Legendrian and transverse knots in cabled knot types

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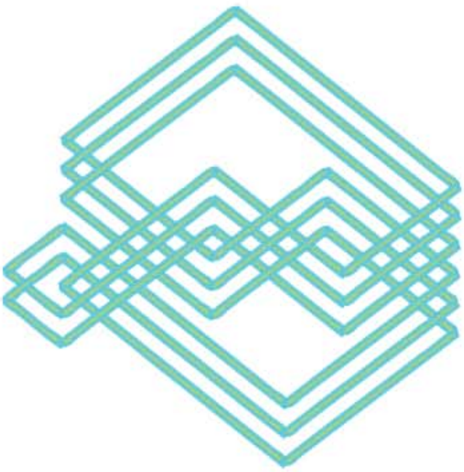
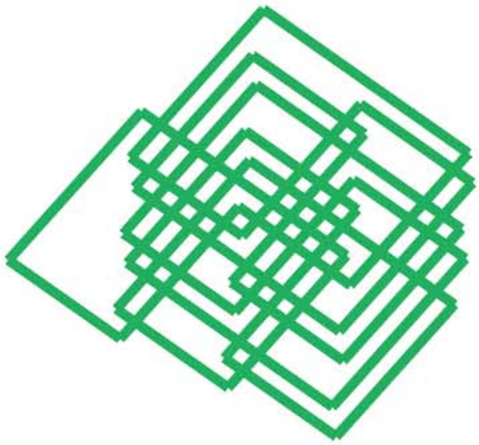
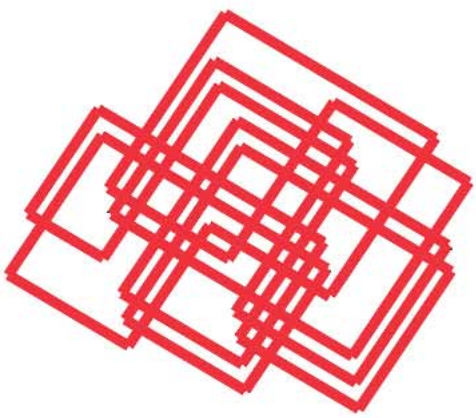






Main problems

(A) Classify Legendrian knots of some particular topological type

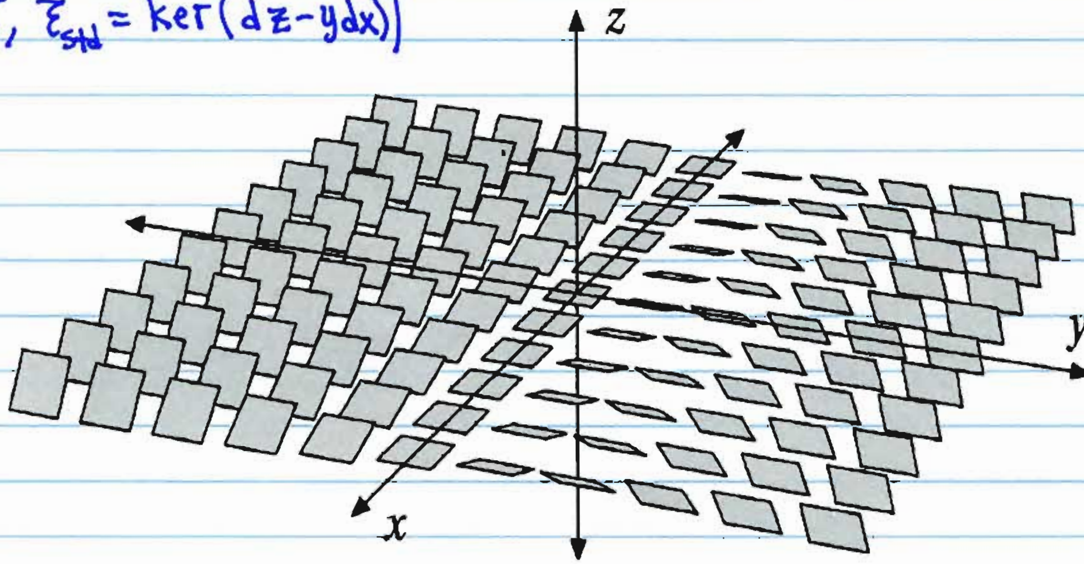
(B) Classify solid tori representing some particular knot type

(C) See what (A) and (B) have to say about classification of contact structures and fillability.

M cooriented contact 3-manifold with contact structure

$$\boxed{\xi = \ker\alpha}$$

E.g. $(\mathbb{R}^3, \xi_{std} = \ker(dz - y\,dx))$



Thm (Darboux): All contact structures are locally diffeomorphic to ξ_{std}

Defn: A knot $L \subset M$ is Legendrian if $T_xL \subset \xi_x$ for all $x \in L$. Two Legendrian knots are Legendrian isotopic if they are isotopic through Legendrian knots.

Legendrian classification problem:

Classify Legendrian knots upto Legendrian isotopy

Contact geometry

- Important in understanding symplectic geometry

  E.g.

  - topological characterization of Stein 4-mflds
  - Understanding of (Broken) Lefchetz fibrations

- Important in low-dimensional topology

  E.g. Ozsváth-Szabó proved that

  If L is ,  or .

  Then $S^3_r(K) \cong S^3_r(L)$

  implies $K = L$

Legendrian Knots

- Interesting in their own right

- Important in classifying contact structures

  E.g. • Eliashberg-Bennequin proved that
  Any one knot can determine if a contact structure ξ on M is tight or overtwisted

  • Kanda proved that
  There are 3 knots in T^3 that classify tight contact structures on T^3 (there are infinitely many)

- Important in constructing tight contact strs

E.g. Lisca-Stipsicz proved that

A Seifert fibered manifold M has a tight contact str.

$$\Longleftrightarrow$$

M is not $S^3_{2n-1}(T_{2,2n+1})$; $n \in \mathbb{N}$

Outline

- Definitions and background results

- Cabling and main results

- Convex surface theory and solid tori

- Idea of the proofs

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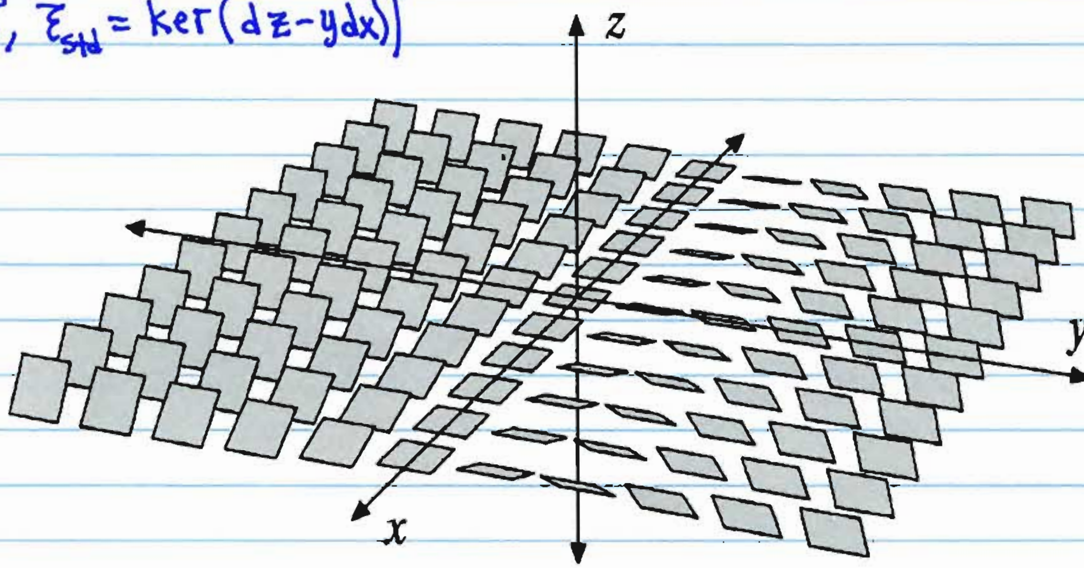

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Legendrian classification problem:

Classify Legendrian knots upto Legendrian isotopy

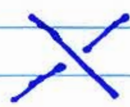
Front Projection

Let $L \subset (\mathbb{R}^3, \xi_{std} = \ker(dz - y\,dx))$ be a Legendrian

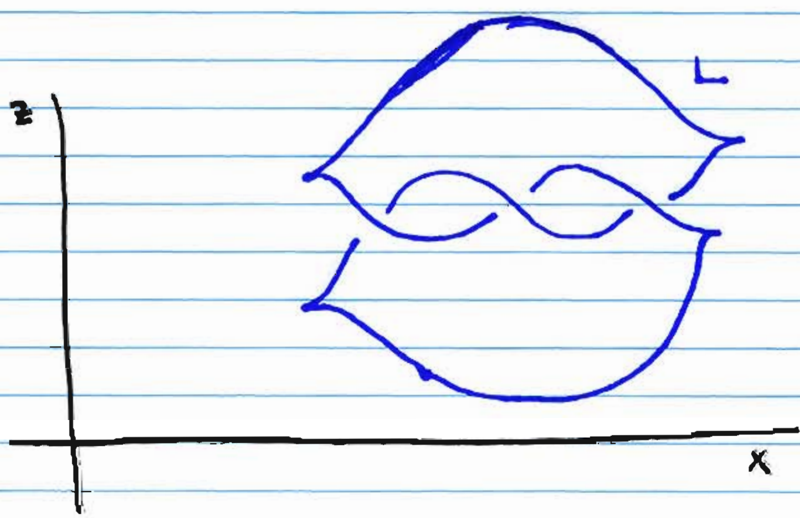
$$\phi_L : S^1 \hookrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (x(t), y(t), z(t)) \longmapsto (x(t), z(t))$$

$$\boxed{\phi_L'(t) \in \xi_{\phi_L(t)} \iff z'(t) - y(t)\,x'(t) = 0}$$

- No vertical tangencies
- At each crossing you see



E.g.



