# Conway mutations and Alternating links

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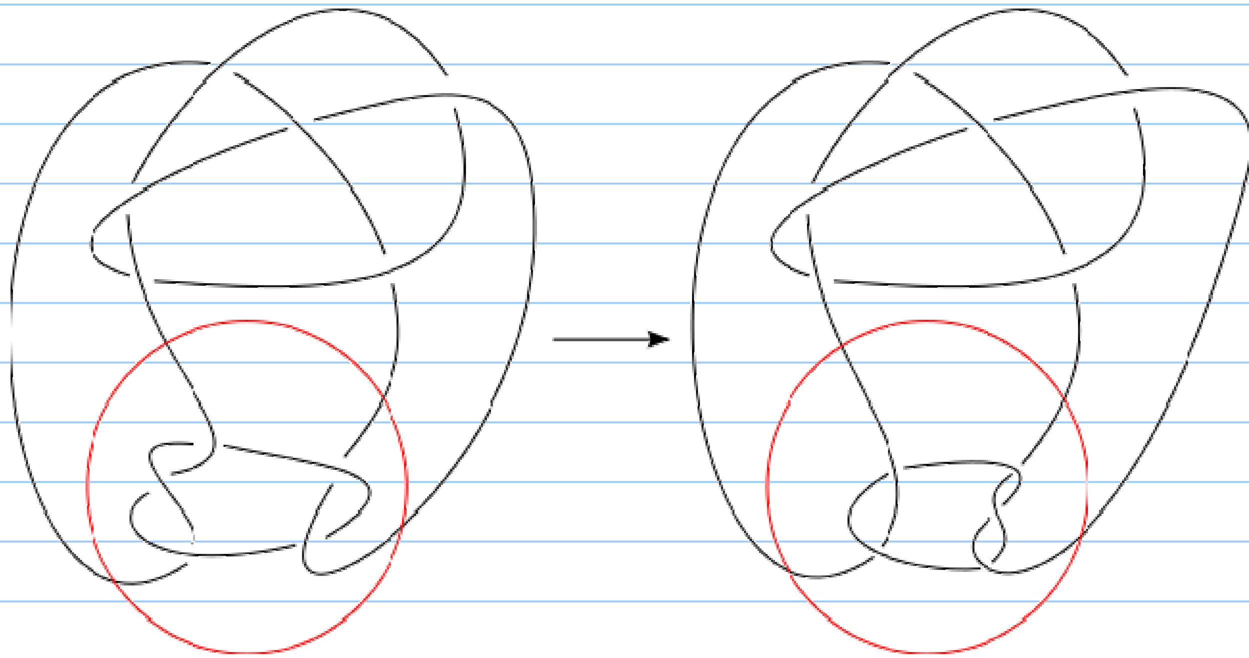
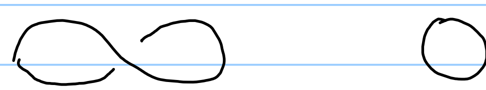
Kinoshita – Terasaki knot $\Delta(KT) = 1$

Conway knot $\Delta(C) = 1$



## Definitions:

Two diagrams links are mutants if they are generated by a sequence of these moves an isotopies (of diagrams links)



Mutant links are tough to distinguish:

They have the same

HOMFLY, Kauffman polynomials
A

Knot signature
hyp. volume
odd Khovanov

They could have different

- genera
  Gabai showed $g(C) \neq g(KT)$
- sliceness
- $\widehat{HFK}$

**Prop (Viro):** The homeomorphism of $\Sigma(L)$ - the double branch cover of $S^3$ along $L$ - does not change under mutation.

Converse is not true

E.g. $L = T(3,7)$, $L' = P(2,-3,7)$

$\Sigma(L) \cong \Sigma(L')$ but $L$ and $L'$ are not mutants

**Fact (Reidemeister-Franz, Schubert):**

For 2-bridge links $L, L'$

$$\Sigma(L) \cong \Sigma(L') \text{ - lens spaces}$$

$$\Longleftrightarrow$$

$$L \simeq L' \quad \left(\Longleftrightarrow \; L \simeq \mu L'\right)$$

Lens spaces are distinguished by their Reidemeister torsion. So they're distinguished by their $d$-invariants.

**MANTRAM:** Every question (answer) that you have about 2-bridge links, you should generalize to alternating links.

**Thm (G.)** $D, D'$ are reduced alternating diagrams for links $L, L'$

- $D, D'$ mutants

$\Downarrow$ $\Uparrow$ Menesco (He also proved that $\underset{\text{alt.}}{L} \underset{M}{\cong} L'' \Rightarrow L''$ alternating)

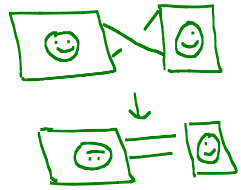
- $L, L'$ mutants

$\Downarrow$

- $\Sigma(L) \cong \Sigma(L')$

$\Downarrow$

- $d$ invariants of $\Sigma(L)$ and $\Sigma(L')$ are the same



—— ∘ ——

Bonahon, Hodgsen – Rubinstein

↳ If a lens space $= \Sigma(L')$

$\Rightarrow$ L is 2-bridge

Conjecture: If $\Sigma(L) \cong \Sigma(L')$, then either both or neither of $L, L'$ are alternating.

