Knot Contact homology and unknot detection

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Low-dimensional topology, knot theory $\xrightarrow{\text{"symplectify"}}$ Symplectic/contact geometry

Today: Cotangent/conormal construction

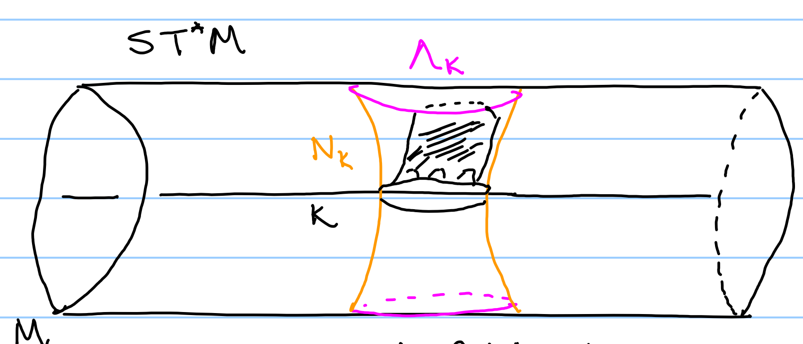
$\rightsquigarrow$ knot contact homology, invt. of topological knots

M smooth $\rightsquigarrow$ T^*M cotangent bundle, naturally symplectic, ST^*M cosphere bundle, naturally contact

$K \subset M$ submanifold $\rightsquigarrow$ $N_K \subset T^*M$ Lagrangian submanifold, conormal bundle

$N_K = \{(x,\xi) \in T^*M \mid x \in K, \langle \xi, v \rangle = 0, \forall v \in T_x K\}$

, $\Lambda_K = N_K \cap ST^*M$ Legendrian submanifold



$N_K \cap M = K$

$M = \mathbb{R}^3$, $ST^*M = J^1(S^2)$, K = knot in $\mathbb{R}^3$

We can associate a Legendrian invariant to $\Lambda_K \subset ST^*M$.

Legendrian contact homology (LCH): counts holomorphic disks in ST^*M with boundary on Λ_K.

Definition: The knot contact homology $HC_*(K)$ is the LCH of $\Lambda_K \subset ST^*M$ and knot invt. of K

- $HC_*(K) = H_*(\mathcal{A}, \partial)$, $(\mathcal{A}, \partial)$ = differentially graded algebra over $\hat{R} = \mathbb{Z}[\lambda^{\mp 1}, \mu^{\mp 1}]$
  $\mathcal{A}$ = generated by Reeb chords of Λ_K, supported in deg ≥ 0
- $\exists$ combinatorial formulation for $(\mathcal{A}, \partial)$ [Ng - 2004, Ekholm-Etnyre-Ng-Sullivan-2011]
- $HC_*(K)$ encodes the Alexander polynomial
- $HC_*(K)$ detects the unknot (Ng - 2004) uses work of Dunfield-Garoufalidis
  i.e. if $HC_*(K) \cong HC_*(\text{unknot})$ then K = unknot
- gives an effective invariant of transverse knots

Broken closed strings:

$M = \mathbb{R}^3$, $K \subset M$ knot, $N = N_K =$ tubular nhd. of K

$\rightsquigarrow M \cup_K N$, fix $* \in N$

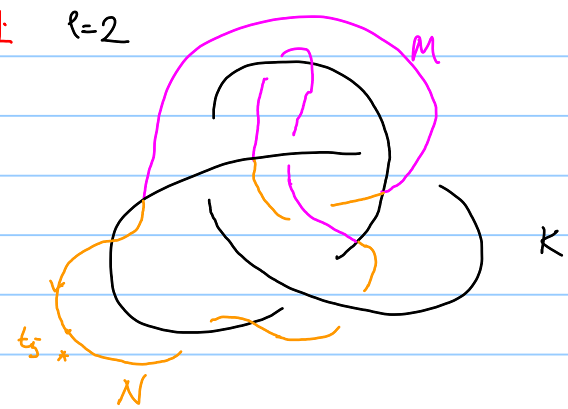
Definition: A broken closed string of length $\ell \in \mathbb{Z}^{\geq 0}$ is a C^1-map $\gamma : [0,1] \longrightarrow M \cup_K N$ s.t. $\exists$