Classical Invariants:

Fix a topological knot type $\mathcal{K}$

$\mathcal{L}(\mathcal{K}) :=$ The set of all $L \in \mathcal{K}$ which are Legendrian w.r.to ξ_{std}

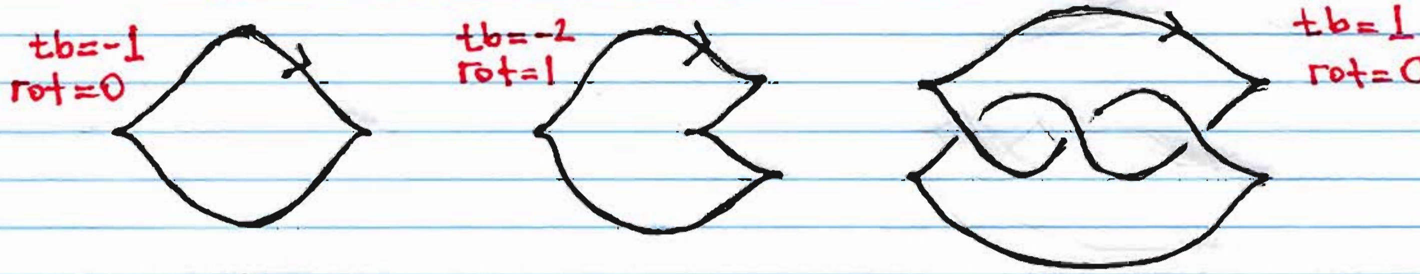
$L \in \mathcal{L}(\mathcal{K})$ is oriented

Thurston–Bennequin number

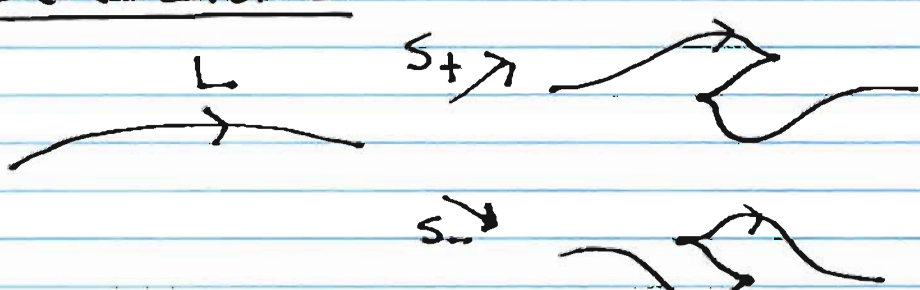
$$\boxed{tb(L) = w(L) - \frac{1}{2}(\#\text{ of cusps})}$$

Rotation number

$$\boxed{rot(L) = \frac{1}{2}(\downarrow \text{cusps} - \uparrow \text{cusps})}$$



Stabilization:



$$\boxed{\begin{array}{l} tb(S_\pm(L)) = tb(L) - 1 \\ rot(S_\pm(L)) = rot(L) \pm 1 \end{array}}$$

Transverse knots

$\mathcal{T}(\mathcal{K}) :=$ The set of all $T \in \mathcal{K}$ which are transverse to ξ_{std}

$T \in \mathcal{T}(\mathcal{K})$ is oriented and $\partial \Sigma_T = T$

$$\xi_{std}\big|_{\Sigma_T} = \mathbb{R}^2 \oplus \Sigma_T$$

- The self-linking number

$$\boxed{sl(T) = lk(T, T')}$$

- The (positive) transverse push-off of a Legendrian L

$$\boxed{sl(T_+(L)) = tb(L) - rot(L)}$$

- $\{\text{transverse knots}\} \xleftrightarrow{1\text{-}1} \{\text{Legendrian knots}\}\big/ \substack{\text{negative} \\ \text{Legendrian stab.}}$

☐ Questions:

For a fix knot type $\mathcal{K}$ we have

$$f_{\mathcal{K}} : \mathcal{L}(\mathcal{K}) \longrightarrow \mathbb{Z} \times \mathbb{Z}$$
$$L \longmapsto (rot(L), tb(L))$$

$\boxed{1.}$ Is $f_{\mathcal{K}}$ injective?

If so, then $\mathcal{K}$ is called Legendrian simple

$\boxed{2.}$ Is $f_{\mathcal{K}}$ onto?

$\boxed{3.}$ What is $\mathrm{Im} f_{\mathcal{K}}$?

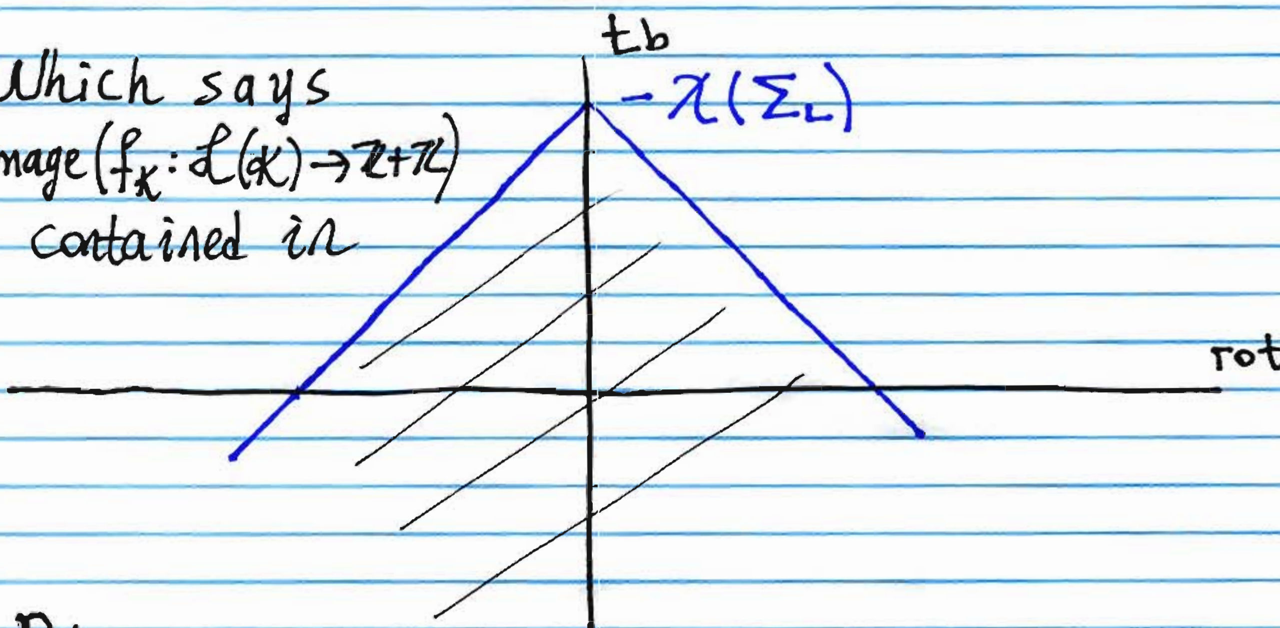
Eliashberg – Bennequin Inequality

Thm: If (M,ξ) is tight

Then

$$\boxed{tb(L) + |rot(L)| \leq -\chi(\Sigma_L)}$$

Which says $\text{Image}(f_K : \mathcal{L}(K) \to \mathbb{Z}+\mathbb{Z})$ contained in



Rks:

- Not sharp e.g. Negative torus knots

Hence define

$$\overline{tb}(K) = \max\{tb(L) \mid L \in \mathcal{L}(K)\}$$

- $tb + |rot|$ is odd

- Not true in overtwisted world

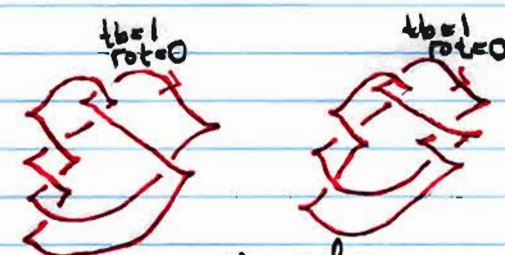
Classification Results

- (Eliashberg–Fraser, '95)

Unknot is Legendrian/transversely simple

- (Etnyre–Honda, '01)

Torus knots & Figure eight are Leg/trans simple



- (Chekanov, '02)

5_2 knot is Legendrian non-simple

- (Birman–Menasco, '02)

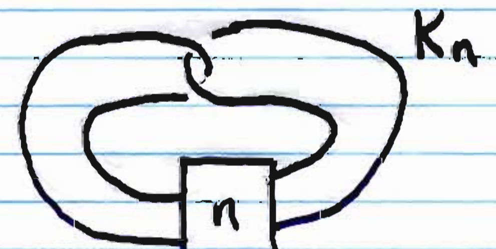
Transversely non-simple examples

- (Etnyre–Honda, '03)

