The Geometry of Character Varieties

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The character variety, $X(M)$, of a finite volume hyperbolic 3-manifold M acts as a moduli space of hyperbolic structures of M. I will discuss how the geometry of this algebraic set mirrors the topology of M and the algebraic structure of $\pi_1(M)$.

Organization:

1. Definition of $X(M)$
2. $X(S^3 - \text{(figure 8)})$ Example
3. Introduce some basic structural questions
4. Discuss some answers to these questions

Parts of this work are joint with Ken Baker, Melissa Macasieb, and Ronald van Luijk.

Let Γ be a finitely presented group (with n generators and k relations). The $\mathrm{SL}_2(\mathbb{C})$ representation variety of Γ is

$$R(\Gamma) = \{\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})\}.$$

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We define $X(\Gamma)$, the $\mathrm{SL}_2(\mathbb{C})$ character variety of Γ to be

$$\{\chi_\rho : \rho \in R(\Gamma)\}$$

where the character $\chi_\rho : \Gamma \to \mathbb{C}$ is the map defined by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$ for all $\gamma \in \Gamma$.

$X(\Gamma)$ is also an affine algebraic set, and can be defined by a finite number of characters.

A representation $\rho(\Gamma)$ is called reducible if up to conjugation $\rho(\Gamma)$ is upper triangular .

The set of reducible representations $X_{red}(\Gamma)$ is itself an algebraic set. (The upper triangular condition can be expressed algebraically by specifying that the $(2,1)$ entries are zero.)

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Essentially, the $SL_2(\mathbb{C})$ character variety of Γ is $R(\Gamma)$ modulo conjugation. (More collapsing can occur for reducible representations.)

We let $X_{irr}(\Gamma)$ be the Zariski closure of $X(\Gamma) - X_{red}(\Gamma)$. (Every component of $X_{irr}(\Gamma)$ contains the characters of some irreducible representation.)

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$$\Gamma = \langle \gamma_1, \ldots, \gamma_N : r_1, \ldots r_k \rangle$$

are linearly determined by traces of elements of the form $\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_n}$ where $i_j < i_{j+1}$.
The finite set of variables $\chi_\rho(\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_n})$ suffice to determine $X(\Gamma)$.

If Γ is generated by only two elements, γ_1 and γ_2, the variables $\chi_\rho(\gamma_1)$, $\chi_\rho(\gamma_2)$ and $\chi_\rho(\gamma_1\gamma_2^{\pm 1})$ suffice to determine $X(\Gamma)$.

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We call an irreducible component of $X(M)$ a canonical component if it contains the character of a discrete faithful representation, and write $X_0(M)$ for such a component.

Figure-8 knot complement example

$$\pi_1(S^3 - 4_1) \cong \Gamma_8 = \langle \alpha, \beta : w\alpha = \beta w, w = \alpha^{-1}\beta\alpha\beta^{-1} \rangle$$

$X(\Gamma_8)$ is determined by $\chi_\rho(\alpha)$, $\chi_\rho(\beta)$ and $\chi_\rho(\alpha\beta^{-1})$. Since α and β are conjugate $\chi_\rho(\alpha) = \chi_\rho(\beta)$.

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A reducible representation is conjugate to

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The matrix relation is $WA = BW$ where $W = A^{-1}BAB^{-1}$, and reduces to

$$WA - BW = \begin{pmatrix} 0 & (t-s)(a^2 + a^{-2} - 3) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $(t - s)(x^2 - 5) = 0$ so $s = t$ or $x^2 = 5$.

$$X_{red}(\Gamma_8) \text{ is given by } \{(x, y) : y = 2\} \cong \mathbb{A}^1.$$

Figure-8 Irreducible representations

Up to conjugation an irreducible representation is

$$\rho(\alpha) = A = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix} \qquad \rho(\beta) = B = \begin{pmatrix} a & 0 \\ 2-r & a^{-1} \end{pmatrix}$$

with $x = a + a^{-1}$, $y = r$. The relation collapses to

$$AW - WB = \begin{pmatrix} 0 & \star \\ (r-2)\star & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\star = (a^2 + a^{-2})(1-r) + 1 - r + r^2 = (x^2 - 2)(1-r) + 1 - r + r^2$.