$$0 = t_0 < t_1 < \cdots < t_{2\ell+1} = 1$$

for which

- $\gamma(0) = \gamma(1) = *$, $\gamma(t_i) \in K$, $\forall i \neq 0, 2\ell+1$
- $\gamma|_{[t_{2i}, t_{2i+1}]} \subset N$, $\gamma|_{[t_{2i-1}, t_{2i}]} \subset M$
- $\dot{\gamma}(t_i) = \dot{\gamma}(t_i^*)$ $\forall i$ (may be $\pitchfork$ to K)

E.g. $\ell = 2$



Define $\Sigma_\ell = \{\text{b.c.s of length } \ell\}_{\ell \geq 0}$

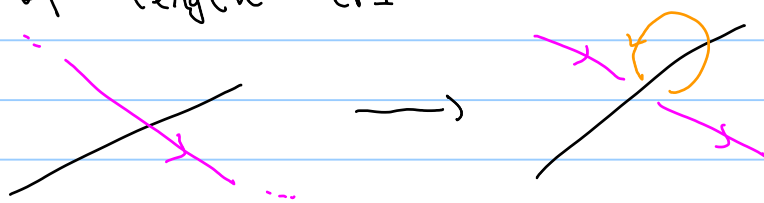
$C_n(\Sigma_\ell) = \{n\text{-chains in } \Sigma_\ell\}$

$C_n = \bigoplus_\ell C_n(\Sigma_\ell)$

Three operations:

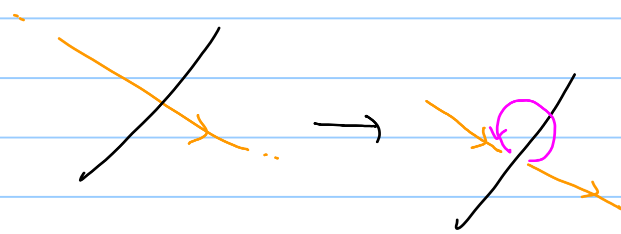
1. $\partial: C_n(\Sigma_\ell) \longrightarrow C_{n-1}(\Sigma_\ell) \longrightarrow \partial: C_n \longrightarrow C_{n-1}$

2. If a b.c.s of length ℓ has an M-string passing through K in its interior, we can define a new b.c.s of length $\ell+1$



$\leadsto$ "$\delta_M: C_n(\Sigma_\ell) \longrightarrow C_{n-1}(\Sigma_{\ell+1})$"

3. same but with M and N switched



$\leadsto$ "$\delta_N: C_n \longrightarrow C_{n-1}$"

"Thm":

$d := \partial + \delta_M + \delta_N : C_n \longrightarrow C_{n-1}$ satisfies $d^2 = 0$

$\leadsto$ define string homology

$HS_*(K) = H_*(C_*, d)$

(Transversality problem)

"Thm": $HS_*(K) \cong HC_*(K)$

Idea: Count holomorphic curves in DT^*M with puncture at a Reeb chord in Λ_K and "mixed" boundary on $N_K \cup M$ (see this on the picture on the first page)