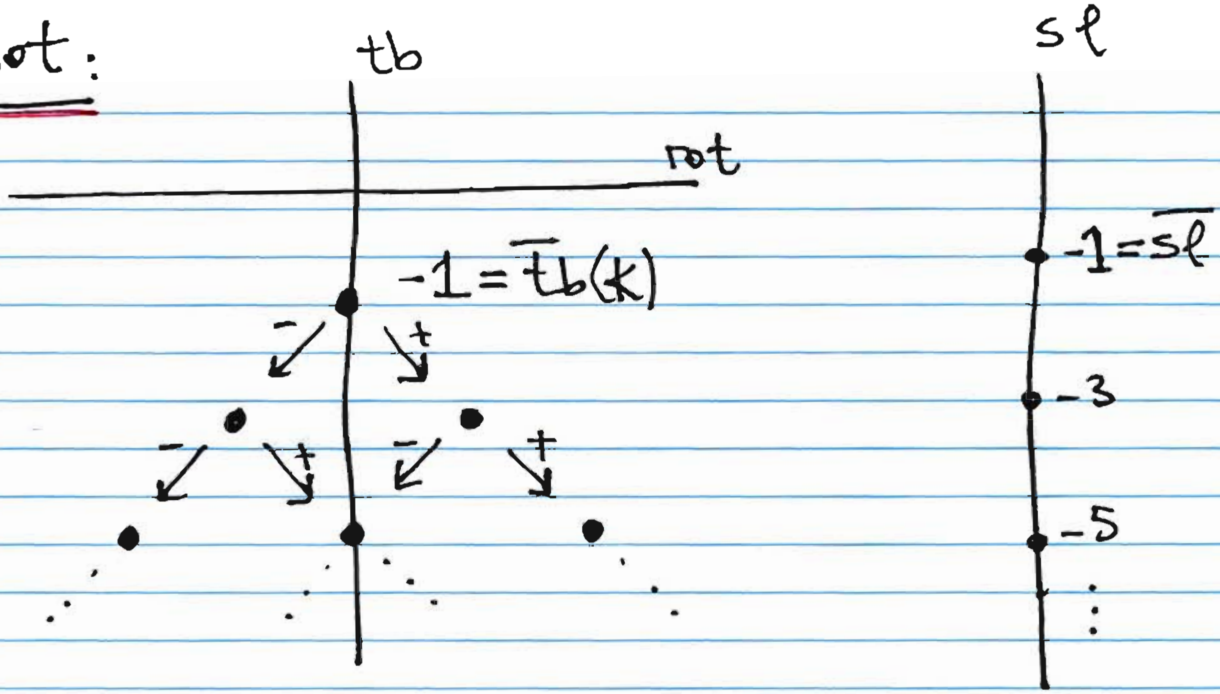
The (2,3)-cable of the (2,3)-torus knot is trans. non-simple

- (Etnyre–Ng–Vértesi) '10)

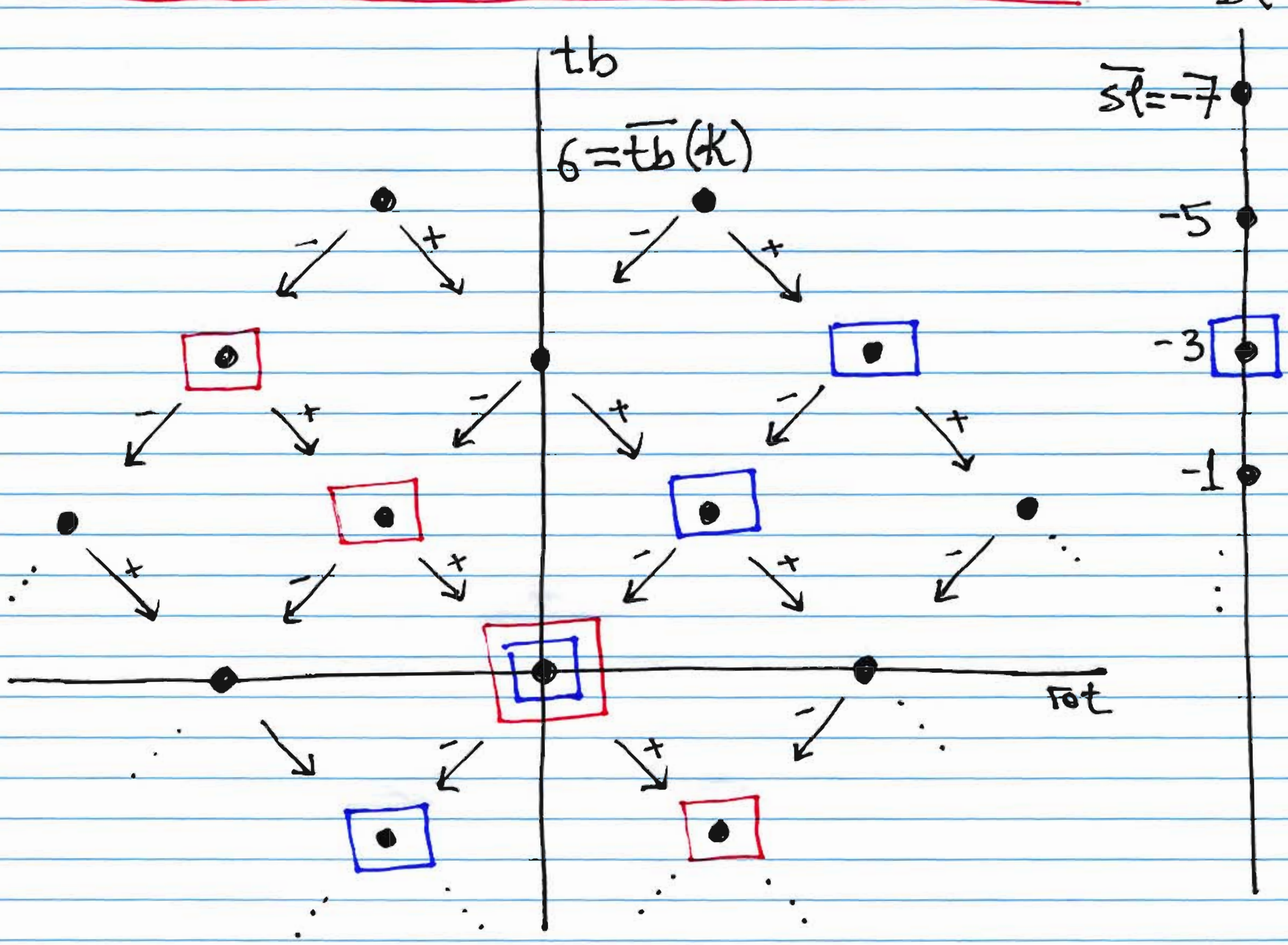
Classify twist knots



K - unknot:



K - the (2,3)-cable of the (2,3)-torus knot:

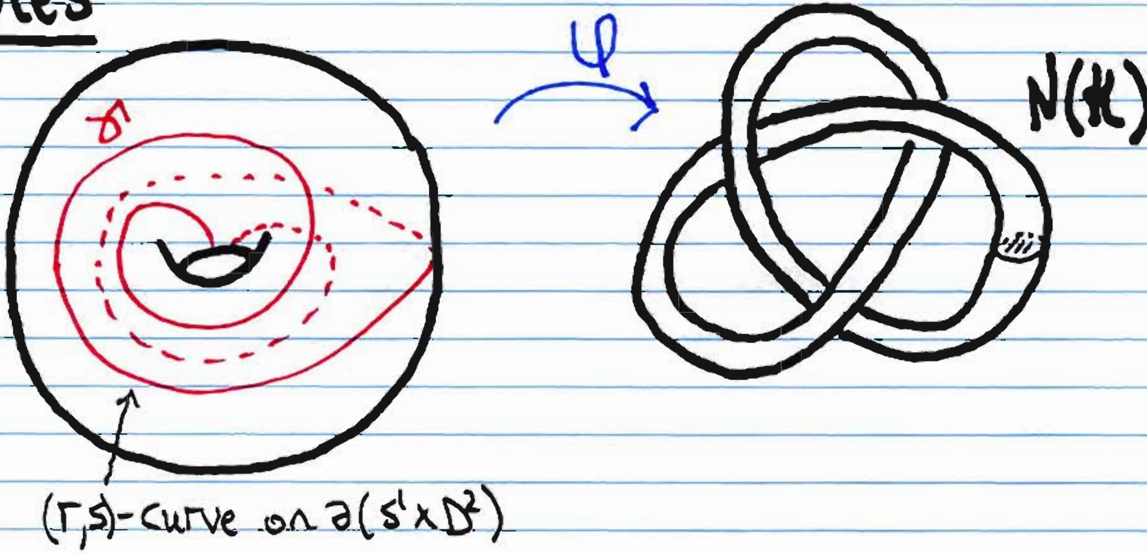


Cabling

and

Main Results

Cables



The (r,s)-cable of K is $\varphi(\gamma)$

[or equivalently, it is a simple closed curve of "slope" s/r]

Contact embeddings of $N(K)$

- Contact width

$$\boxed{w(K) = \sup_{\substack{N(K) \\ \partial N(K) \text{ convex}}} \frac{1}{\text{slope}(\Gamma_{\partial N(K)})}}$$

- $\overline{tb}(K) \leq w(K) \leq \overline{tb}(K)+1$

E.g.

