Filtering smooth concordance classes of topologically slice knots

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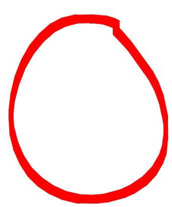
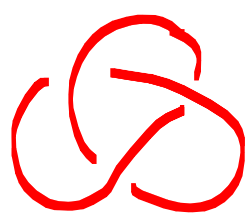
Def: A knot is a smooth embedding

$$f : S^1 \longrightarrow S^3$$



unknot



trefoil

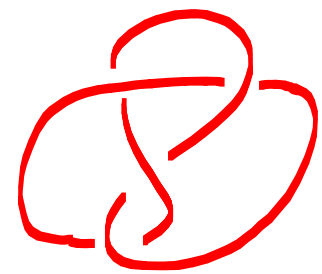
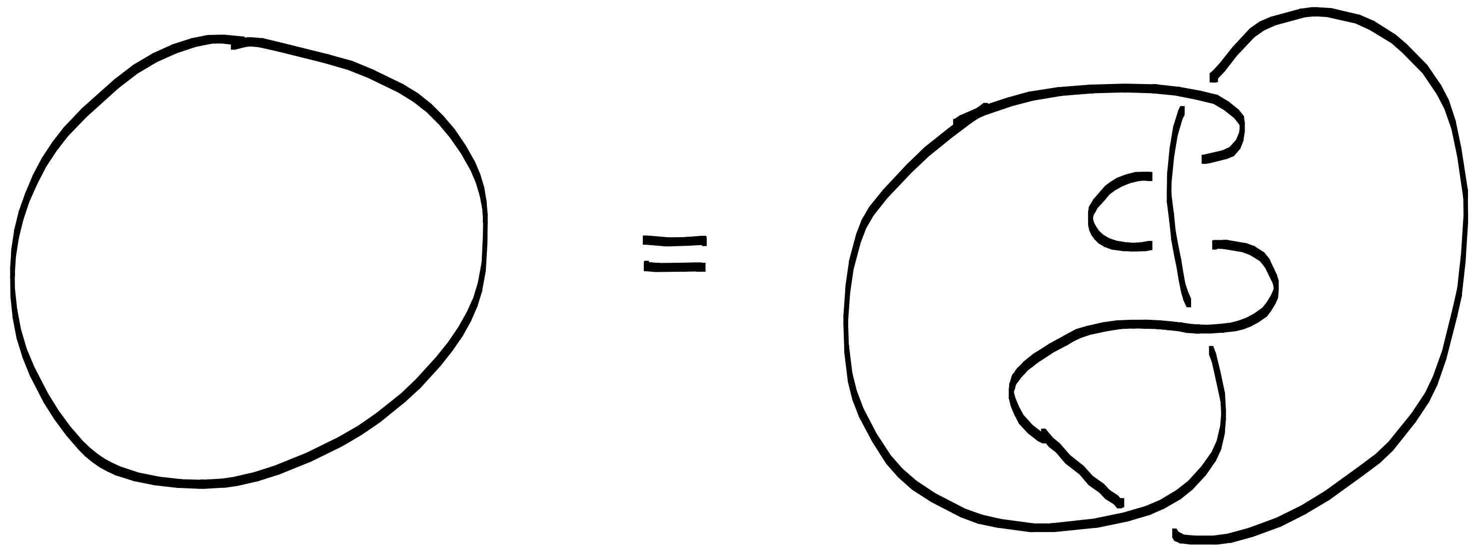


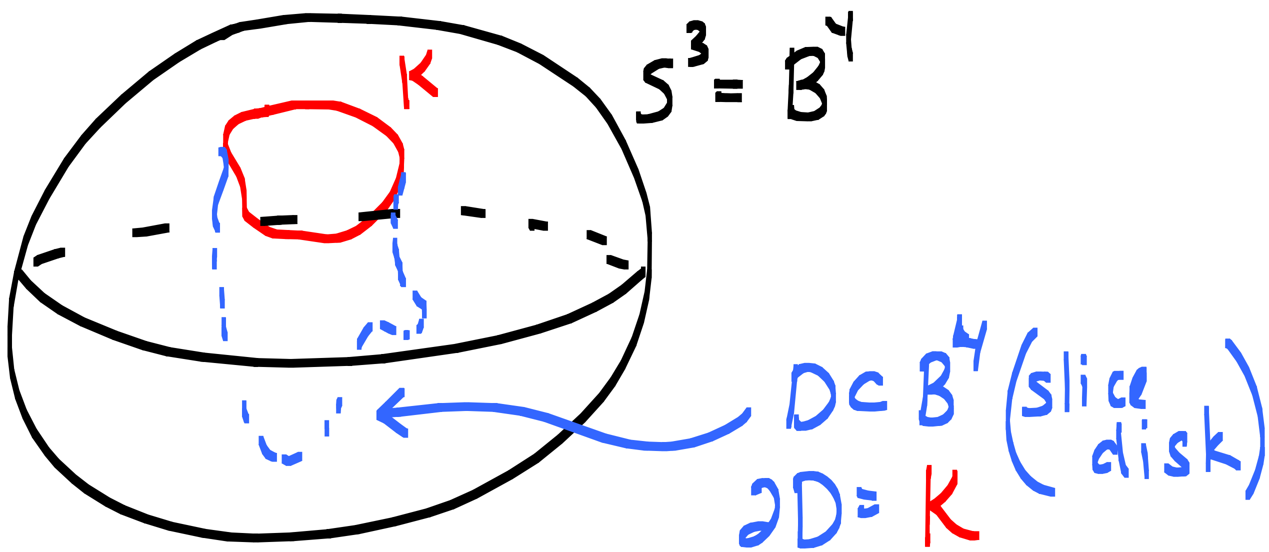
figure-eight



<u>Note</u>: In S^3, a knot K is trivial if and only if it bounds an embedded disk

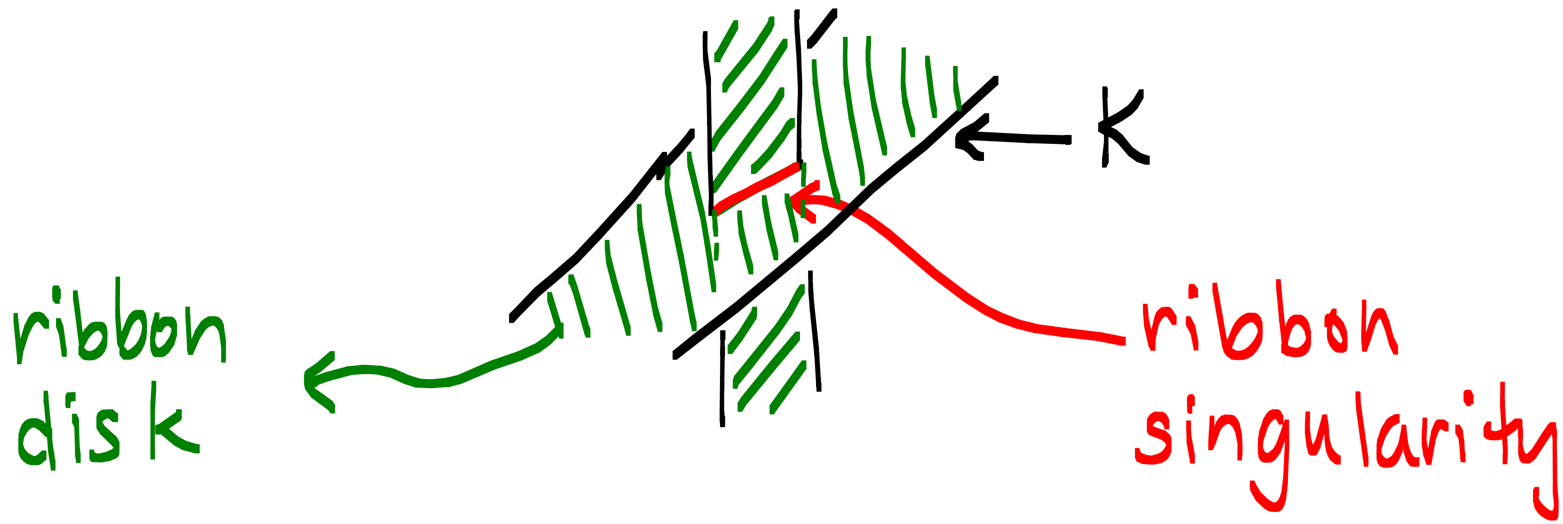
However, if we allow the disk to move into $\mathbb{R}^4$, we can get more interesting knots that bounds disks.

Def: A knot $K \subset S^3$ is slice if the boundary of a smoothly embedded disk in B^4