—— ∘ ——

- d-invariant is a complete invariant of the homeomorphism type of $\Sigma(L)$, L-alternating.

Question: Is there some information coming from HF that distinguishes isotopy types of alternating links?

—— ∘ ——

The d-invariant following Ozsváth –

$(S, z, \alpha, \beta, 0)$ in Heegaard Floer homology.

Heegaard diagram

Y – closed, oriented rational homology sphere

$$d : \mathrm{Spin}^c(Y) \longrightarrow \mathbb{Q}$$

using d-invariants, $\exists$ lots of applications to low dim. topology:

concordance, unknotting, Dehn surgery, a classification result.

In a lucky situation, you can calculate d-invt in lattice theoretic terms:

Suppose $Y = \partial X$

$H_1(Y) = 0$, intersection pairing on $X$ is positive-definite form

Every $t \in Spin^c(Y)$ extends to a spin$^c$-str $s$ on $X$

$$d(Y,t) \leq \min\left\{ \frac{\langle c_1(s), c_1(s)\rangle - b_2(X)}{4} \;\middle|\; \begin{array}{l} s \in Spin^c(X) \\ s|_Y = t \end{array} \right\}$$

$X$ is sharp if $(\leq)_t$ is attained for all $t \in Spin^c(Y)$.

For $\Sigma(L)$, $L$-alternating, there is a very natural sharp 4-mfld $X$ with $\partial X = \Sigma(L)$

$D$: alternating diagram

Spaning surface $F$

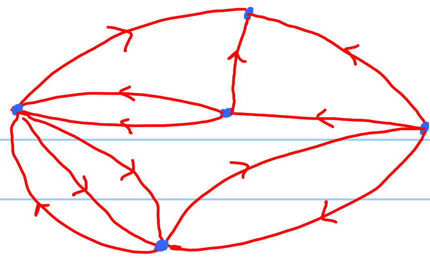
push $int(F)$ into $int(D^4)$

$X(D) = \Sigma(D^4, F)$ is a sharp 4-mfld and

$\partial X(D) = \Sigma(L)$

The intersection pairing on $X(D)$

$H_2(X(D), Q_{X(D)})$ is the lattice of flows on the Tait graph $G(D)$



$\partial : C_1(G) \to C_0(G)$

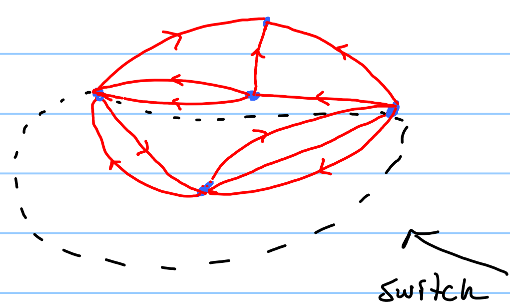
$\ker(\partial) = \mathcal{F}(G)$ lattice of flows

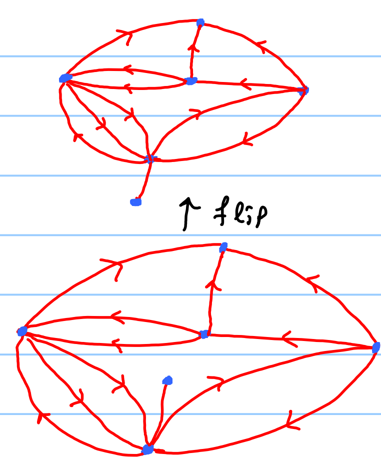
$\mathrm{Im}(\partial^*) = C(G)$ lattice of cuts

If $D$ is alternating diagram with Tait graph $G(D)$, then Floer theoretic $d$-invariant of $\Sigma(L)$ is expressed solely in terms of the lattice theoretic $d$-invt of $\mathcal{F}(G(D))$.

When can a pair of graphs $G, G'$ have flow lattices w/ same $d$-invariant?

$\longrightarrow$ Mutant diagram $D'$ will have a Tait graph $G(D')$



**Thm (Whitney + E):** The Tait graph construction gives a 1-1 correspondence

$$\{\text{alternating diagrams}\}/\text{mutation} \longmapsto \{\text{planar graphs}\}/\text{flips \& switches}$$

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Two graphs are 2-isomorphic if $\exists$ bijection $f: E(G) \to E(G')$ that preserves cycles

**THM (G.)** TFAE for a pair of graphs $G$ and $G'$:

- $G$ and $G'$ are 2-isomorphic
- $\mathcal{F}(G) \cong \mathcal{F}(G')$
- $d$-invts of $\mathcal{F}(G)$ and $\mathcal{F}(G')$ are the same.