- K is unknot, $\overline{tb}(K) = -1$ but $w(K) = 0$
- K is torus knot, $w(K) = \overline{tb}(K)$

The uniform thickness property (UTP)

Def$\underline{^n}$:

A knot type $\mathcal{K}$ satisfies the UTP

If

- $\overline{tb}(\mathcal{K}) = w(\mathcal{K})$

- Every solid torus representing $\mathcal{K}$ can be thickened to a standard nbd of $L \in \mathcal{L}(\mathcal{K})$ with $tb(L) = \overline{tb}(\mathcal{K})$

E.g. Negative torus knots, figure eight are uniformly thick but unknot is not

Thm (Etnyre-Honda)

If $\mathcal{K}$ is Legendrian simple and has the UTP

Then $\mathcal{K}_{(r,s)}$ is Legendrian simple

Results

Lower width

$$\boxed{lw(K) = \inf_{\substack{N \text{ s.t.} \\ \bullet\ \partial N \text{ convex} \\ \bullet\ N \text{ is non-thickenable}}} \frac{1}{\operatorname{slope}(\Gamma_{\partial N})}}$$

Thm (T.) Let K be Legendrian simple

- If $w(K) \in \mathbb{Z}$ and $r/s > w(K)$

  Then $K_{(r,s)}$ is Legendrian simple

- If $lw(K) \in \mathbb{Z}$ and $r/s < lw(K)$

  Then $K_{(r,s)}$ is Legendrian simple

Question

$$\ell w(\mathcal{K}) < r/s < w(\mathcal{K}) \, ?$$

Thm (Etnyre–LaFountain–T.)

- The positive (p,q)-torus knot $\mathcal{K}$ is not uniformly thick
- $w(\mathcal{K}) = \overline{tb}(\mathcal{K}) = pq - p - q$ and $\ell w(\mathcal{K}) = 0$
- Complete classification of $\mathcal{L}(\mathcal{K}_{(r,s)}) \,/\, \mathcal{T}(\mathcal{K}_{(r,s)})$ for all (r,s)

• What is new?

Legendrian classification problem:

⊡ gives the first example of a knot type with non-destabilizable Legendrian knots with Thurston-Bennequin (TB) number arbitrarily far from the maximal TB number

[Twist knots]

⊡ gives yet another family of prime knot types which have arbitrarily many Legendrian knots with fixed classical invariants

⊡ gives the first set of prime Legendrian knots with the same classical invariants that require arbitrarily many stabilizations before becoming Legendrian isotopic

Transverse classification problem:

- gives the first set of examples where there are arbitrarily many non-destabilizable transverse knot with fixed self-linking number which is arbitrarily far from the maximal self-linking number.

[Twist knots]

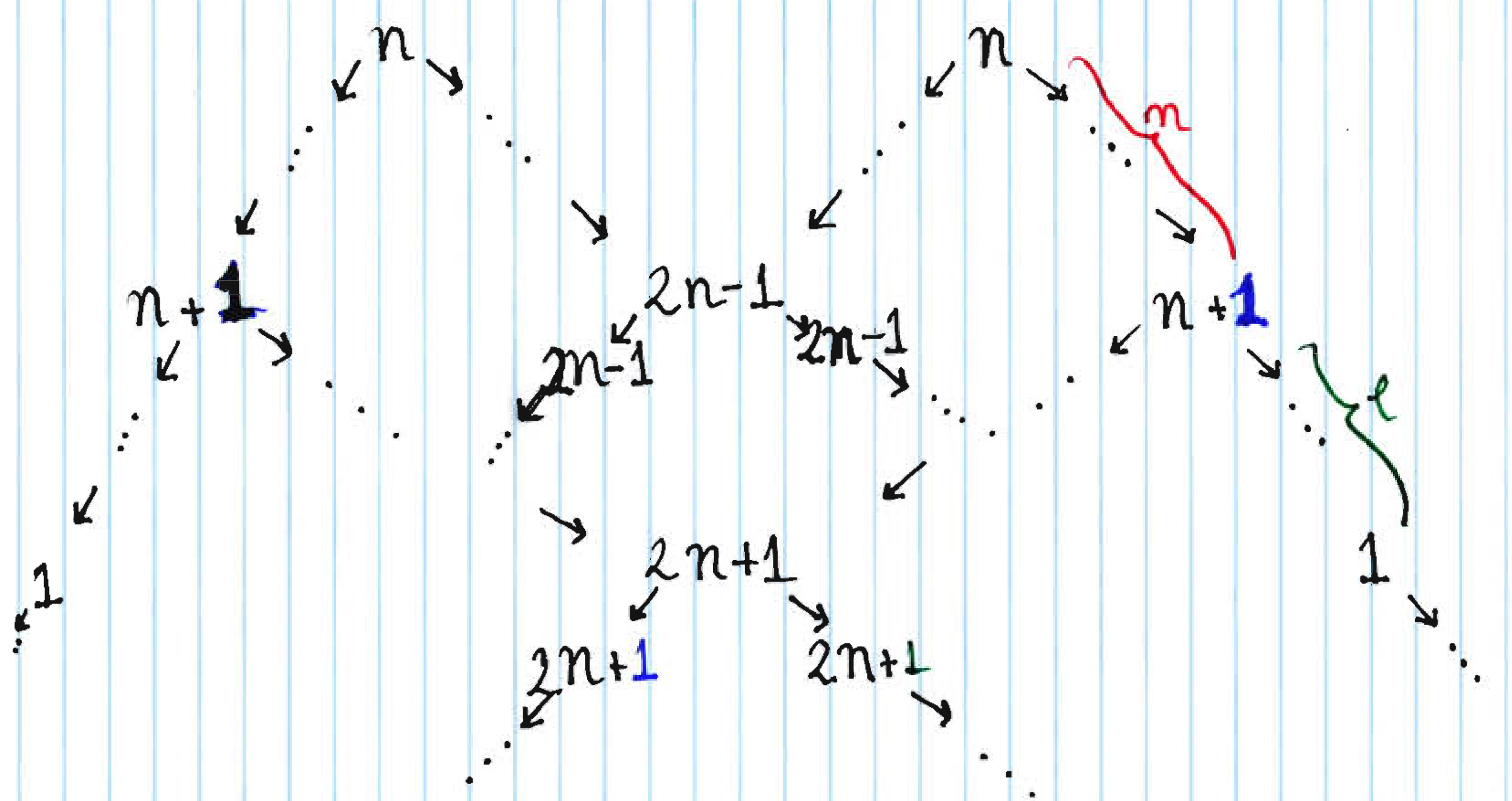
- gives yet another infinite family of transversely non-simple prime knot types