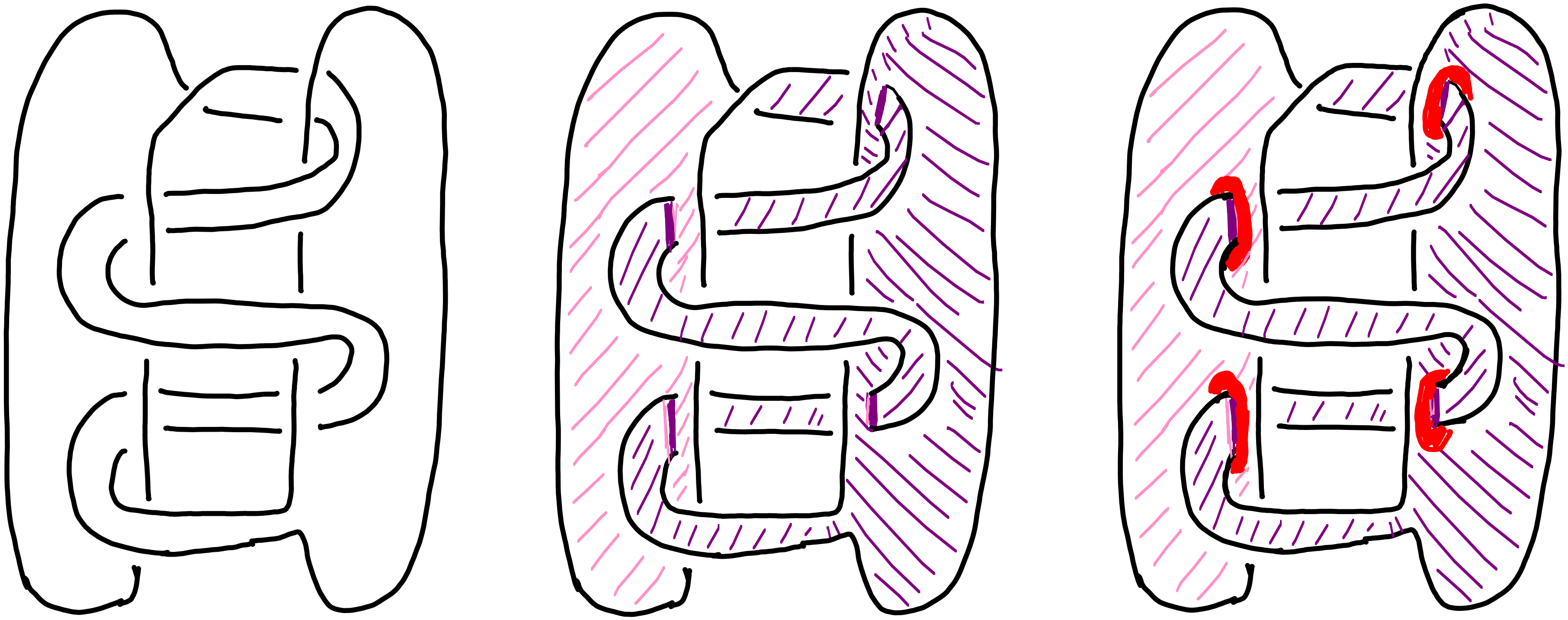


Examples of slice knots:

Def: K is ribbon if it bounds an immersed disk in $\mathbb{R}^3$ with only ribbon singularities.



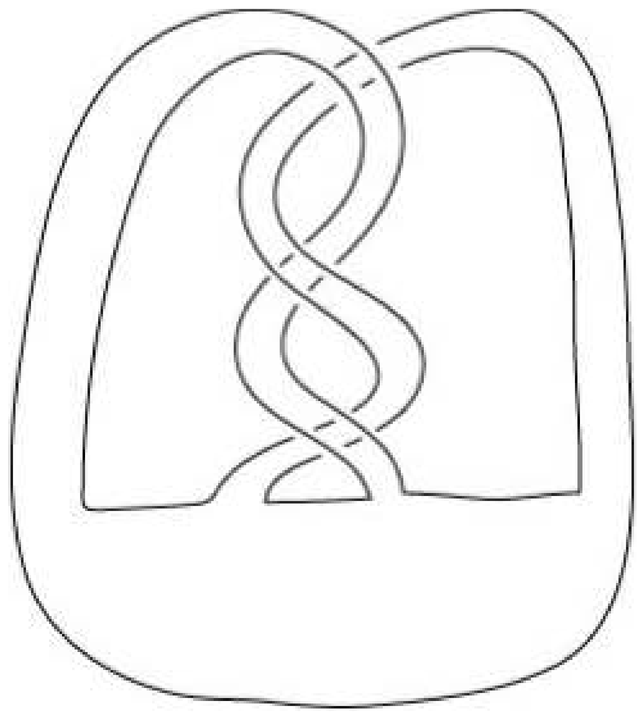
A ribbon knot is slice



Pf: To obtain a disk <u>embedded</u> in $\mathbb{R}^4_+$, push the interior of red disks into interior of $\mathbb{R}^4_+$.

Conjecture: A knot is (smoothly) slice $\Leftrightarrow$ it is ribbon.

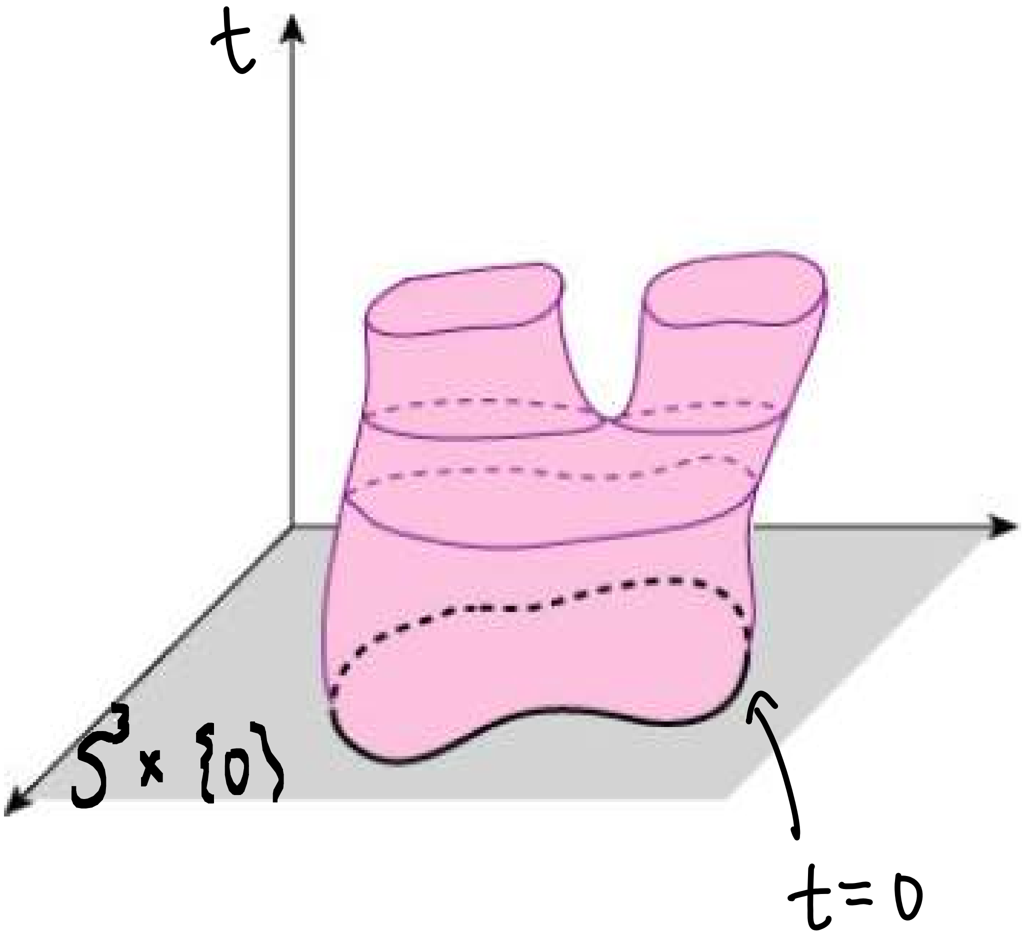
Important example for later:

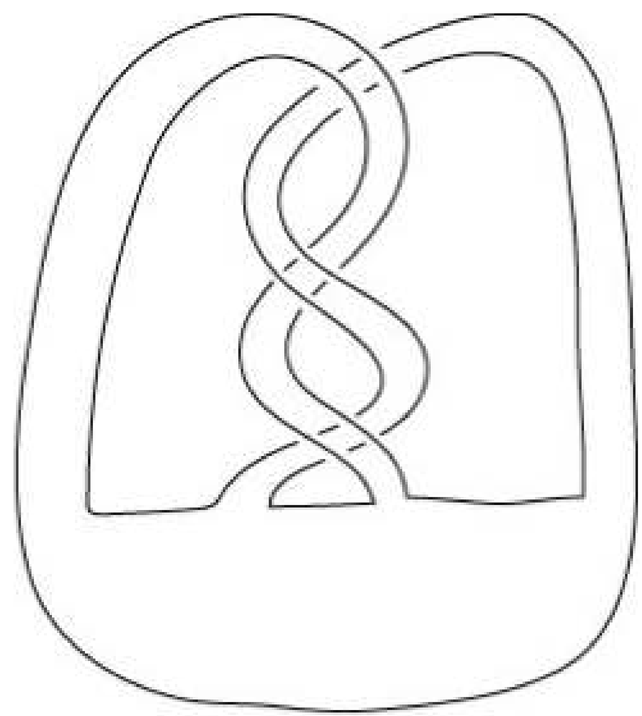
 $=: 9_{46}$

The 9_{46} knot is slice.

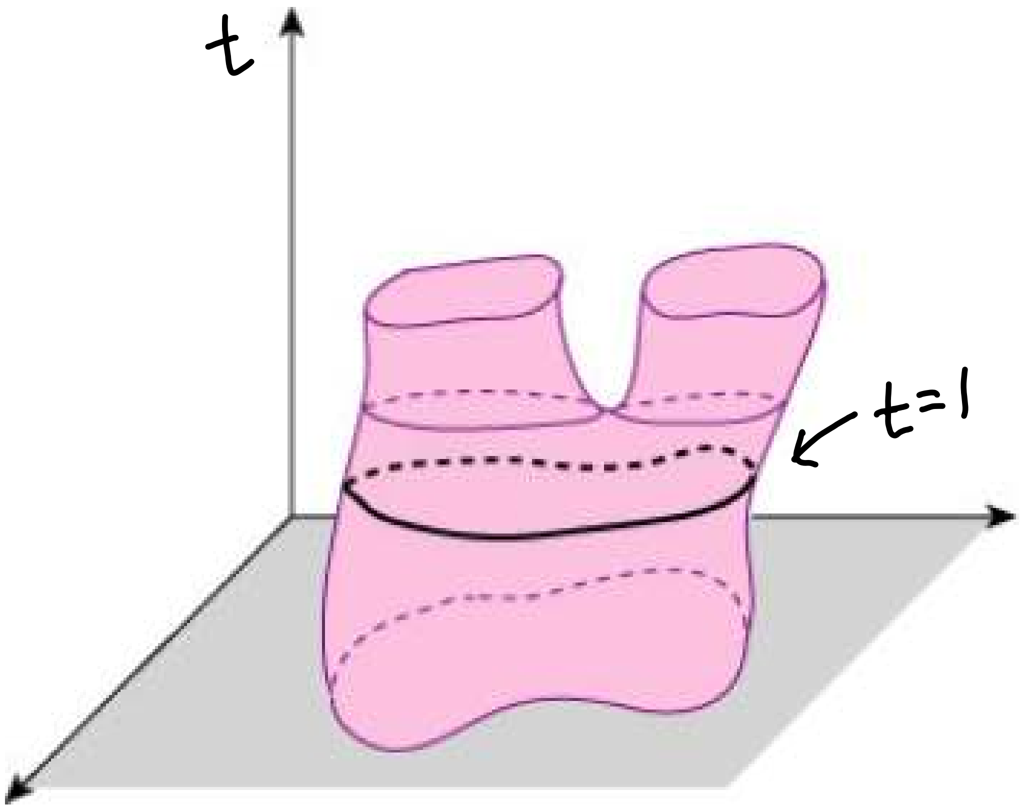
How to build a slice disk:

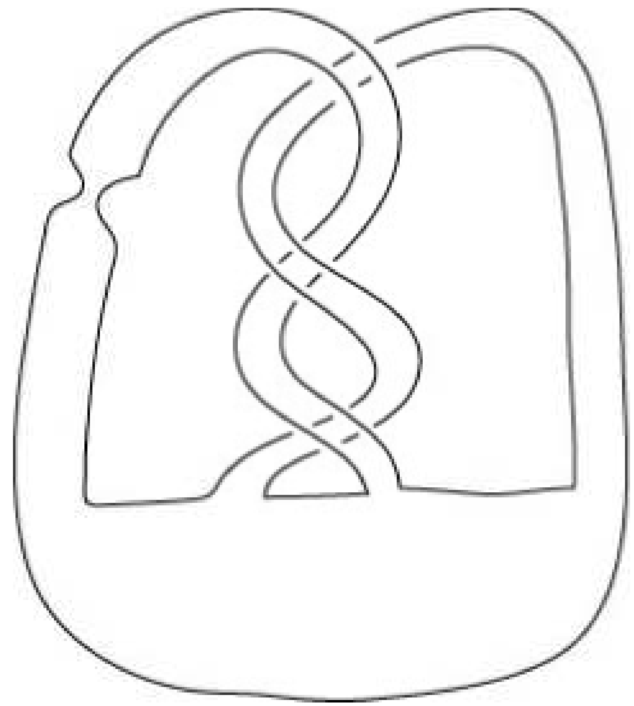
- look at slices of the disk in time



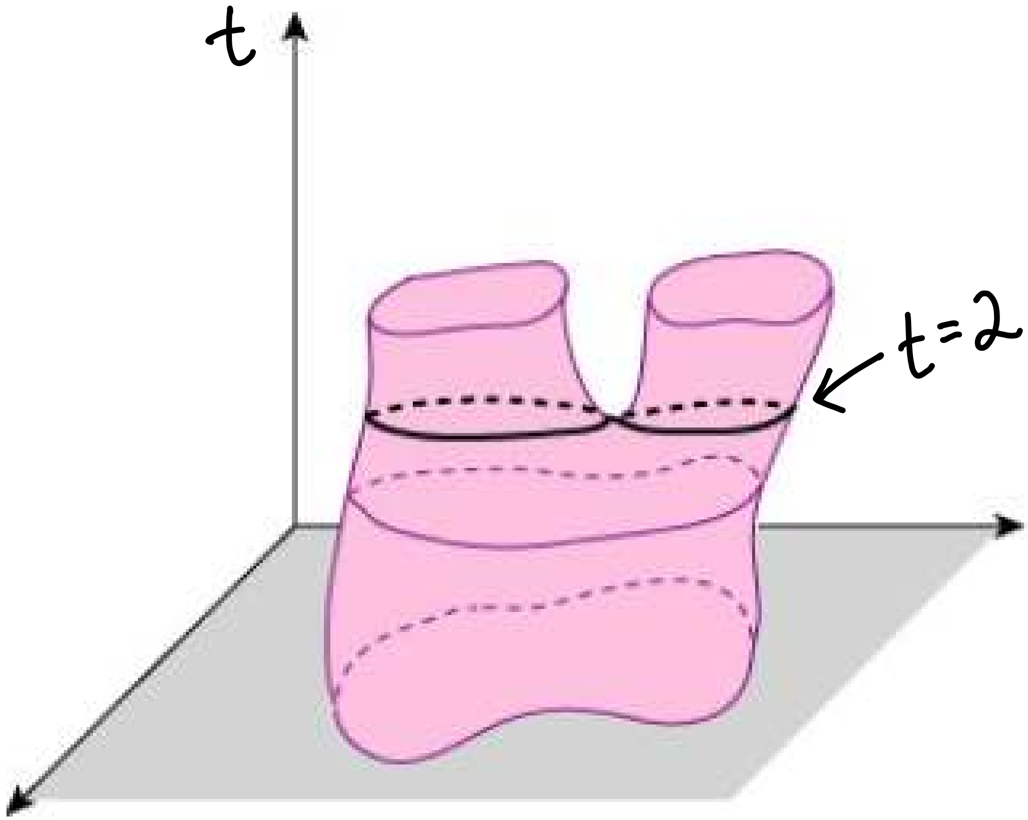


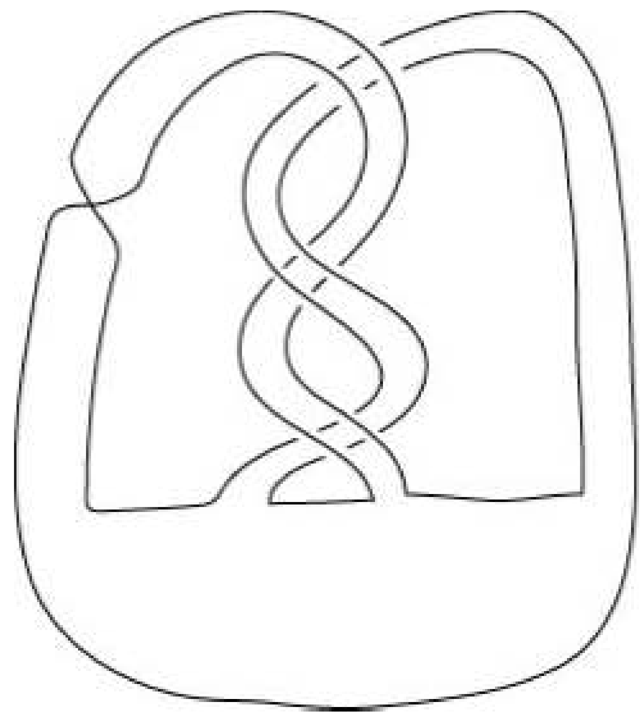
t=0



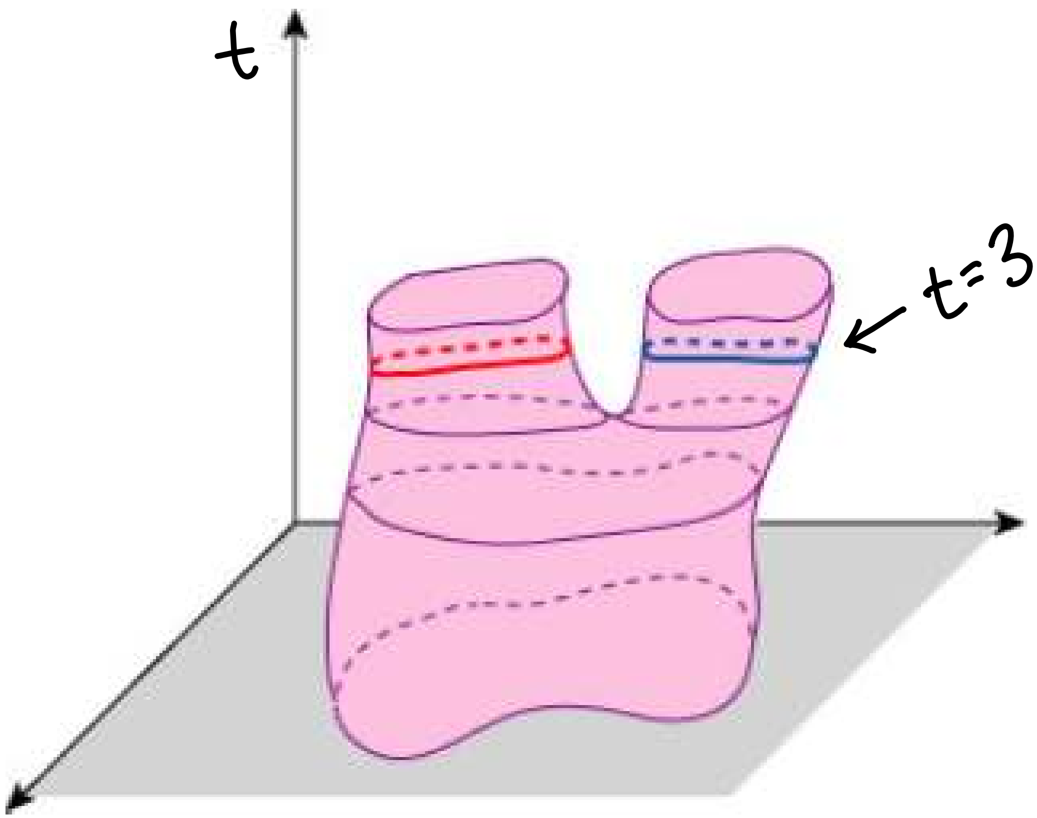


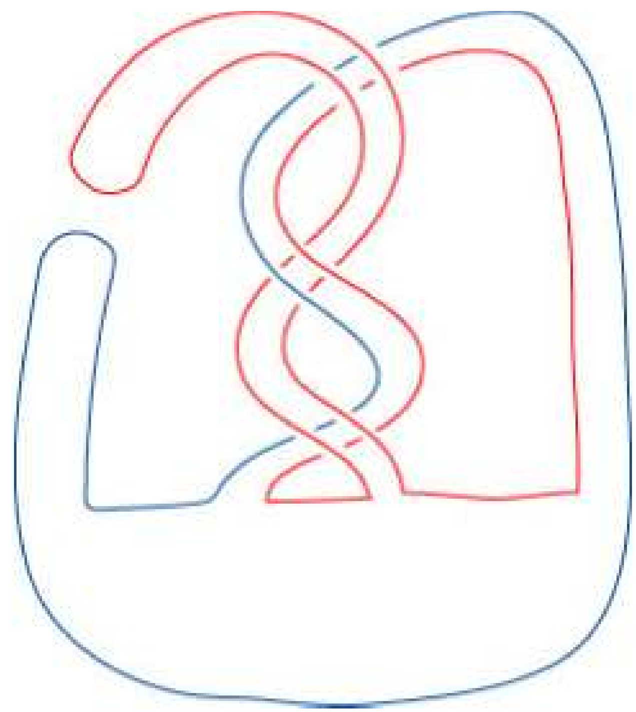
t = 1



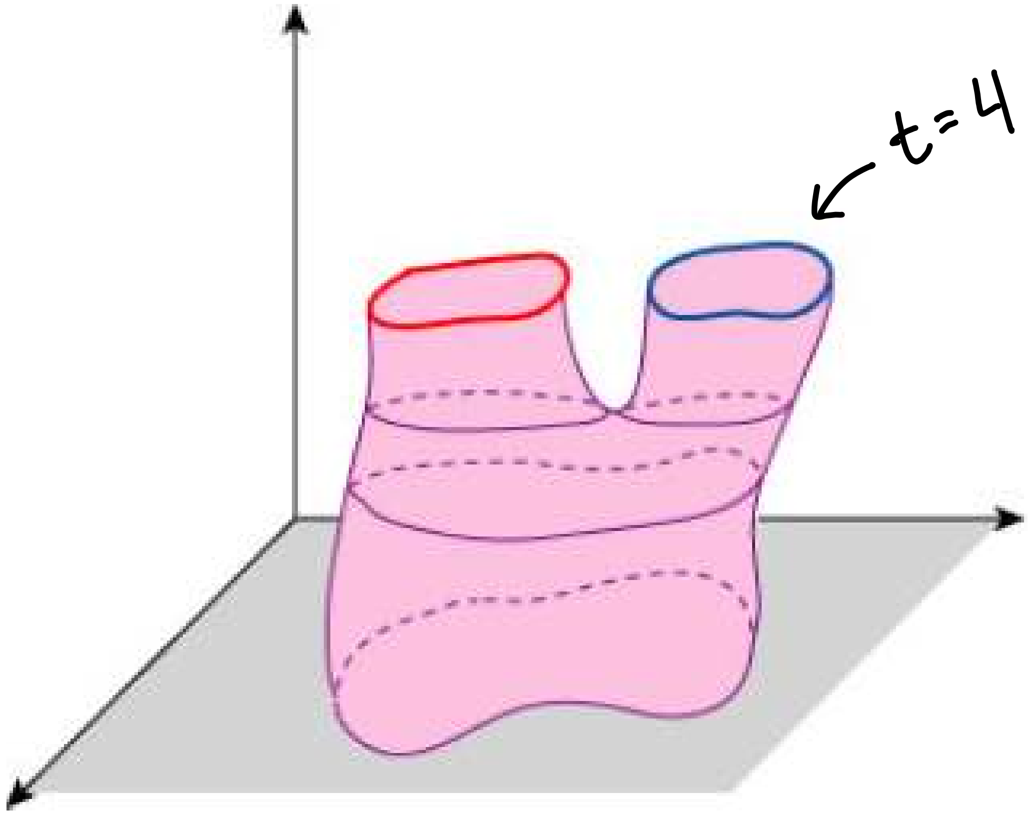


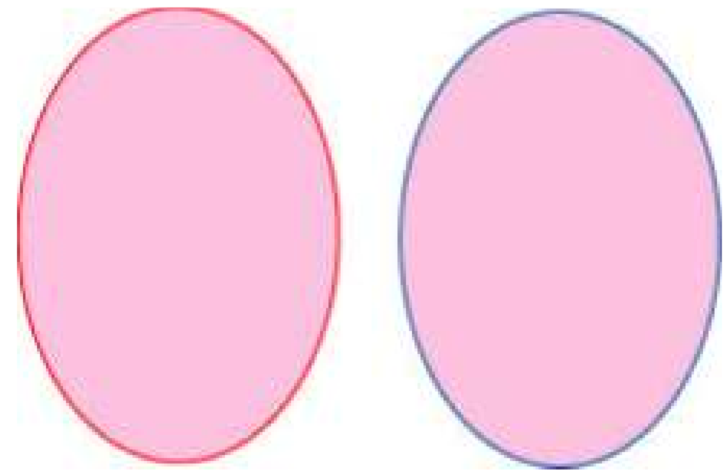