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Writing this as

$$(x(r-1))^2 = (r^2 + r - 1)(r - 1)$$

we let $z = x(r-1)$ and have the Weierstrass equation of the elliptic curve

$$z^2 = r^3 - 2r + 1.$$

Some Structural Questions About $X(M)$

Question 1) How many irreducible components does $X(M)$ have?

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• Thurston's hyperbolic Dehn surgery theorem implies that if M is a finite volume hyperbolic 3-manifold

$$\dim_{\mathbb{C}} X_0(M) = \#(\text{cusps of } M).$$

• Culler and Shalen have shown that if M is small (does not contain a closed essential surface) then any component of $X_{irr}(M)$ must have $\mathbb{C}$-dimension at most 1.

• If Γ is generated by N elements, then the dimension of any component of Γ is bounded by $\binom{N}{1} + \binom{N}{2} + \dots \binom{N}{N}$. If Γ is generated by two elements g_1 and g_2 then the dimension is bounded by 3, as the necessary variables are $\chi_\rho(g_1), \chi_\rho(g_2)$ and $\chi_\rho(g_1 g_2)$.

• Long and Reid proved that if M_1 and M_2 are one-cusped commensurable hyperbolic 3-manifolds that cover a common orbifold with a flexible cusp then $Y_0(M_1)$ is birational to $Y_0(M_2)$.

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• Let F_N be the free group on N generators $g_1, \ldots, g_N$. Let $G_n = \rho(g_n)$. Up to conjugation we can take

$$G_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix}, G_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2^{-1} \end{pmatrix}, G_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

with $a_n d_n - b_n c_n = 1$ for $n = 3, \ldots, N$ and $b_2 c_2 = 0$ and $b_1 = 0$ or 1.
Therefore, for $n > 2$ each matrix has 3 degrees of freedom, and G_1 and G_2 have a total of 3 degrees of freedom. So $\dim_{\mathbb{C}}(F_N) \geq 3(N-2) + 3 = 3N - 3$.

Consider $H \triangleleft G$.
The inclusion $\varphi : H \hookrightarrow G$ then φ induces a map

$$\varphi^* : R(G) \to R(H)$$

by $\varphi^*(\rho) = \rho \circ \varphi$. We let φ^* also denote $\varphi^* : X(G) \to X(H)$ defined by

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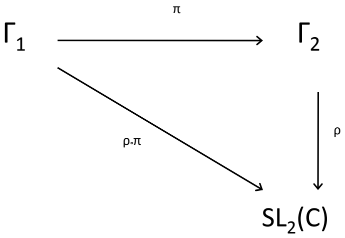
Boyer, Luft, and Zhang use this to show the existence of families of manifolds $\{M_n\}$ where the number of components of $X_{irr}(M_n)$ is arbitrarily large.

One such family is the finite cyclic covers of the figure-8 knot complement.

If $\Gamma_1 \twoheadrightarrow \Gamma_2$ is any representation $\rho : \Gamma_2 \longrightarrow \mathrm{SL}_2(\mathbb{C})$ induces $\pi \circ \rho : \Gamma_1 \longrightarrow \mathrm{SL}_2(\mathbb{C})$.