⋮

$K_{(r,s)}$: the (r,s)-cable of the $(2,3)$-torus knot $1 \leq k < s/r < k+1$

$$\boxed{\begin{aligned}\overline{tb} &= rs\\ rot &= \mp(s-r)\end{aligned}}$$

<u>Legendrian</u>



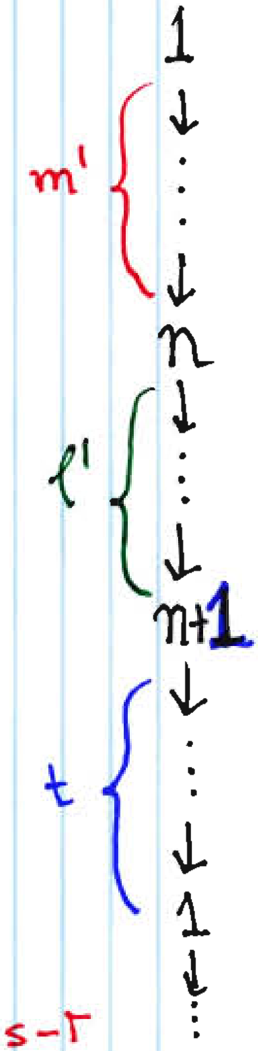
$n = \lfloor s/r \rfloor = k$

$m = r\lceil s/r \rceil - s = r(k+1) - s$

$\ell = s - rk$

<u>Transverse</u>

$$\boxed{\overline{s\ell} = rs + s - r}$$



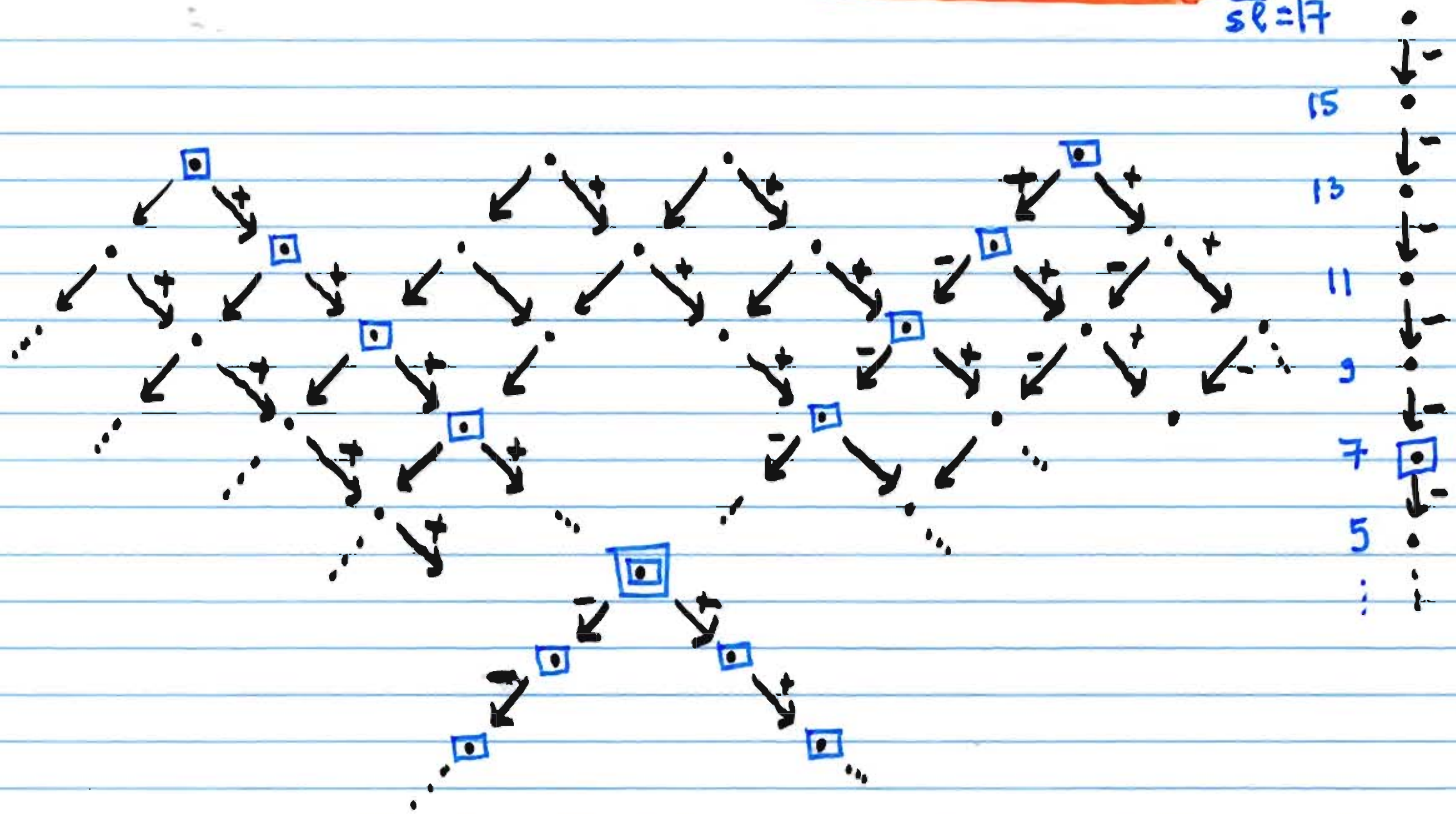
$m' = s - r$

$\ell' = r(s-k) - m$

$t' = rs - s - r$

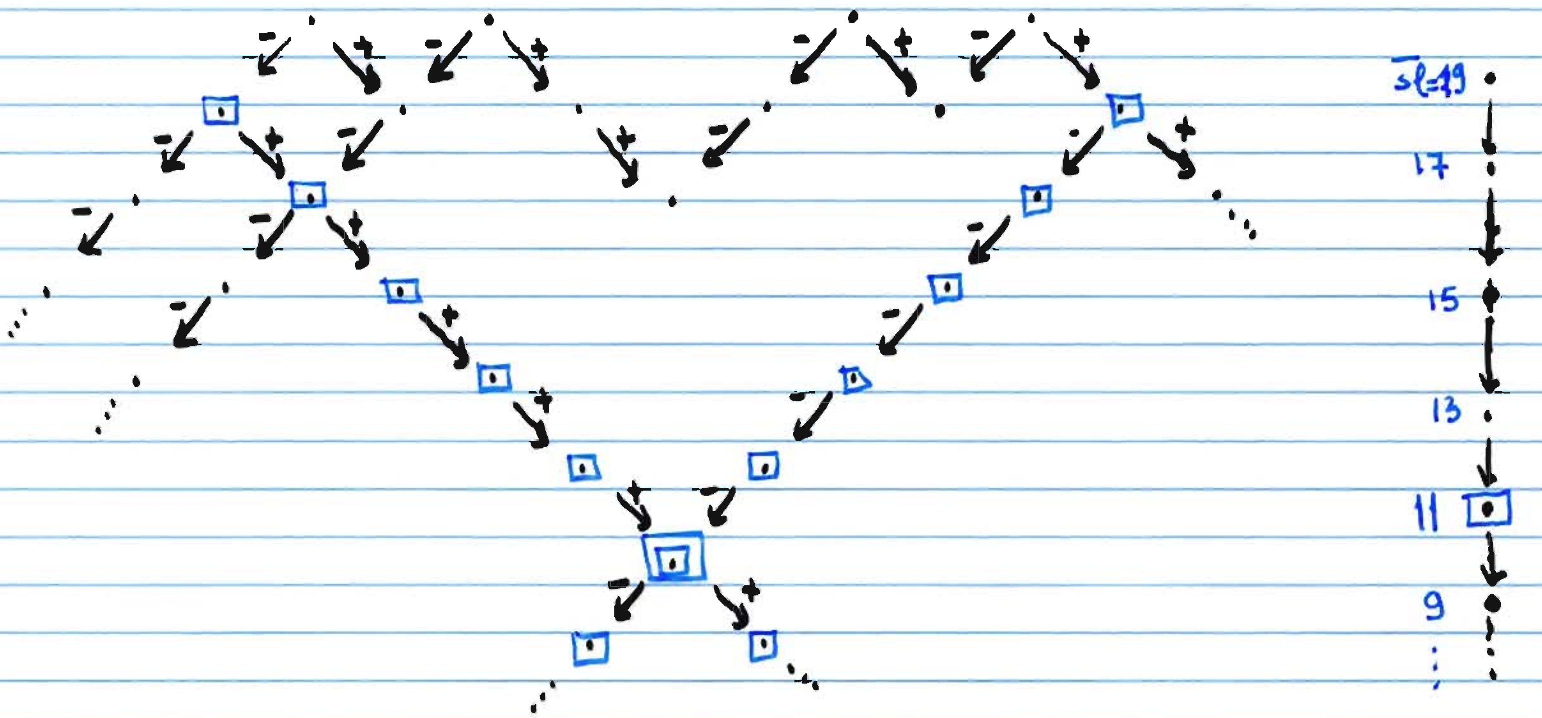
The (4,3)-cable of the (2,5)-torus knot

$\boxed{\overline{tb}=12 \text{ and } rot=\mp 5, \mp 1}$



The (5,3)-cable of the (2,5)-torus knot

$\boxed{\overline{tb}=15 \text{ and } rot=\mp 4, \mp 2}$



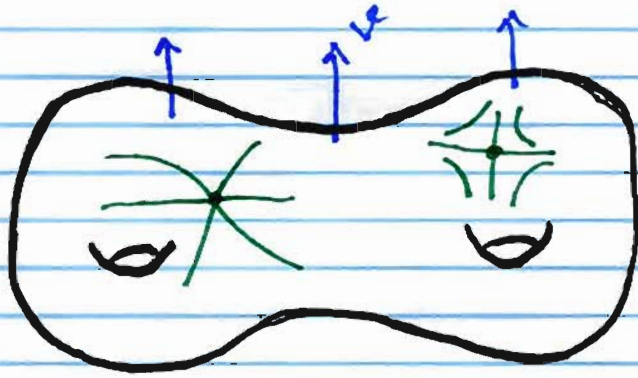
Convex Surface Theory

and

Solid Tori

⊡ Convex surfaces

Σ



$T_x\Sigma \cap \xi_x = \Sigma_\xi$ - char. fol$^{\underline{n}}$

⊡ Σ_ξ determines ξ in the neighborhood of Σ

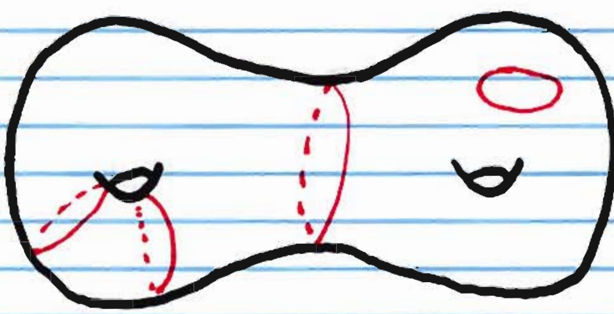
A properly emb. surface $\Sigma \subset (M,\xi)$ is convex if $\exists$ a contact vector field $v \pitchfork \Sigma$

($\Leftrightarrow$ Σ has a nbhd $\Sigma \times (-\epsilon,\epsilon)$ s.t. ξ is invariant in $(-\epsilon,+\epsilon)$ direction)

Fact: Any closed surface is C^∞- close to a convex surface.

⊡ Dividing set:

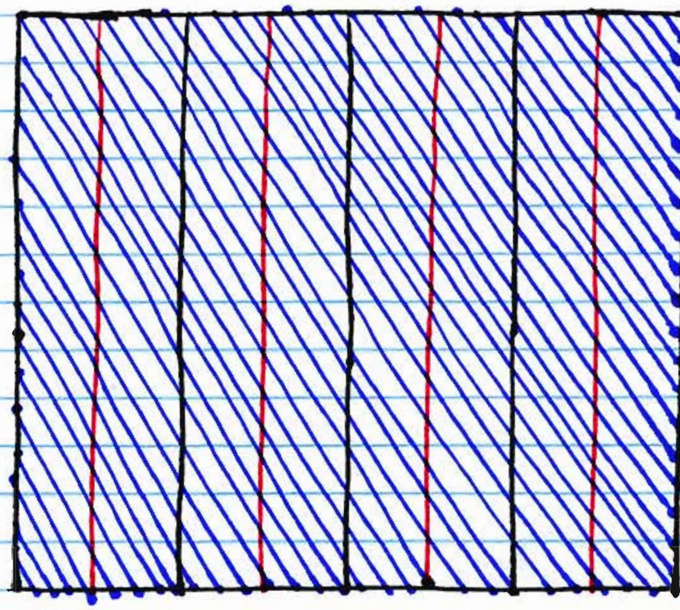
$$\Gamma_\Sigma = \{x \in \Sigma \mid v(x) \in \xi_x\}$$



- Γ_Σ is embedded 1-manifold
- $\Gamma_\Sigma \neq \phi$
- Γ_Σ is independent, up to isotopy, of v