t = 2



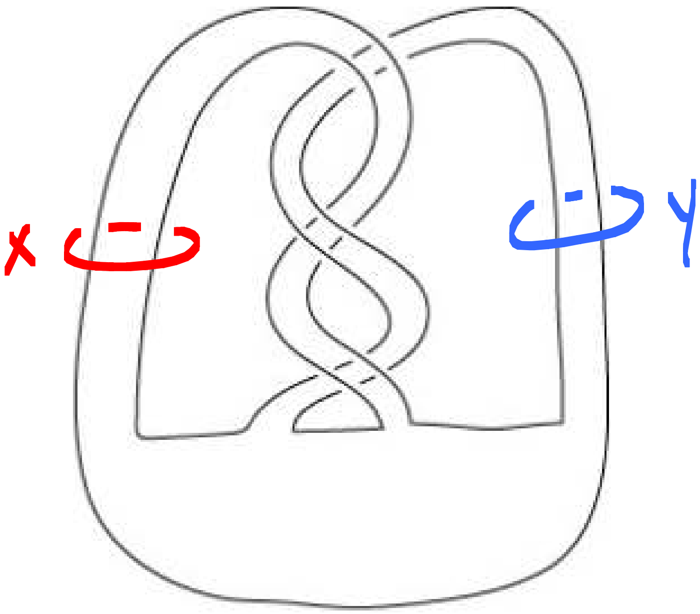


t = 3



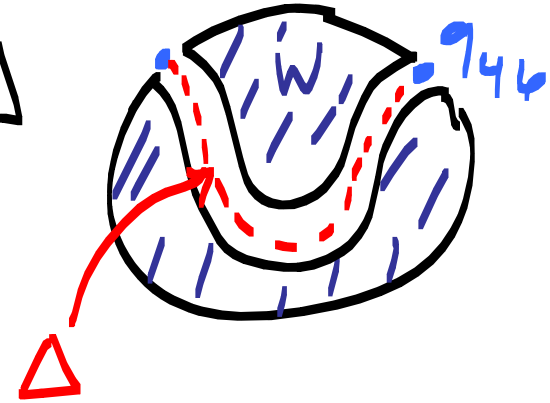


$t=4$



$\Rightarrow \quad 9_{46} = \partial \Delta \qquad \Delta = \text{disk} \subset B^4$

Let $W = B^4 - \Delta$

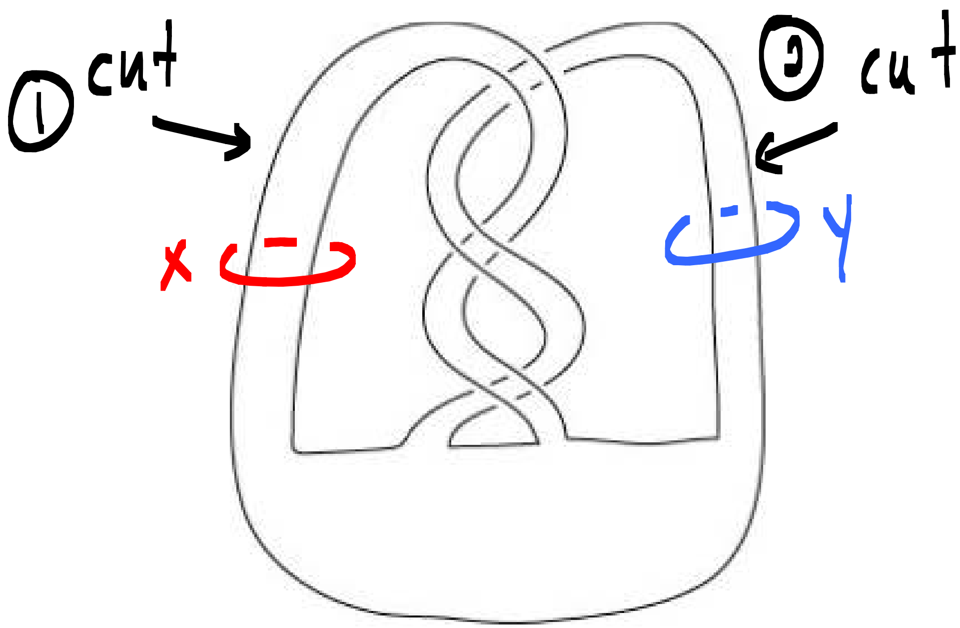


Note: $x \in \ker(\pi_1(S^3 - K) \longrightarrow \pi_1(W))$.