This gives a chain map $(\mathcal{A}, \partial) \longrightarrow (C_*, \partial)$

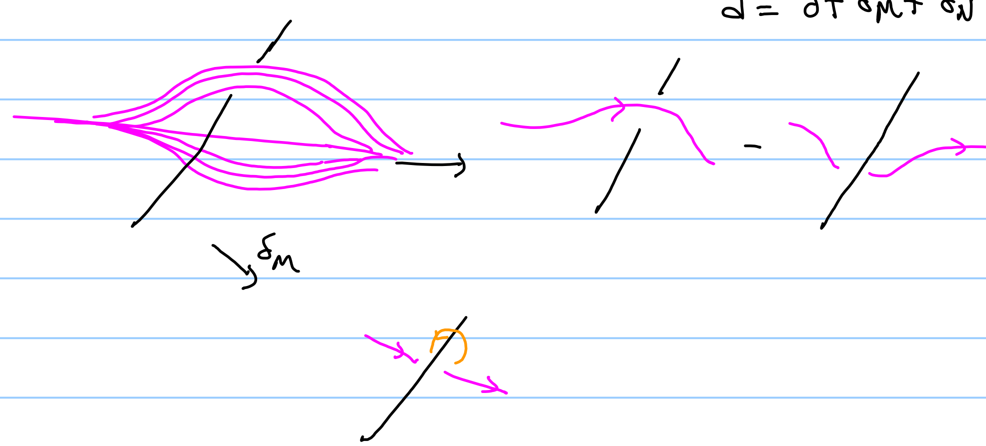
Can make this rigorous for $*=0$:

What does $HS_0(K)$ look like?

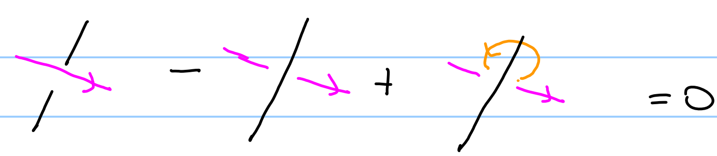
$$C_0 / (\operatorname{Im} d : C_1 \to C_0)$$

$$C_1 \xrightarrow{d} C_0$$

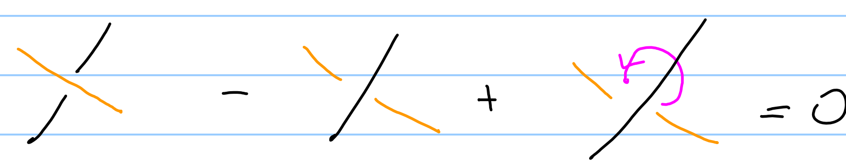
$$d = \partial + \delta_M + \delta_N$$



So in $HS_0(K)$



Similarly



Definition: A generic b.c.s is a b.c.s whose M-strings, N-strings don't pass through K in their interior.

$$\rightsquigarrow HS_0(K) \cong \mathbb{Z}\langle \text{htpy classes of generic b.c.s} \rangle / \text{two "skein relations"}$$

"Thm": $HS_*(K) = HC_*(K)$

Rigorously true for $* = 0$

(interpret $HC_0(K)$ in terms of "chord algebra")

$\exists$ homotopical interpretation for $HS_0(K)$ in terms of

$\pi := \pi_1(\mathbb{R}^3 - K)$ knot group

$\cup$

$\hat{\pi} := \pi_1(\partial N(K)) = \pi_1(T^2)$ peripheral subgroup

$R := \mathbb{Z}\pi$

$\lambda, \mu \in \hat{\pi} \subset \pi$

Proposition: $HS_0(K) = \mathbb{Z}[\lambda^{\pm 1}] \oplus R$

$(HC_0(K))$ as $\mathbb{Z}[\lambda^{\pm 1}]$-bimodules

For $K =$ unknot $\mathcal{U}$ $\quad HS_0(\mathcal{U}) \cong \underbrace{\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]}_{\hat{R}} / (\lambda-1)(\mu-1)$

Proposition: The map $(\lambda - 1)\cdot : HS_0(K) \longrightarrow HS_0(K)$

is injective $\Longleftrightarrow K \neq \mathcal{U}$

Follows directly from $\hat{\pi} \hookrightarrow \pi$ if $K \neq \mathcal{U}$ (Loop theorem)

Corollary: $HC_0(K)$ detects the unknot.

Question: Is knot contact homology is a complete knot invariant?

Reason to ask this! ☺

Proposition (Ng-2004):

If $K \neq \mathcal{U}$, then $HS_0(K) \cong$ subring of $\mathbb{Z}\pi$ generated by $\hat{\pi}(\lambda^{\pm 1}, \mu^{\pm 1})$ and $\mathrm{im}((1-\mu)\cdot : \mathbb{Z}\pi \to \mathbb{Z}\pi)$

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