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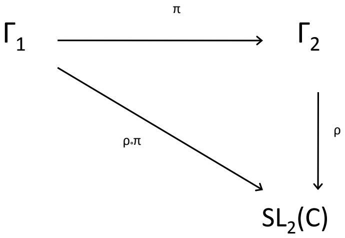

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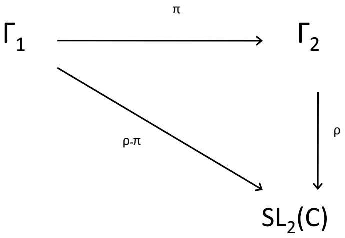

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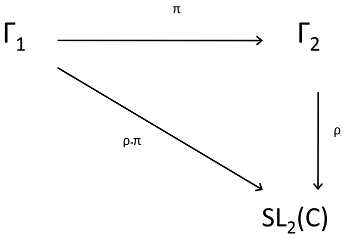

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- If $\dim_{\mathbb{C}} X(\Gamma_1) = \dim_{\mathbb{C}} X(\Gamma_2) = 1$ then

$$X_0(\Gamma_1) \cong X_0(\Gamma_2) \qquad \text{or} \qquad \#(\text{components of } X(\Gamma_2)) < \#(\text{components of } X(\Gamma_1))$$

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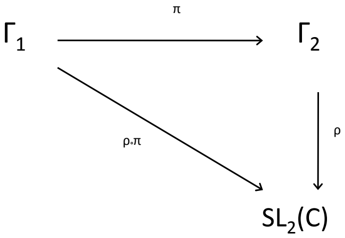

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- Ohtsuki Riley and Sakuma use this idea to show that for all N there is a two-bridge knot K_N so that

$$\#(\text{components of } X(S^3 - K_N)) > N$$

They find towers of surjections of two-bridge knots

$$\pi_1(S^3 - K_N) \twoheadrightarrow \pi_1(S^3 - K_{N-1}) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(S^3 - K_1).$$

Genera of components

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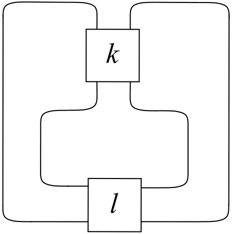

Theorem (Macasieb, -P, van Luijk)

Let $X(k,l)$ be the $SL_2(\mathbb{C})$ character variety of $S^3 - J(k,l)$. If $k \neq l$ then with $m = \lfloor \frac{|k|}{2} \rfloor$ and $n = \lfloor \frac{|l|}{2} \rfloor$

$$genus\Big(X_0(k,l)\Big) = 3mn - m - n - b$$

where $b \in \{0, \pm 1\}$.

Corollary

For all $N > 0$ there is a twist knot K such that $genus(X_0(S^3 - K)) > N$.

Symmetries

If $\phi : M \to M$ is a symmetry, then ϕ induces an automorphism ϕ_* on $\pi_1(M)$, for loops α and β

$$\phi_*([\alpha]) = [\phi(\alpha)].$$

This, in turn, induces an automorphism ϕ^* on $X(\Gamma)$, so

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Consider $X(\Gamma) \subset \mathbb{A}^N$.

If ϕ acts trivially on the set of free homotopy classes of un-oriented loops, the action is trivial on $\mathbb{A}^N$, and therefore the action is trivial on $X(\Gamma)$. (The conjugacy classes of elements are fixed up to inverses, and conjugate matrices and inverses have same trace.)

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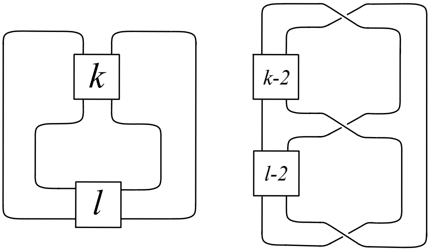
The set $S(\Gamma)$ of fixed points of ϕ^* is an algebraic subset of $X(\Gamma)$.

If M has only one cusp and ϕ fixes a framing of ∂M, then ϕ^* fixes the Dehn surgery characters so $X_0(M) \subset S(M)$.

For most two bridge knot complements M, the symmetry group of M is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Those with palindromic twist regions have extra symmetries.



Riley and Ohtsuki studied the effect of this extra symmetry on varieties of representations, and characters.

By studying the action at infinity of $X(M)$, Ohtsuki showed that $X(M)$ factors when M has this additional symmetry.

