\mathrm{Conf}_n(M)$

$H^i(\mathrm{Conf}_n(M), \mathbb{Q})$

Example (Arnold '69): For $M = \mathbb{C}$

computed $H^*(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q})$

$$\left\{\begin{array}{l}\text{monic deg } n \text{ square free} \\ \text{polynomials}\end{array}\right\} \longleftrightarrow \mathrm{Conf}_n(\mathbb{C})/S_n$$

$$(z-\lambda_1)\cdots(z-\lambda_n) \longmapsto (\lambda_1, \dots, \lambda_n)$$

Big Idea:

$$\mathrm{Conf}_{n+1}(M) \rightrightarrows \mathrm{Conf}_n(M)$$

$$(z_1, \dots, z_{n+1}) \longmapsto (z_1, \dots, \hat{z}_i, \dots, z_n)$$

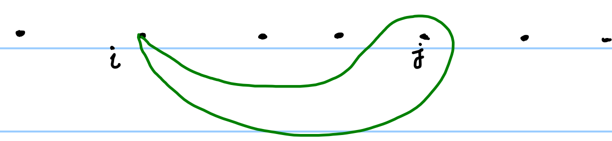
$$V_n = H^i(\mathrm{Conf}_n(M), \mathbb{Q})^{\circlearrowleft S_n} \longrightarrow H^i(\mathrm{Conf}_{n+1}(M), \mathbb{Q})^{\circlearrowleft S_{n+1}}$$

is this an $\approx$?

Homological stability: NEVER!!!

E.g.

$$H^1(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}^{\frac{n(n-1)}{2}}$$



$$\lim_{n\to\infty} \left(H^1(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q}) \right) = \infty$$

Reason: Symmetry!!!

$$V_n \longrightarrow \underset{\circlearrowleft S_{n+1}}{V_{n+1}} \longrightarrow \underset{\circlearrowleft S_{n+2}}{V_{n+2}}$$

Vector space V_n | $\dim(V_n)$

$V_n = H^i(\mathcal{M}_{g,n}, \mathbb{Q})$? $i \geq 5$

$V_n =$ space of degree i polys on the variety of $n \times n$ matrices of rank $\leq d$

Algebraic Combinatorics

$$\mathbb{C}[x_1, \ldots, x_n] \big/ (\mathrm{Sym}) = \bigoplus R_i$$

co-invariant algebra

$$\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \big/ \left(\substack{\text{multi-}\\ \text{sym}}\right) = \bigoplus_{i,j} R_{i,j}(n)$$

Weyl '35

Heiman:

$$\dim\left(\bigoplus_{i,j} R_{i,j}(n)\right) = (n+1)^{n-1}$$

$$\dim(R_{2,0}(n)) = (n-2) + \binom{n-1}{2}$$

$$\dim(R_{2,1}(n)) = ?$$

Thm: For eah V_n above, $\exists$ polynomial $f \in \mathbb{Q}[x]$ $\exists\, N \in \mathbb{Z}^{\geq 0}$ such that

$$\dim(V_n) = f(n) \qquad \forall\, n \geq N$$

- can sometimes choose $N = 1$
- $\deg(f)$ computable

Representations:

Sequences of S_n-reps $V_n \circlearrowleft S_n$

Real question: What is V_n as an S_n-representation?

Definition: The character of an S_n-reps V is

$$\chi_V : S_n \longrightarrow \mathbb{C}$$
$$\sigma \longmapsto \mathrm{Trace}(\rho(\sigma))$$
$$\rho : S_n \to GL(V)$$

- class function
- generalization: $\dim(V) = \chi_V(\mathrm{Id})$
- $V \cong W \iff \chi_V = \chi_W$

Real Real question: What is χ_{V_n}?

Character polynomials:

Let $X_i : \coprod S_n \longrightarrow \mathbb{Z}$

$$X_i(\sigma) = \#\text{ of } i\text{-cycles in } \sigma$$

E.g.

$$\chi_{\mathbb{Q}^n}(\sigma) = \#\text{ of fixed} \underset{\text{basis vectors}}{} = \#\text{ of } 1\text{-cycles}$$

$$\sigma \cdot e_i = e_{\sigma(i)}$$

$$\chi_{\mathbb{Q}^n} = X_i$$

E.g. $\chi_{\Lambda^2 \mathbb{Q}^n} = \binom{X_1(\sigma)}{2} - X_2(\sigma)$

$$\sigma \cdot (x \wedge y) = \sigma_x \wedge \sigma_y$$

$$x_1 \wedge x_2 \begin{bmatrix} & \\ & \end{bmatrix}$$

$(i)(j) \qquad x_i \wedge x_j \longmapsto x_i \wedge x_j$

$(i\,j) \qquad x_i \wedge x_j \longmapsto -x_i \wedge x_j$

Thm $\forall$ sequences V_n above

$\exists N$, $\exists$ polynomial $f(x_1, \dots, x_r)$ some r s.t.

$$\chi_{V_n} = f(x_1, \dots, x_r) \quad \forall n \geq N$$

- $\dim(V_n) = \chi_{V_n}(\mathrm{Id}) = f(n, 0, \dots, 0)$

Underlying structure:

FI - modules:

FI = category of finite sets and injections

$$\underset{\substack{\circlearrowleft \\ S_n}}{\{1,\dots,n\}} \;\rightrightarrows\; \underset{\substack{\circlearrowleft \\ S_m}}{\{1,\dots,m\}}$$

Definition: An FI-module is a functor

$$FI \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$$

$$\{1,\dots,n\} \longmapsto \underset{\substack{\circlearrowleft \\ S_n}}{V_n} \;\rightrightarrows\; \underset{\substack{\circlearrowleft \\ S_{n+1}}}{V_{n+1}}$$

E.g.

$$\{1,\dots,n\} \longmapsto \mathbb{C}[X_1,\dots,X_n]$$
$$\downarrow$$
$$\mathbb{C}[X_1,\dots,X_{n+1}]$$

FI-space: Functor FI $\longrightarrow$ topological spaces

get a single object

$$V : FI \longrightarrow \mathrm{Vect}$$

Definition An FI-module is finitely generated if

$$V_1 \longrightarrow \underset{u_1,u_2}{\overset{\circlearrowleft}{V_2}} \;\rightrightarrows\; \underset{u_3}{\overset{\circlearrowleft}{V_3}} \;\rightrightarrows\; V_4 \longrightarrow \cdots \longrightarrow \underset{u_r}{V_{15}} \longrightarrow \cdots$$

$\exists\; u_1,\dots,u_r$

E.g.

$$\mathbb{C}[X_1] \longrightarrow \mathbb{C}[X_1, X_2] \rightrightarrows \mathbb{C}[X_1, X_2, X_3]$$

$X_1 \in$

is fin. generated

Thm: Let V = an FI-module, then

V is fin. gen. $\Longleftrightarrow$ $\left(\begin{array}{l}\{V_n\} \text{ is unif.} \\ \text{repr. stable}\end{array}\right)$

in the sense of Farb–Church

$\Longleftrightarrow$ χ_{V_n} is a polynomial in the X_i = # i-cycles

Key: The category of FI-modules is Notherian

Remark: A preprint of Church (2011) was very helpful in the formulation of the thm above