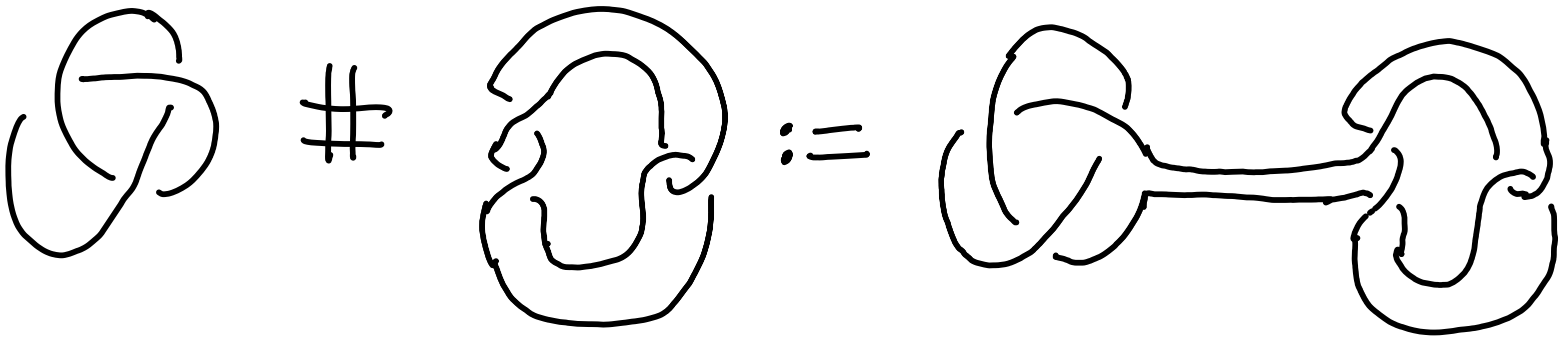
We could "cut the other band" and create W' with $y \in \ker(\pi_1(S^3 - K) \longrightarrow \pi_1(W'))$.

Fundamental Idea:

There are "two ways of slicing" the 9_{46} knot.



There is a binary operation on (oriented) knots:

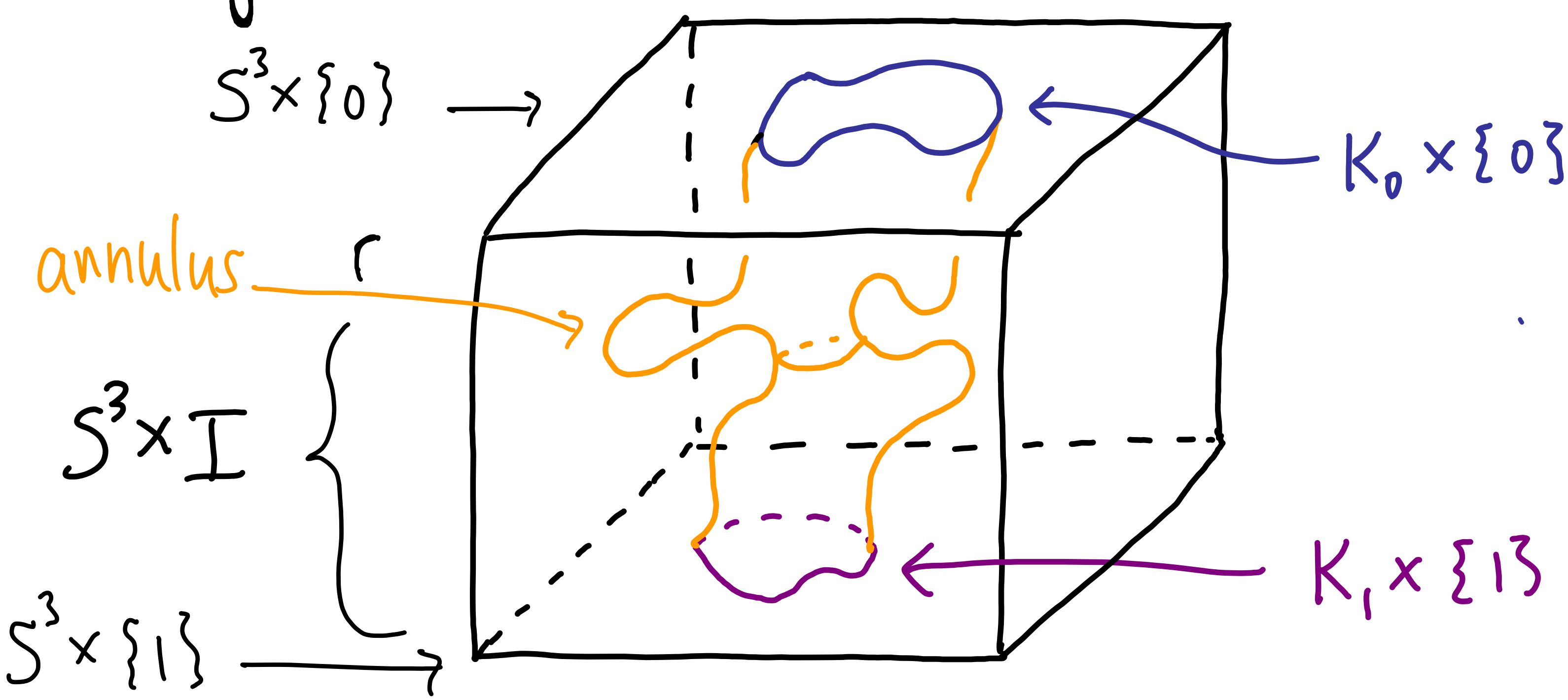


K_1 K_2 $K_1 \# K_2$

connected sum

We can form a group using $\#$ if we change our equiv. relation on knots

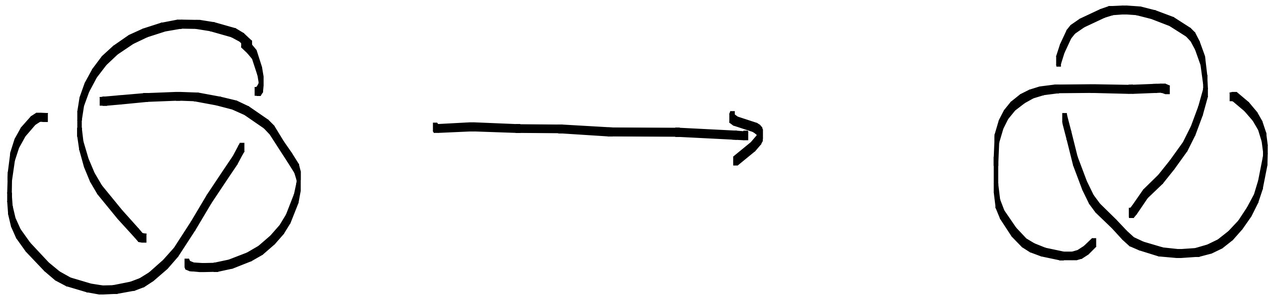
Def: Knots K_0 and K_1 are **concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$



The relation, is concordance to, is an equivalence relation. We denote it by $\sim_c$.

Def: $\mathcal{C} := \{\text{equivalence classes of knots}\}$
$$= \{\text{knots}\}/\sim_c$$

Claim: $\mathcal{C}$ is an abelian group under the operation of connected sum.



K $\qquad\qquad$ $\bar{K}$

Mirror image (change all crossings)

$\boxed{\text{For any knot } K,\ K \# r\bar{K} \text{ is slice}}^{*}$

* need to orient knots and $r\bar{K}$ is $\bar{K}$ with the reversed orientation.

- Identity $= [0] = \{\text{slice knots}\}$
- Inverse of $[K]$ is $[r\bar{K}]$:

$$K \# r\bar{K} = \text{slice} \Rightarrow [K \# r\bar{K}] = [0].$$

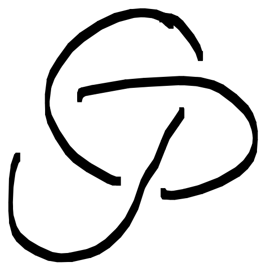
$$\Rightarrow \quad -[K] = [r\bar{K}]$$

What is this group?

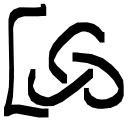
Is it ... trivial?

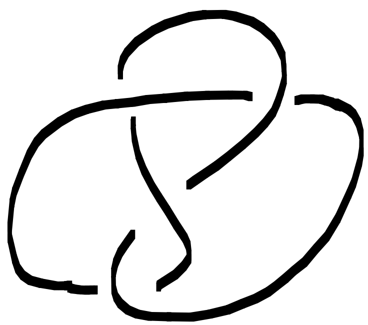
finitely generated?

torsion-free?

Ex:  is not slice

$\Rightarrow$ $\mathcal{C}$ is non-trivial

In fact, [] has infinite order in $\mathcal{C}$.