Giroux's Flexibility Thm: Σ convex and $\mathcal{F}$ is any singular foliation on Σ "adapted" to Γ_Σ. Then $\exists$ an admissible C^0-small isotopy of Σ to Σ' such that $\Sigma'_\xi = \mathcal{F}$

$\Sigma = T^2$ - standard torus

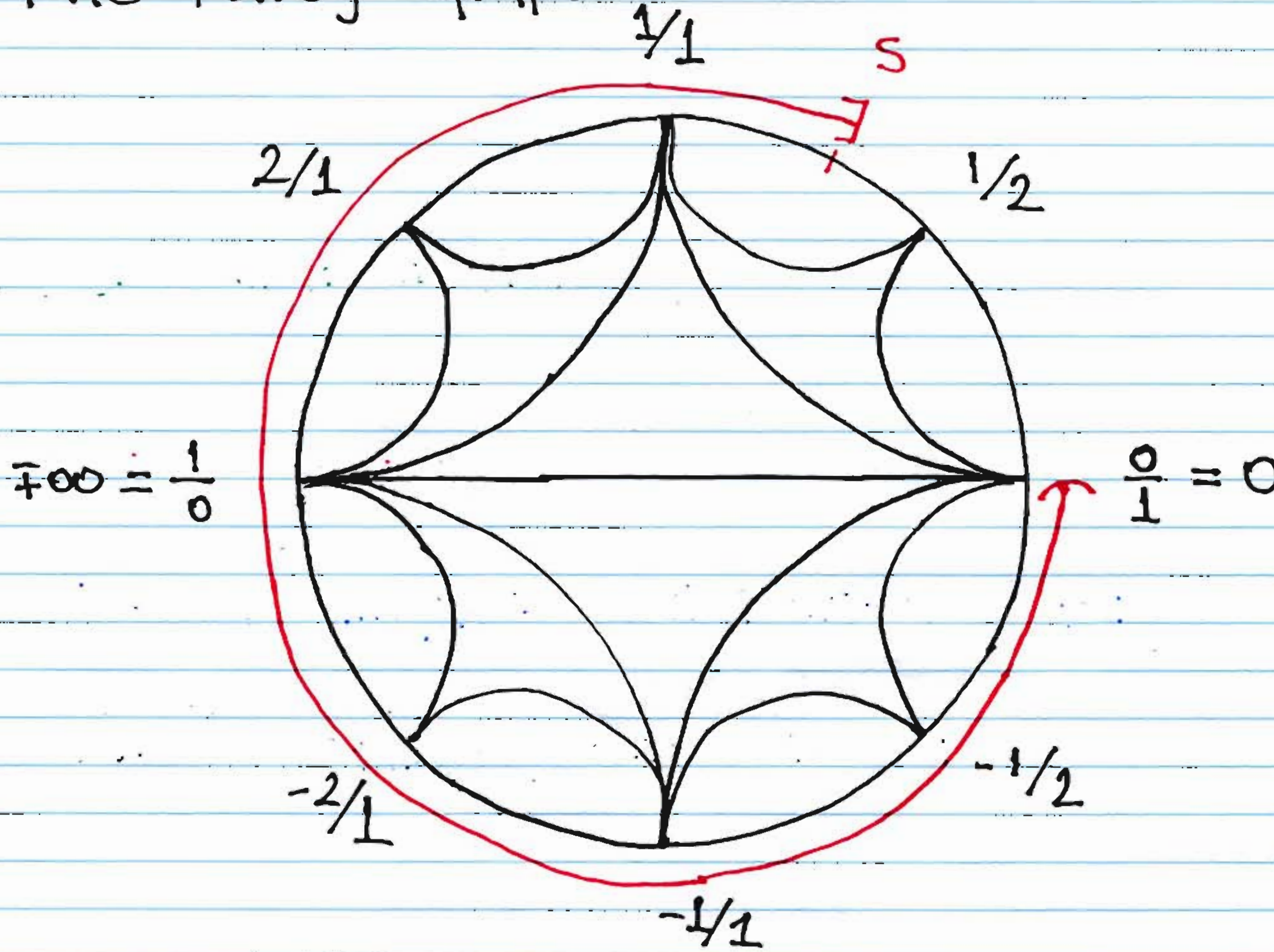


Fact:

- By Giroux's flex. thm., any convex torus T with slope(T) = slope$(\Gamma_T) = s$ in a tight manifold can be put in a std. form with rulling slope $r \neq s$
- Let N be a solid torus with
  - ∂N convex
  - $\#\Gamma_{\partial N} = 2$
  - slope$(\Gamma_{\partial N}) = 1/n$

Then $\exists$ unique tight contact str. on N and such N is called sth. nghd.

The Farey Graph



<u>Fact:</u> If $\text{slope}(\Gamma_{\partial N}) = s$. Then we can realize all slopes in $[s, 0)$

<u>Question:</u> Given a solid torus N with $\text{slope}(\Gamma_{\partial N}) = s$

What can you say about $N' \supset N$ and $\text{slope}(\Gamma_{\partial N'})$?

Contact embeddings of $N(\mathcal{K})$

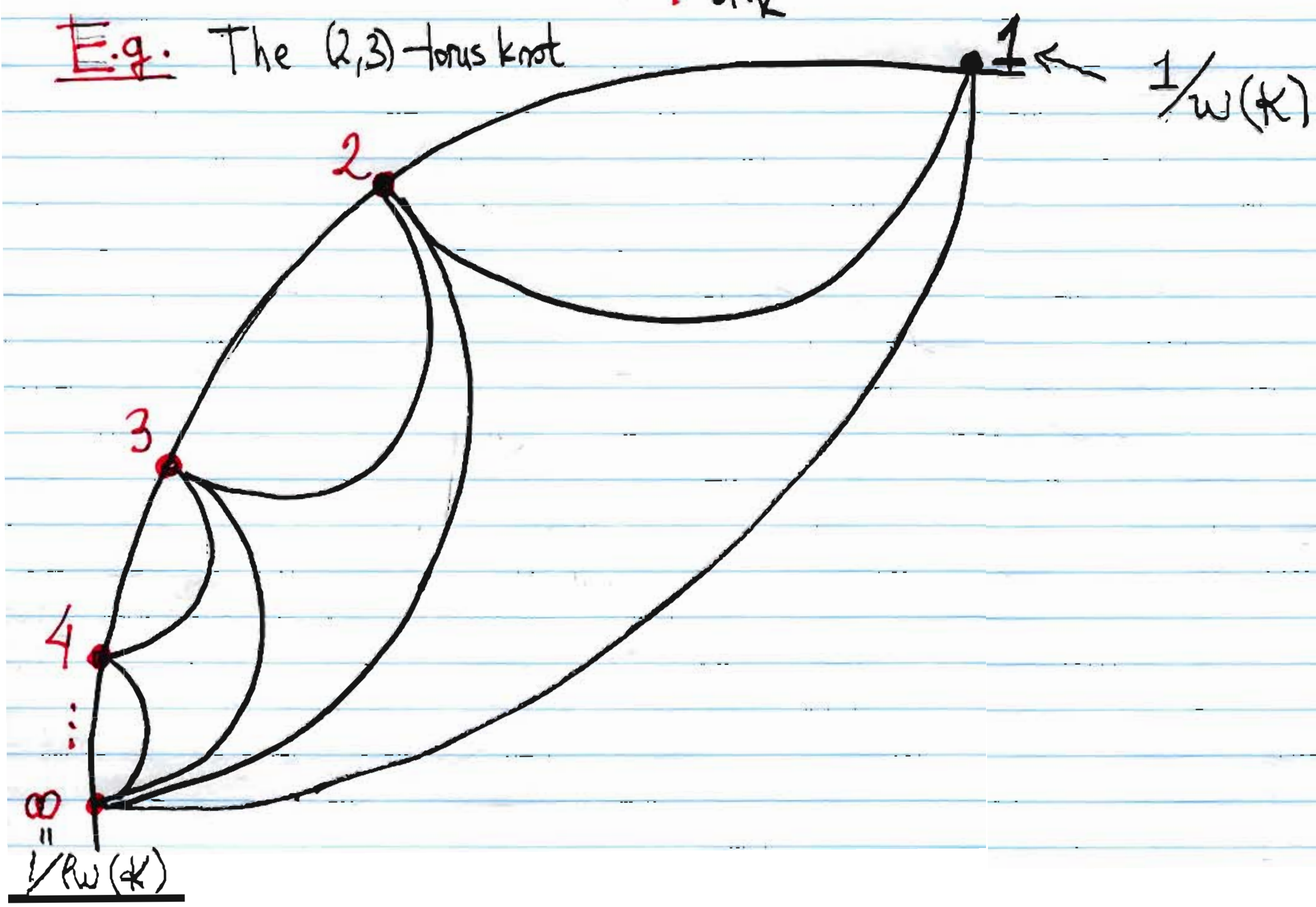
($\mathcal{K}$ is the positive (p,q)-torus knot)

1. Non-thickenable embeddings

There are non-thickenable solid tori N_k representing $\mathcal{K}$ such that

- ∂N_k convex
- $\text{slope}(\Gamma_{\partial N_k}) = k/pq-p-q$, $k=1,2,$
- $\#\Gamma_{\partial N_k} = 2\gcd(k, pq-p-q)$