An algebraic view of the action of a symmetry is as follows. If $\phi : M \to M$ then

$$\pi_1(M) \twoheadrightarrow \pi_1^{orb}(\mathcal{O})$$

where $\mathcal{O} = M/\phi$ is the orbifold quotient, and $\pi_1^{orb}(\mathcal{O})$ is the orbifold fundamental group.

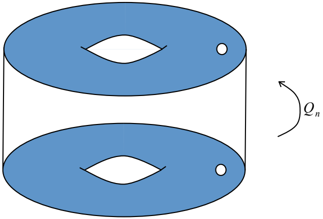
$$X(\mathcal{O}) \subset X(M)$$

To ensure that $X_{irr}(M)$ factors, it suffices to find an irreducible character not contained in $X(\mathcal{O})$. (You need to be a bit careful here about dimension.)

For two-bridge knots, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements act trivially on unoriented free homotopy classes.

The additional symmetry always factors $X_{irr}(M)$.

One component of $X_{irr}(M)$ consists of those characters of representations that factor through $\pi_1(\mathcal{O})$ and the other component is those characters that do not.



Let M_n be the once-punctured torus bundle with tunnel number one, $(T \times I)/Q_n$.

Theorem (Baker, -P)

Let g_n be the genus of $X_0(M_n)$ and d_n be the dilatation of the pseudo-Anosov map corresponding to M_n. Then

$$\lfloor d_n \rfloor = 2g_n + \alpha$$

where $\alpha \in \{-3, -2, -1, 0, 1\}$ is explicitly determined.

This is a consequence of a precise computation of the character varieties, and showing that each component is smooth and irreducible.

Once-punctured torus bundles of tunnel number one

Up to homeomorphism, the monodromy of $M_n = (T \times I)/Q_n$ is $Q_n = \tau_c \tau_b^{n+2}$ where b and c are curves forming a basis for the fiber T and τ_a means a right handed Dehn twist about the curve a.

M_n is hyperbolic if $|n| > 2$. We will consider only these n.

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The fundamental group is isomorphic to

$$\Gamma_n = \langle \alpha, \beta : \beta^{-n} = \alpha^{-1}\beta\alpha^2\beta\alpha^{-1} \rangle.$$

The abelianization is $\Gamma_n^{ab} \cong \mathbb{Z} \times \mathbb{Z}_{n+2}$. Therefore $X(\mathbb{Z} \times \mathbb{Z}_{n+2}) \subset X(\Gamma_n)$.

$$X(\mathbb{Z} \times \mathbb{Z}_{n+2}) \cong \{(x, y, z) : x^2 + y^2 + z^2 - xyz = 4, y \in 2\mathrm{Re}(\zeta_{n+2})\}.$$

In fact, $X_{red}(\Gamma_n)$ exactly corresponds to these conics.

Irreducible Representations

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$$A = \rho(\alpha) = \begin{pmatrix} a & 0 \\ t & a^{-1} \end{pmatrix}, \quad B = \rho(\beta) = \begin{pmatrix} b & s \\ 0 & b^{-1} \end{pmatrix}$$

and the variables

$$x = \chi_\rho(\alpha) = a + a^{-1}, \; y = \chi_\rho(\beta) = b + b^{-1}, \; z = \chi_\rho(\alpha\beta) = ab + a^{-1}b^{-1} + st.$$

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$$A = \rho(\alpha) = \begin{pmatrix} a & 0 \\ t & a^{-1} \end{pmatrix}, \quad B = \rho(\beta) = \begin{pmatrix} b & s \\ 0 & b^{-1} \end{pmatrix}$$

and the variables

$$x = \chi_\rho(\alpha) = a + a^{-1}, \; y = \chi_\rho(\beta) = b + b^{-1}, \; z = \chi_\rho(\alpha\beta) = ab + a^{-1}b^{-1} + st.$$