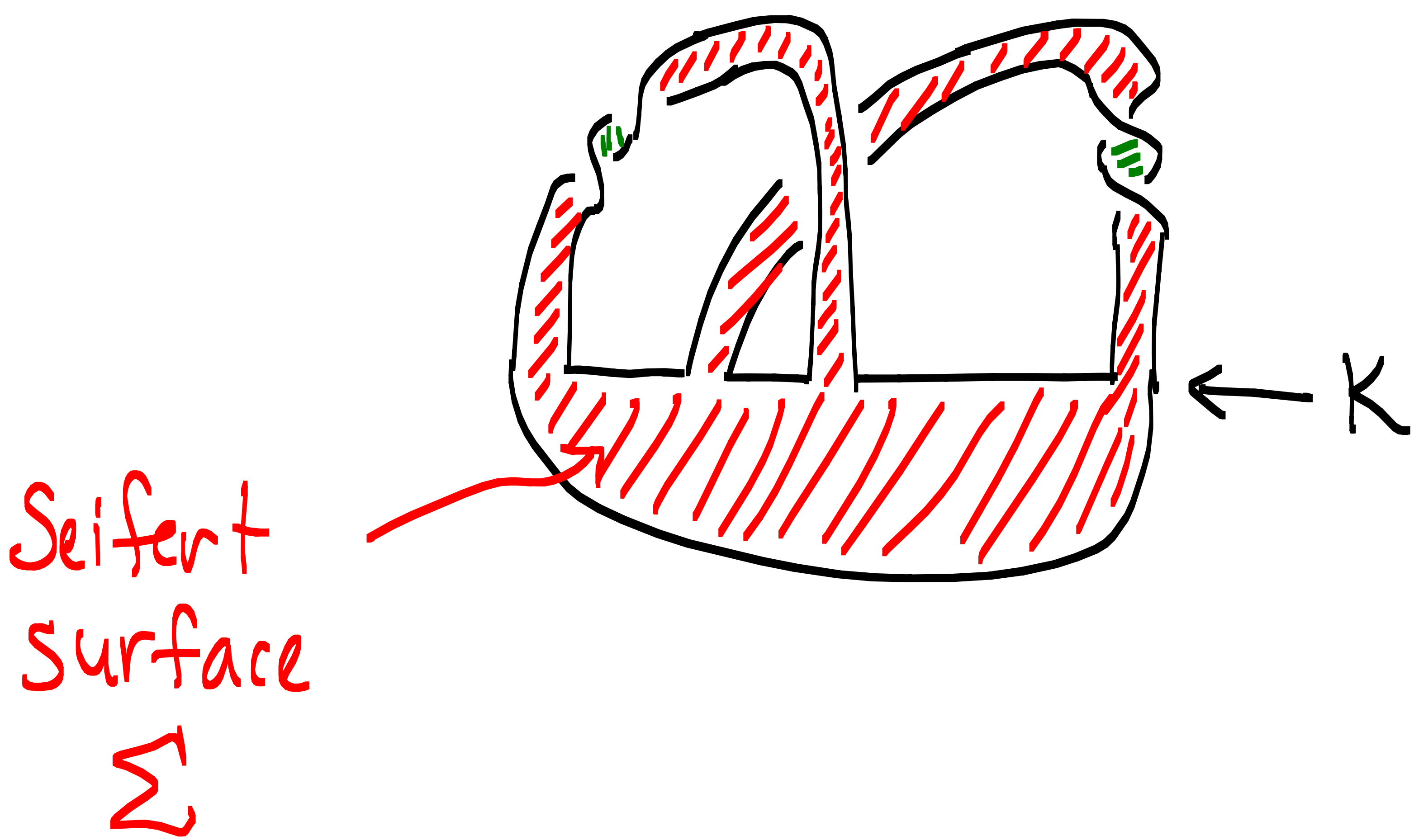
Ex: $K =$  is not slice but

$K = r\bar{K} \Rightarrow [K] \neq [0]$ but

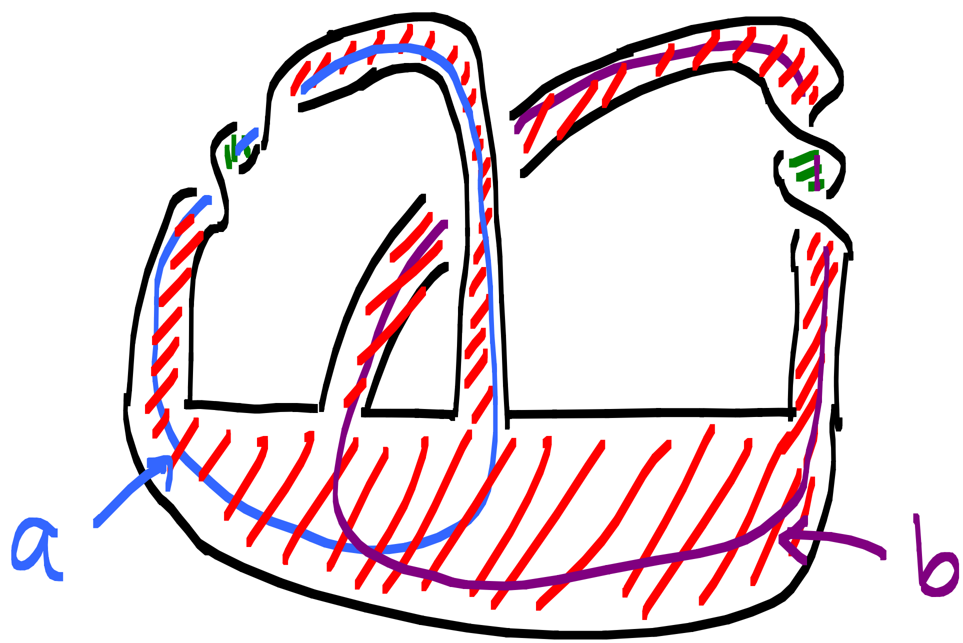
$$2[K] = [K \# K] = [K \# r\bar{K}] = [0]$$

Q. How can one show that a particular knot K is not slice? i.e. there is <u>no</u> possible way for K to bound <u>any</u> disk in B^4.

Def: A Seifert surface Σ for K is a 2-sided surface embedded in S^3 with $\partial\Sigma = K$.



From a Seifert surface ⇝ Seifert matrix



$$V = \begin{pmatrix} lk(a,a^+) & lk(a,b^+) \\ lk(b,a^+) & lk(b,b^+) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

b^+ = push b off Σ into + direction

$lk(a,b^+)$ = linking number of a and b^+.

For $\omega \in \mathbb{C}$,

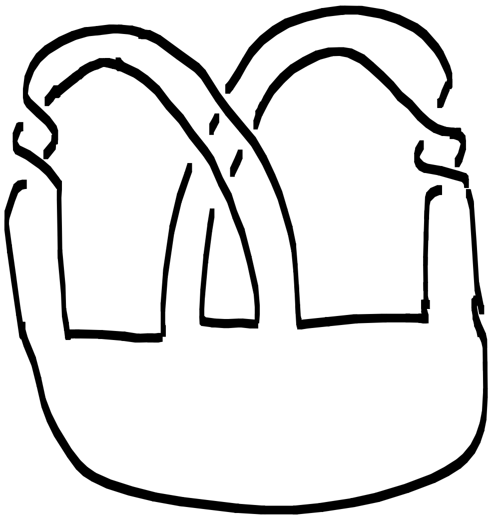
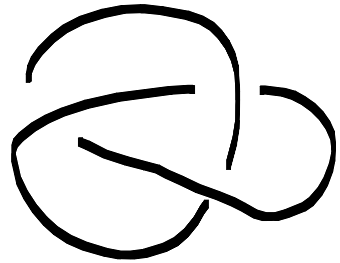
$(1-\omega)V + (1-\bar{\omega})V^T$ is a Hermitian matrix

Def: $\sigma_\omega(K) :=$ signature of $\left((1-\omega)V + (1-\bar{\omega})V^T\right)$
$\in \mathbb{Z}$.

If K is slice and $\omega = (p^k)^{th}$ root of unity $\Rightarrow \sigma_\omega(K) = 0$.

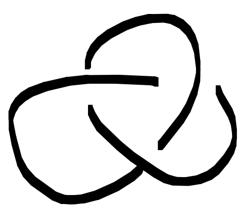
$\rightsquigarrow \quad \oplus \sigma_\omega : \mathcal{C} \longrightarrow \mathbb{Z}^\infty$

Ex:

 = 

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$w = -1 \Rightarrow \sigma_w = -2$$

$\Rightarrow$  is not slice

There are many other invariants of knots that obstruct sliceness.

Q. What do these tell us about the structure of the group $\mathcal{C}$?

Late 90's: Cochran-Orr-Teichner

defined the (n)-solvable filtration of $\mathcal{C}$:

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

slice

This filtration is highly non-trivial:

Theorem (Milnor, Tristram, Levine, Jiang, Livingston, Casson, Gordon, Friedl, Cochran, Orr, Teichner, Harvey, Leidy, Burke ...)

For each $n \geq 0$,

$$\bigoplus_{\{p(t)\}} \mathbb{Z}^{\infty} \oplus \mathbb{Z}/2 \subset \mathcal{F}_n / \mathcal{F}_{n.5}.$$

However, their definition is fundamentally flawed!

Def$\underline{\text{n}}$: Let $\mathcal{T}$ be the subgroup of $\mathcal{C}$ of topologically slice knots.

Fact: $\mathcal{T}$ is highly non-trivial.

But $\mathcal{T} \subset \bigcap_{n=0}^{\infty} \mathcal{F}_n$!

We refine the (n)-solvable filtration:

$$\begin{array}{ccccccccc}
\subset \cdots \subset & \mathcal{B}_2 & \subset & \mathcal{B}_1 & \subset & \mathcal{B}_0 & \subset & \mathcal{C} \\
 & \cap & & \cap & & \cap & \swarrow & \\
\subset \cdots \subset & \mathcal{F}_2 & \subset & \mathcal{F}_1 & \subset & \mathcal{F}_0 & &
\end{array}$$

and will show that $\mathcal{B}_n$ induces a non-trivial filtration on $\mathcal{T}$, the subgroup of topologically slice knots.

Let W be a smooth 4-dimensional manifold with $\partial W = S^3$. If K is any knot, we can ask if K is slice in W, i.e. does K bound a smoothly embedded disk $\Delta \subset W$ with $[\Delta] = 0 \in H_2(W, \partial W)$.

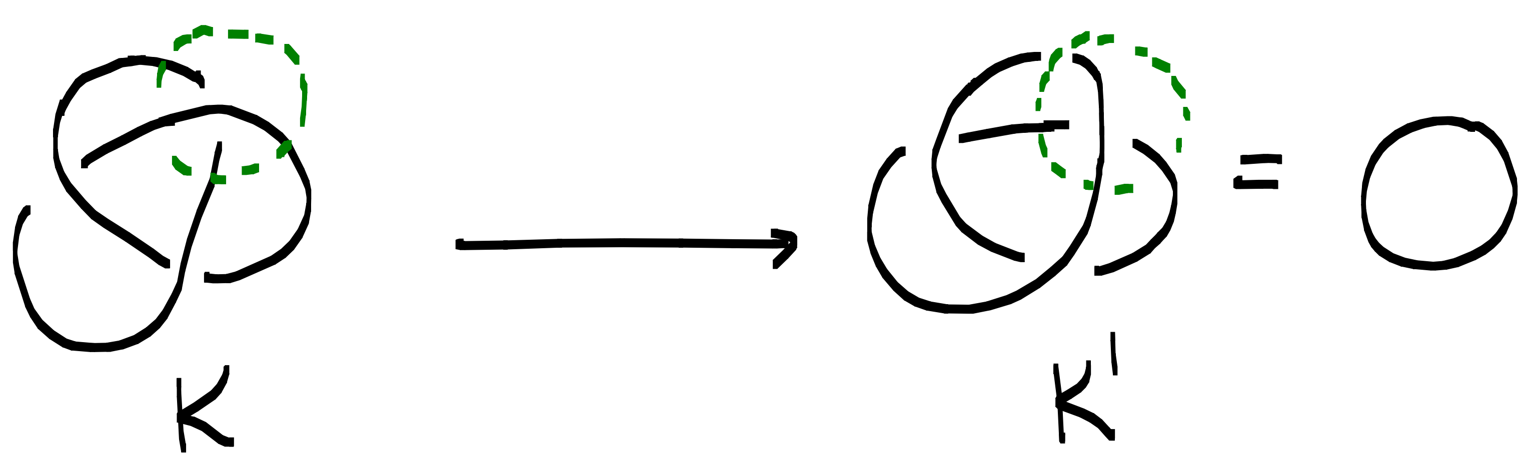
Then we could try to filter $\mathcal{C}$ by considering various 4-manifolds W_i.