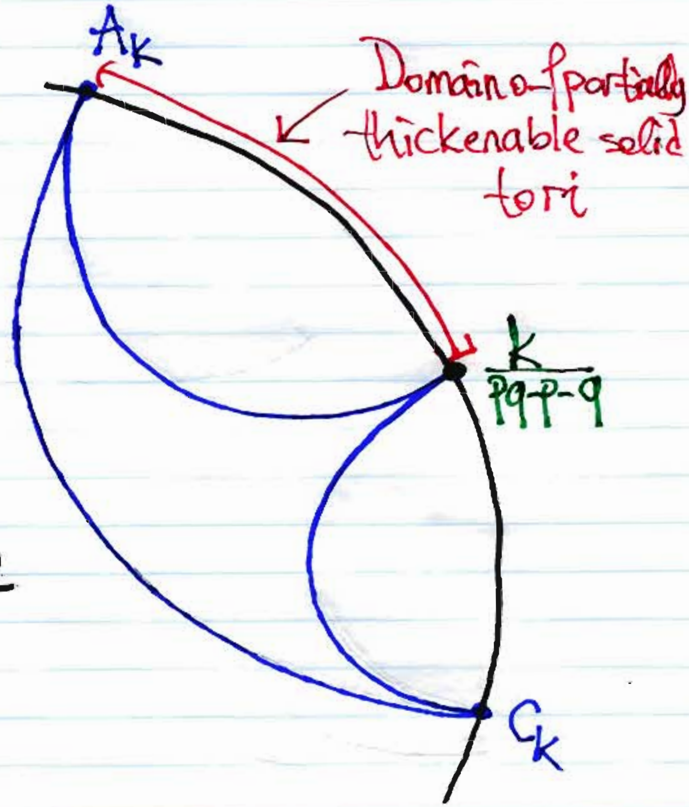
E.g. The $(2,3)$-torus knot



2. Partially thickenable embeddings

Non-simple domains

Let N_k' be non-thickenable solid torus with $\#\Gamma_{\partial N_k'} = 2$



If $\frac{k}{pq-p-q} = [a_0, a_1, \cdots, a_n]$

where $a_0 \geq 1$ and $a_i \geq 2$
$i = 1, 2, \cdots$

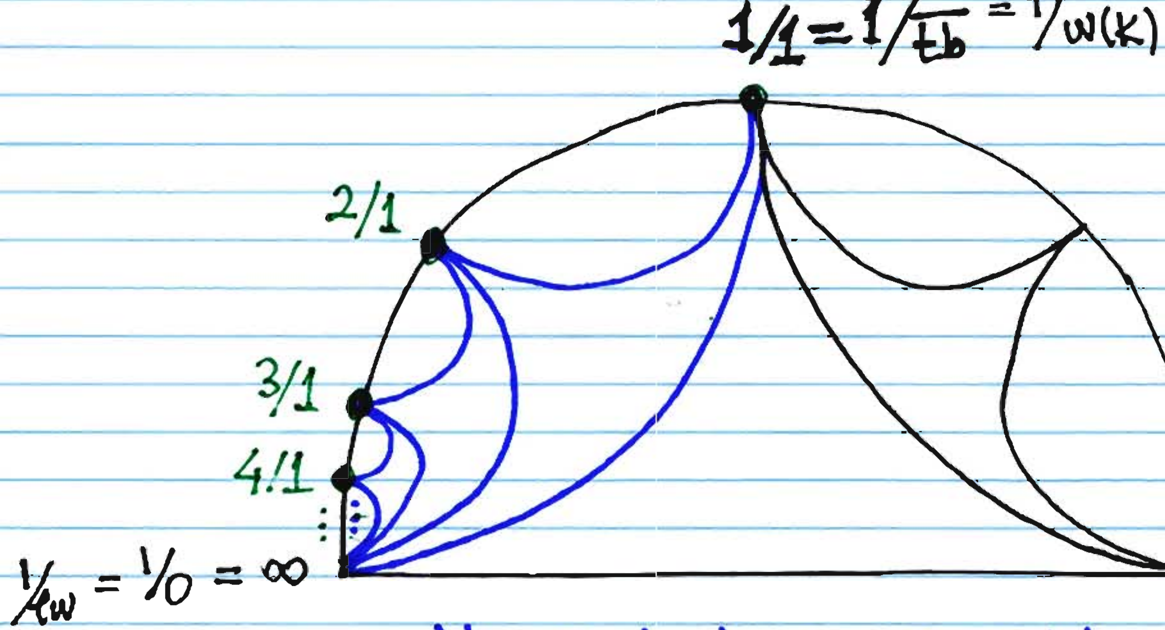
Then

$A_k = [a_0, a_1, \cdots, a_{n-1}]$

$C_k = [a_0, a_1, \cdots, a_n - 1]$

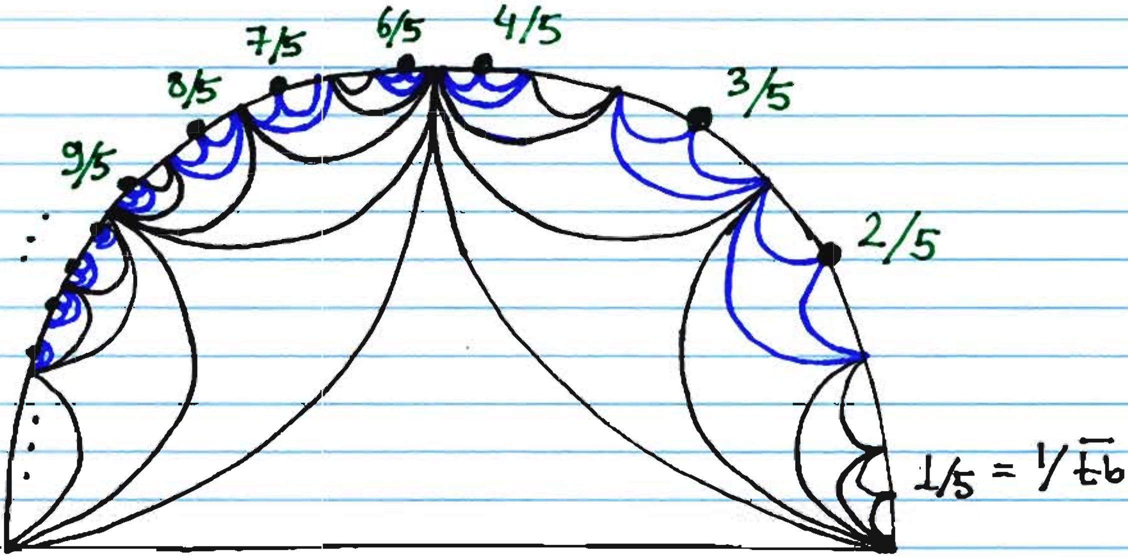
E.g.

- K – the (2,3)-torus knot



Non-simple domains are nested.

- K – the (2,7)-torus knot



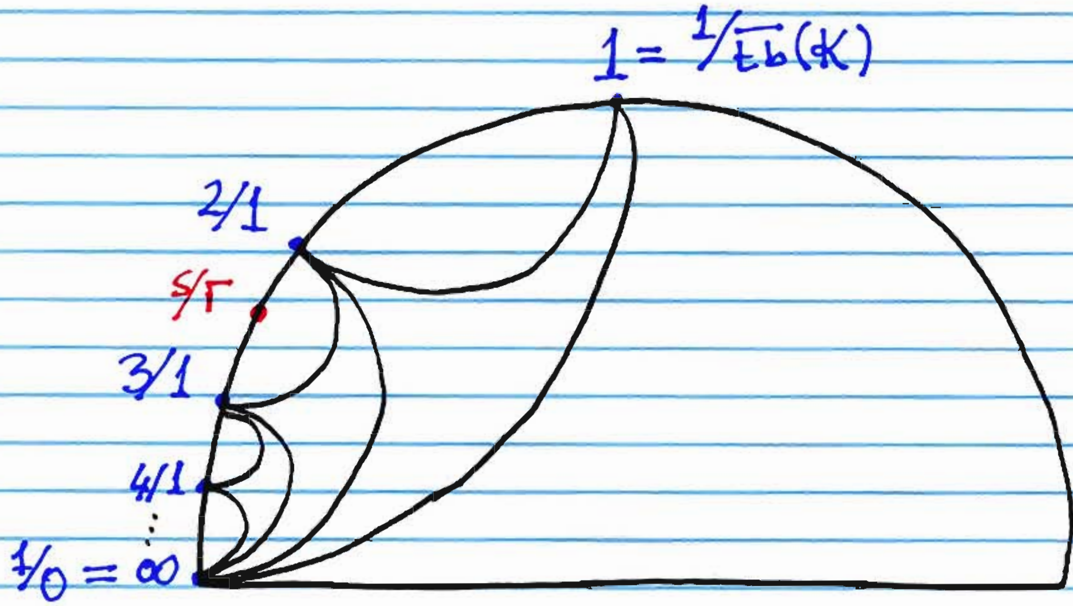
Non-simple domains for the positive torus knot K are nested

$\Longleftrightarrow$

K is the (2,3)-torus knot

Idea of the proofs

Idea of the Proof:



We know

$\exists$ two tori T_1, T_2 with slope s/r that are partially thickenable

$\exists$ a torus T_0 with slope 3 that is non-thickenable

$\Longrightarrow$

$$\left.\begin{array}{l} \pm L_1 \text{ dividing curve on } T_1 \\ \pm L_2 \text{ dividing curve on } T_2 \\ \pm L_0 \text{ ruling curve of slope } s/r \text{ on } T_0 \end{array}\right\} \begin{array}{l} \pm L_i \in \mathcal{L}(K_{(r,s)}) \quad i=0,1,2 \\ \text{the } (r,s)\text{-cable of} \\ \text{the } (2,3)\text{-torus knot.} \end{array}$$

...