The relation is $B^{-n} = A^{-1}BA^2BA^{-1}$. We use the Cayley-Hamilton theorem so that

$$B^{-n} = \begin{pmatrix} b^{-n} & -sf_n(y) \\ 0 & b^n \end{pmatrix}$$

where f_n is the n^{th} Fibonacci polynomial;

$$f_0(y) = 0, f_1(y) = 1, \; f_{k+1}(y) + f_{k-1}(y) = yf_k(y).$$

The (irreducible) representations are determined by the vanishing set of

$$\begin{aligned}\varphi_1 &= x^2 - 1 + f_{n-1}(y)\\ \varphi_2 &= zx - y + f_n(y)\\ \varphi_3 &= x\big(f_{n+1}(y) - 1\big) - zf_n(y).\end{aligned}$$

(The polynomial φ_3 factors as $f_{n+1}(y) - 1$ and $f_n(y)$ share a factor.)

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The character variety can be 'blown up' to be a union of lines (from the blow ups) and the zero set of

$$\psi_1 = x^2 - 1 + f_{n-1}(y), \quad \psi_2 = ux + h_n(y), \quad \psi_3 = x - u\ell_n(y)$$

where $h_n\ell_n = f_{n-1} - 1$.

With $x = u\ell_n(y)$ this is isomorphic to the vanishing set of $u^2\ell_n(y) + h_n(y) = 0$.

The equation

$$u^2 \ell_n(y) = -h_n(y)$$

is irreducible except when $n \equiv 2 \pmod 4$. Here y divides both ℓ_n and h_n.

This extra component corresponds to the characters $x = y = 0$ and representations with A and B as above with $a^2 = b^2 = -1$ in the original model.

These are faithful representations of the group

$$\langle \alpha, \beta : \alpha^2 = \beta^2, \alpha^4 = 1 \rangle$$

The Canonical Component

Writing h_n^* and ℓ_n^* as h_n and ℓ_n except for these factors in the $n \equiv 2 \pmod 4$ case, the curve is

$$z^2 \ell_n(y)^* = -h_n(y)^*$$

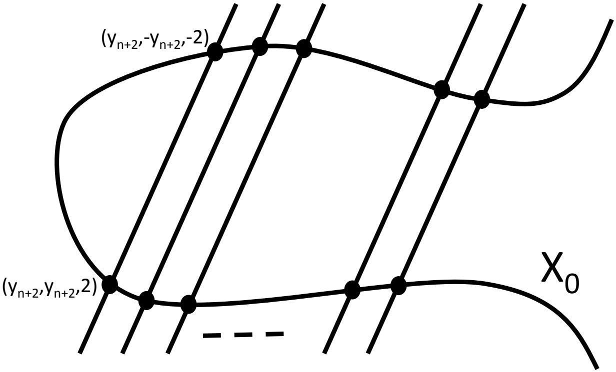
and is birational to

$$w^2 = -h_n(y)^* \ell_n(y)^* = (1 - f_{n-1}(y))^*.$$

The degree of f_{n-1} is $|n-1| - 1$.

This is $X_0(\Gamma_n)$ and is a hyperelliptic curve of genus $\lfloor \frac{1}{2}(|n-1|-1) \rfloor$ if $n \neq 2 \pmod 2$ and $\lfloor \frac{1}{2}(|n-1|-3) \rfloor$ otherwise.

The once-punctured torus bundles with tunnel number one have lens space fillings corresponding to a surjection to $\mathbb{Z}/(n+2)\mathbb{Z}$. Let y_{n+2} be an element of $2\mathrm{Re}(\zeta_{n+2})$.



The $(y_{n+2}, y_{n+2}, 2)$ points correspond to a lens space filling.
M_3 has an additional filling, corresponding to a surjection onto $\mathbb{Z}_{10}$ which corresponds to the $(y_{n+2}, -y_{n+2}, -2)$ points.
The characters corresponding additional fillings of M_3 and M_5 are the same as those above.