Fact: Every knot is slice in $\left(\#_k \mathbb{CP}^2\right) \# \left(\#_\ell \overline{\mathbb{CP}^2}\right) \setminus B^4$.

Recall $\mathbb{CP}^n = \{\text{complex lines in } \mathbb{C}^{n+1}\}$.

$\overline{\mathbb{CP}^n} = \mathbb{CP}^n$ w/ opposite orientation

Pf: Every knot can be changed into the unknot by changing crossings.



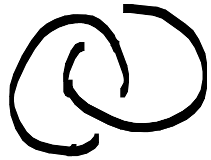
A crossing change gives a homotopy $S^1 \times I \longrightarrow S^3 \times I$ starting at $K \times \{0\}$ and ending at $K' \times \{1\}$ which is an embedding except at one point p.

$t=0$ $\quad$ $t=1/2$ $\quad$ $t=1$

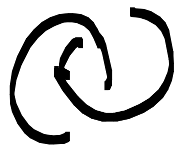
Take a 4-ball $B \subset S^3 \times I$ around p.

Let $A =$ image of $S^1 \times I$. Then $A \cap B$

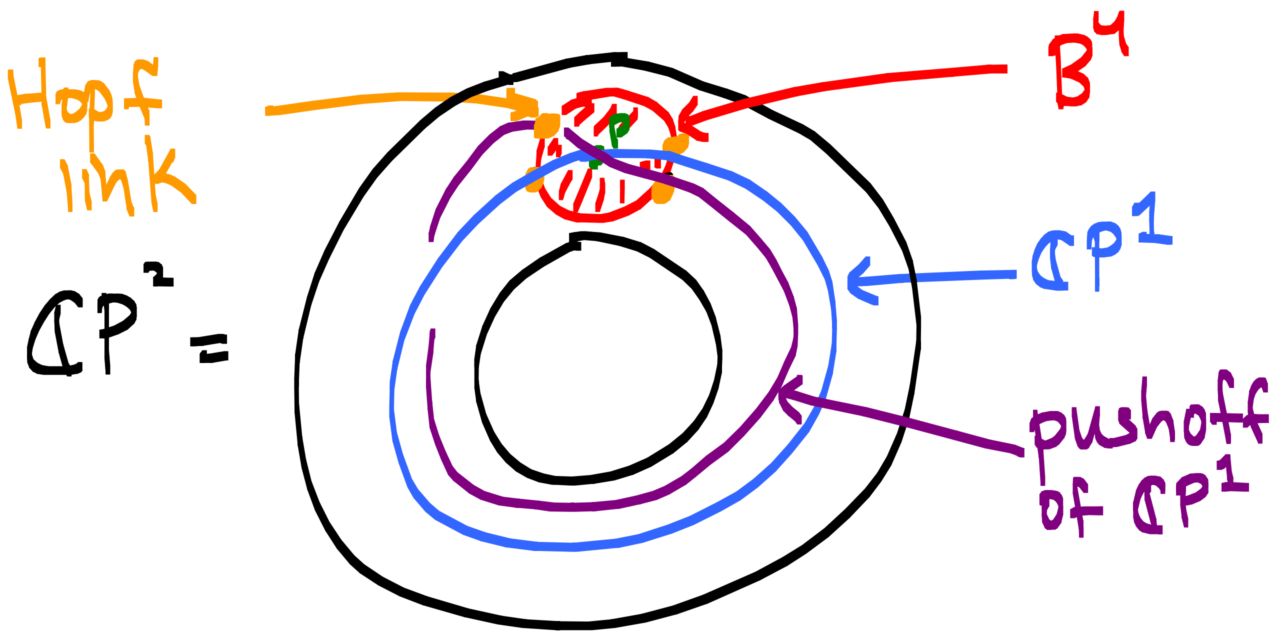
is two complex disks in $B^4 \subset \mathbb{C}^2$ and

$\partial(A \cap B) =$  , Hopf link.

We can replace B with $\mathbb{CP}^2 \setminus B^4$ or $(\overline{\mathbb{CP}}^2 \setminus B^4)$

and then  will bound disjointly

embedded disks (blowing up at p).

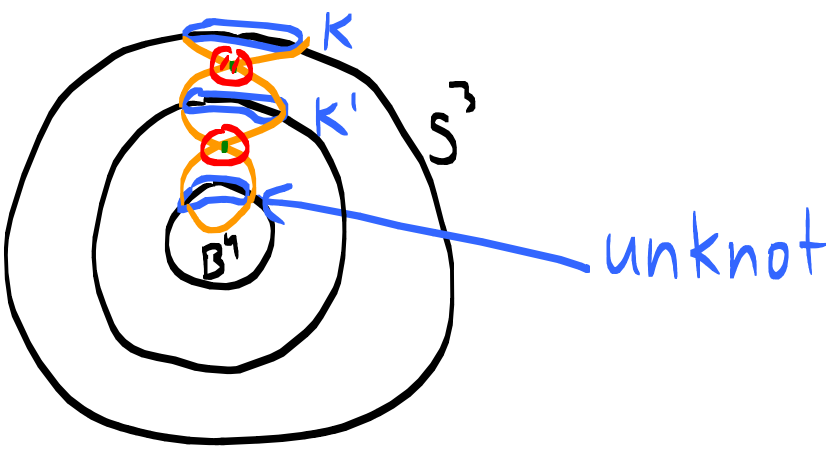


replace B^4 with $\mathbb{CP}^2 \setminus B^4$

Note: Whether you use $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ depends on if the crossing was a + or − crossing.

We have shown that K and K' cobound an embedded annulus in $(S^3 \times I) \# \pm \mathbb{C}P^2$.

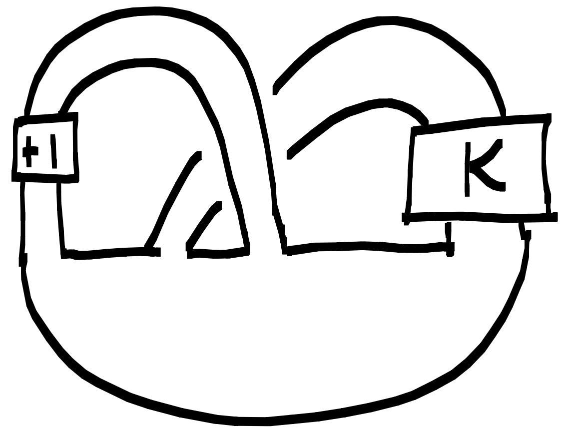
Doing this for every crossing change and capping of S^3 w/ a B^4, we see that K is slice in $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \cdots \# \overline{\mathbb{C}P}^2 \setminus B^4$.



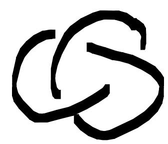
∎

<u>Remarks</u> ① If K bounds a disk Δ in $W = (\#_k \mathbb{C}P^2) \# (\#_\ell \overline{\mathbb{C}P}^2) \setminus B^4$ and Δ misses the copies of $\mathbb{C}P^1$ (the generators of $H_2(W)$) then one can blowdown the $\mathbb{C}P^1$'s (i.e. replace each $\pm\mathbb{C}P^2$ with B^4) to get a disk $\Delta \subset B^4$.

② NOT every K bounds a disk in a punctured $\#_k \mathbb{C}P^2$ (or $\# \overline{\mathbb{C}P^2}$).

Ex: $Wh^-(K) =$ 

Whitehead double of K

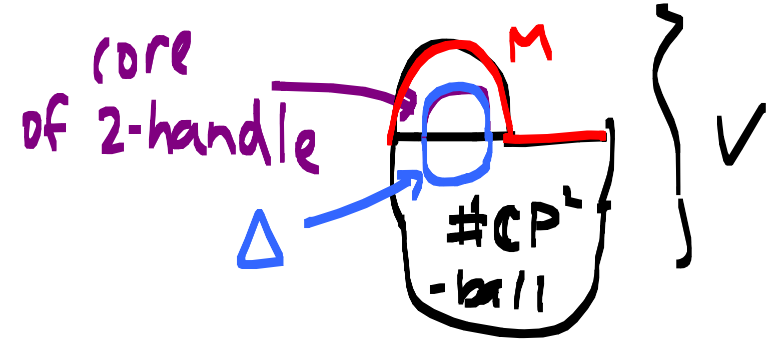
LHT = 

$\boxed{Wh^-(LHT) \text{ does not bound a disk in } \#_k \mathbb{CP}^2\text{-ball}}$

Lemma: If J bounds a disk in $\#\mathbb{CP}^2$-ball then (-1)-surgery on J bound a 4-mfd W with intersection form $Q_W = \begin{pmatrix} +1 & & \\ & \ddots & \\ & & +1 \end{pmatrix}$.

Pf: Let $M = (-1)$-surgery on J. Attach a (-1)-handle to B^4 along $J \rightsquigarrow V$ with $\partial V = M$.

The core of 2-handle $\cup$



Δ = disk in $\#\mathbb{CP}^2$-ball is a sphere with normal bundle of Euler class $-1 \rightsquigarrow$ a connected sum with $\overline{\mathbb{CP}}^2$. Blow down this $\mathbb{CP}^1$ to get W with $Q_W \cong \begin{pmatrix} +1 & & \\ & \ddots & \\ & & +1 \end{pmatrix}$.