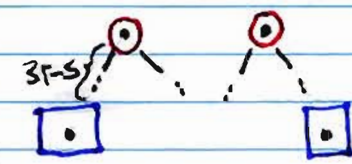
Step 1:

- $tb(\pm L_1) = tb(\pm L_2) = \overline{tb}(K_{(r,s)}) = sr$

- $rot(\pm L_1) = rot(\pm L_2) = \pm(s-r)$
- $\pm L_1$ is not Legendrian isotopic to $\pm L_2$

Step 2:

- $tb(\pm L_0) = sr - |3r-s|$

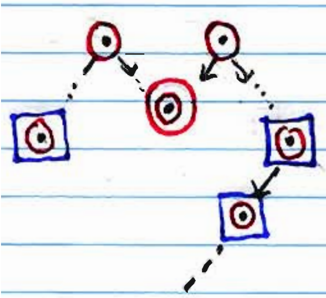


$\Rightarrow \pm L_0$ are non-maximal

- $\pm L_0$ are non-destabilizable

Step 3:

- $S_\pm(\pm L_0)$ is Legendrian isotopic to $S_\pm^r(\pm L_i)$
  $i=1,2$
- $S_-^n(L_1)$ is not Legendrian isotopic to $S_-^n(L_2)$ for all $n \in \mathbb{N}$

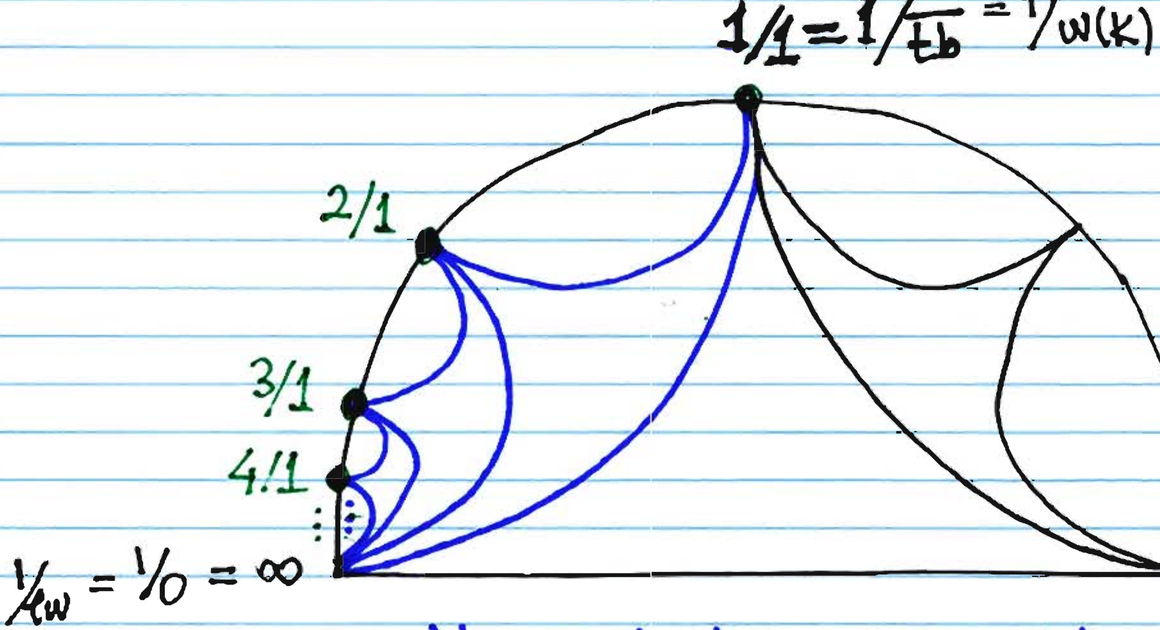


- $S_-^n S_+^{3r-s}(L_1)$, $S_-^n S_+^{3r-s}(L_2)$ and $S_-^n(L_0)$ are pairwise not Legendrian isotopic for all $n \in \mathbb{N}$

⋮

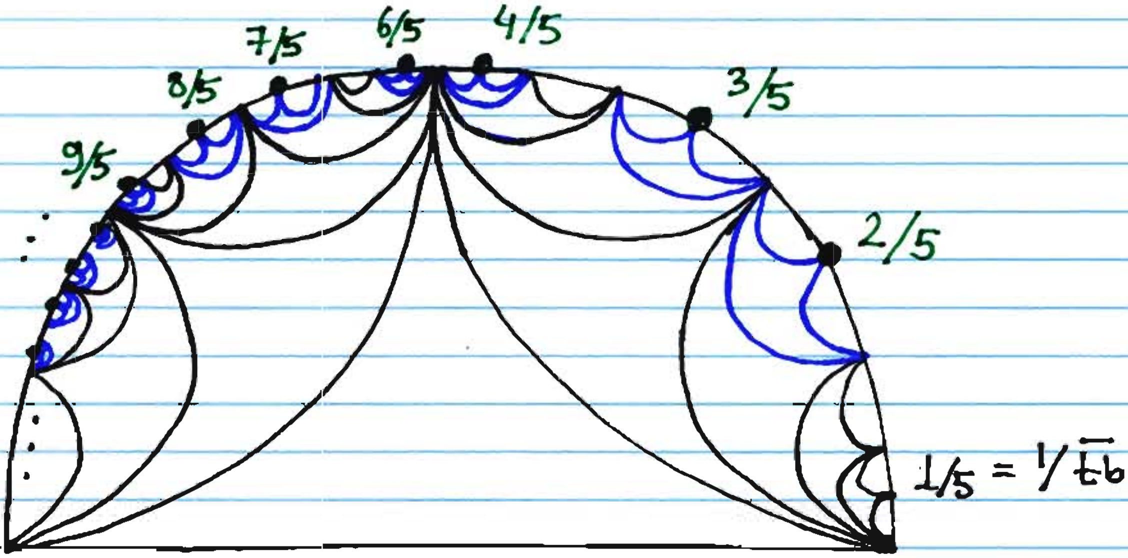
E.g.

- K - the (2,3)-torus knot



Non-simple domains are nested.

- K - the (2,7)-torus knot

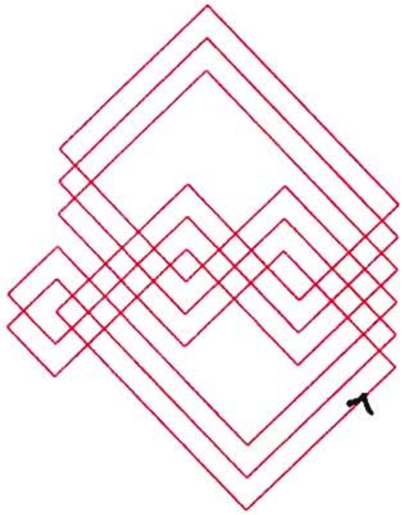


Non-simple domains for the positive torus knot K are nested

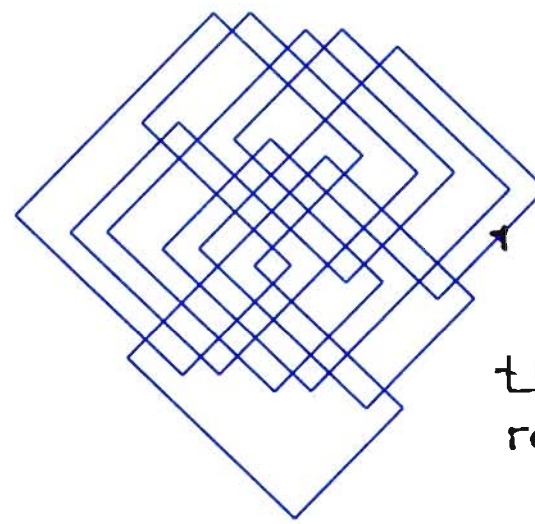
$\Longleftrightarrow$

K is the (2,3)-torus knot

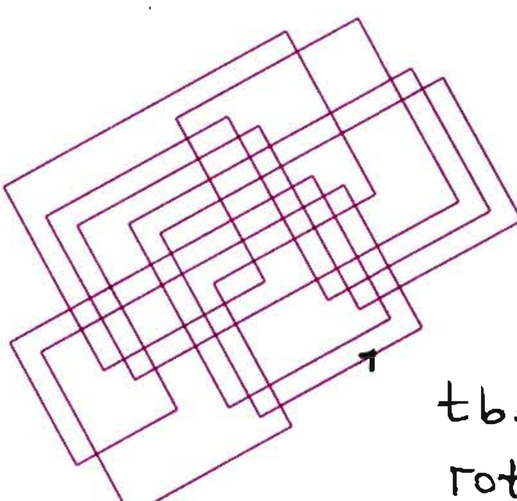
Legendrian (1,3)-cables of the (2,3)-torus knot



tb = 3
rot = 2



tb = 3
rot = 2



tb = 3
rot = 2

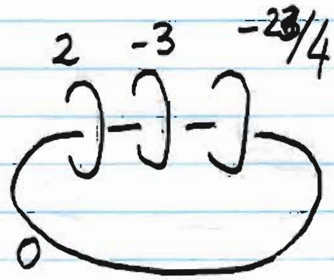
Work in progress

Let $K_{(r,s)}$ denote the (r,s)-cable of the $(2,3)$-torus knot $\exists\ L_j \in \mathcal{L}(K_{(r,s)})$ with fixed classical invariants but pairwise Legendrian non-isotopic

Q: Does Legendrian surgery on L_j result distinct/isotopic tight contact strs?

E.g.

$$S^3_1(K_{(1,2)}) \cong$$



$\#\mathrm{Tight}(S^3_1(K_{(1,2)}))=6$

$\mathcal{L}(K_{(1,2)})$