$\square$

<u>Fact:</u> $M :=$ (-1)-surgery on $Wh^-(LHT)$ bounds a 4-mfld X with $Q_X \cong -E_8$. Gluing $\bar{X}$ to W (from Lemma) gives a closed smooth 4-mfld with non-diagonalizable pos. def int. form $E_8 \oplus (+1) \oplus \cdots \oplus (+1)$.

This contradicts Donaldson's theorem.

So $Wh^-(LHT)$ does not bound a disk in $\#_k \mathbb{C}P^2$-ball ($Wh^-(LHT) \in \Upsilon$).

We filter the condition of being slice in a punctured $\#_k \mathbb{C}P^2$.

Def: For a group G, $G^{(0)} = G$,

$$G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Def: A knot K is n-positive if $\exists$ a smooth 4-manifold W with $\partial W = S^3$ and a disk Δ smoothly embedded in W with $\partial \Delta = K$ s.t.

(0) Δ is trivial in $H_2(W, \partial W)$

(1) $\pi_1(W) = 0$

(2) $\exists$ disjointly embedded surfaces $S_1, \dots, S_j$ freely generating $H_2(W)$ with $S_i \cap S_i^+ = \{pt\}$, a positive intersection $\forall\, i$.

(3) $S_i \cap \Delta = \emptyset \quad \forall\, i$

(4) $\pi_1(S_i) \subset \pi_1(W \setminus \Delta)^{(n)} \quad \forall\, i$

Similarly for n-negative except

$S_i \cap S_i^+$ = negative intersection point

(Euler class of normal bundle is -1).

- $P_n = \{$n-positive knots$\} \subset \mathcal{C}$
- $N_n = \{$n-negative knots$\} \subset \mathcal{C}$.
- $B_n = P_n \cap N_n$ is a filtration by subgroups.

  Called the bipolar filtration of $\mathcal{C}$.

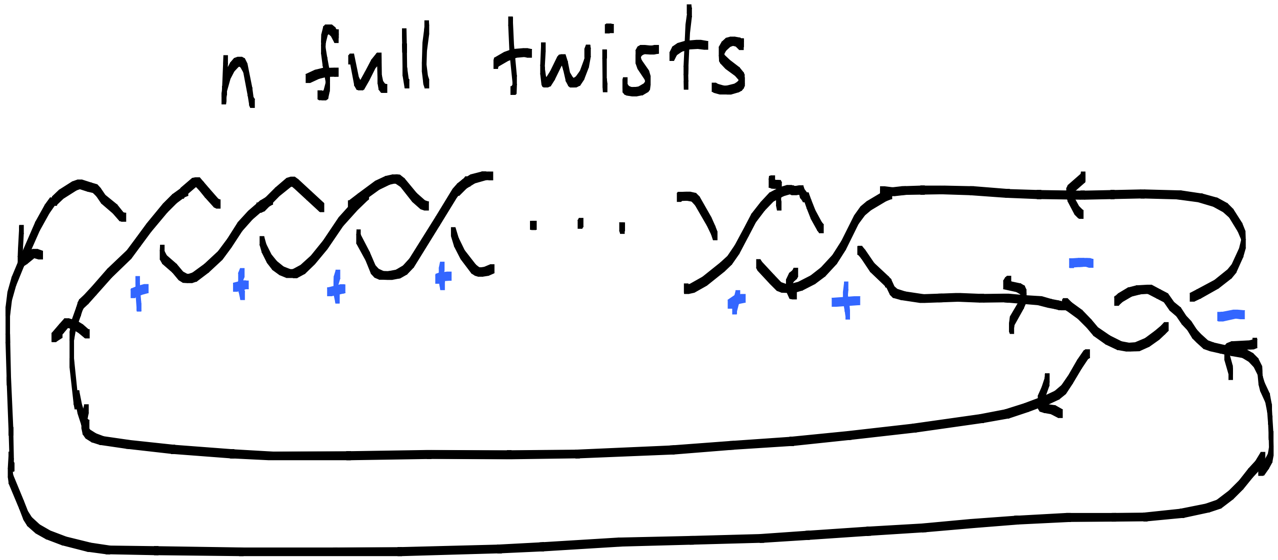
$$\cdots \subset \mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{B}_0 \subset \mathcal{C}$$

Prop (Cochran-H-Horn): $\mathcal{B}_n \subset \mathcal{F}_n^*$

Prop (Cochran-H-Horn): If K can be changed to the unknot by changing only positive crossings $\Rightarrow K \in P_0$.

Idea: K is slice in $\#_k \mathbb{C}P^2$. Can find representative of $H_2(\#_k \mathbb{C}P^2)$ in exterior of Δ.

Ex: Twist knots

$Tw_n =$ 

$Tw_n \in P_0$ since can change half the + crossings to a − crossing to unknot

$Tw_n \in N_0$ since can change one − crossing to + crossing to unknot

$\Rightarrow Tw_n \in \mathcal{B}_0$

Prop (CHH): If $K \in \mathcal{B}_0$ then

(1) $\tau(K) = 0$ (Ozsvath-Szabo)

(2) $s(K) = 0$ (Kronheimer-Mrowka)

(3) $\sigma_\omega(K) = 0$ if $\omega = p^k$-root of unity

$\Rightarrow K$ has finite order in the Algebraic concordance group.

(4) $d(\pm 1\text{-surgery on } K) = 0$

If $K \in \mathcal{B}_1 \Rightarrow K$ is algebraically slice

If $K \in \mathcal{B}_2 \Rightarrow$ Casson-Gordon invariants vanish.

<u>Theorem</u> (Cochran-H-Horn) Suppose $K \in \mathcal{P}_1$ and Y is the p^r-fold cyclic branched cover of S^3 branched over K. There is a subgroup $G < H^2(Y)$ with $|G|^2 = |H^2(Y)|$ and a spin^c-structure $\mathfrak{s}$ on Y s.t. the Ozsváth-Szabó correction terms

$$d(Y, \mathfrak{s} + g) \leq 0$$

$\forall \; g \in G$.

Note: • $\mathfrak{s}$ = spin^c structure on Y that comes from a spin structure on Y and

$G \longleftrightarrow$ poincare dual of classes in $\ker(H_1(Y) \xrightarrow{i_*} H_1(W))$.

• If $K \in N_1$, get $d(Y, \mathfrak{s}+h) \geq 0$ for $h \in H$, some a priori different subgroup of $H^2(Y)$.

Theorem (Cochran-H-Horn):

$$\bigoplus_{p(t)} \left(\mathbb{Z}^\infty \oplus \mathbb{Z}/2\right) \subset \mathcal{B}_n / \mathcal{B}_{n+1} \quad \forall\, n.$$

We are interested in

$\mathcal{T}$ = topologically slice knots $\subset \mathcal{C}$.

Define $\mathcal{T}_n = \mathcal{T} \cap NP_n$.

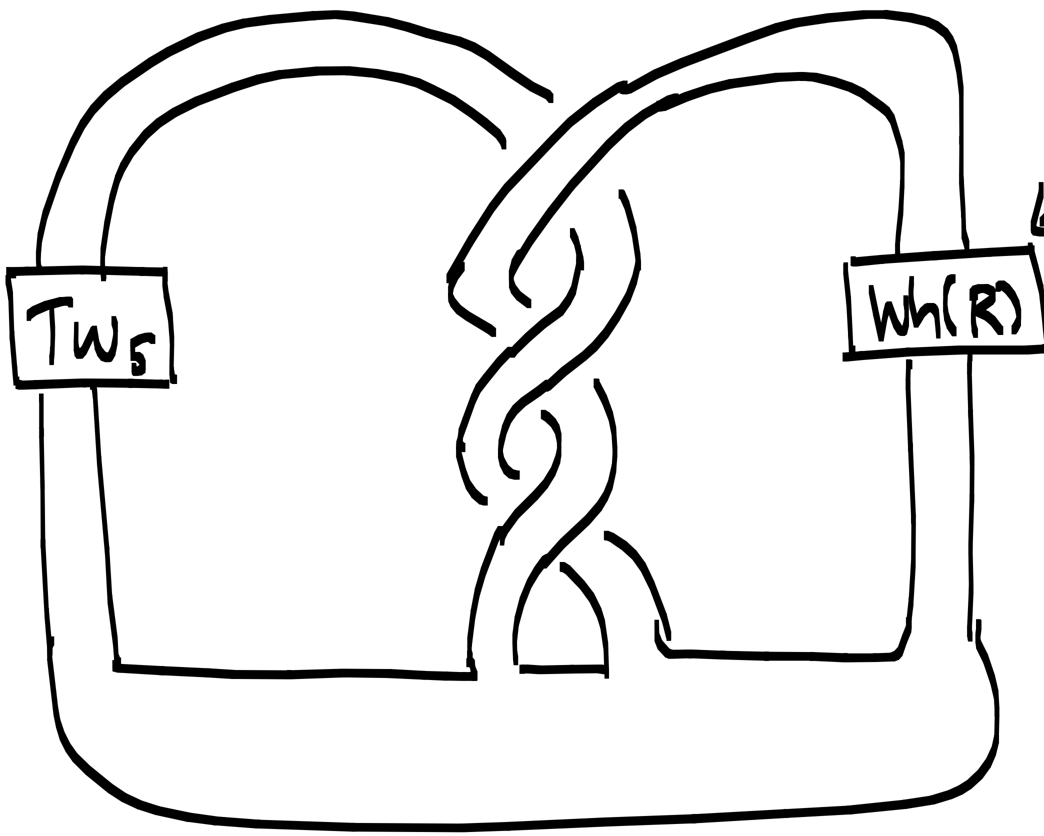
Theorem (Cochran-H-Horn):

$$\boxed{\mathbb{Z} \subset T_1/T_2.}$$

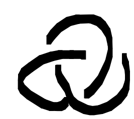
In particular $T_1/T_2 \neq 0$.

Ex:

twist knot w/ 5 twists →



untwisted Whitehead double of



Proof uses d-invariants from Heegaard-Floer homology and Casson-Gordon signature invariants.