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Floer homology and fractional Dehn twists

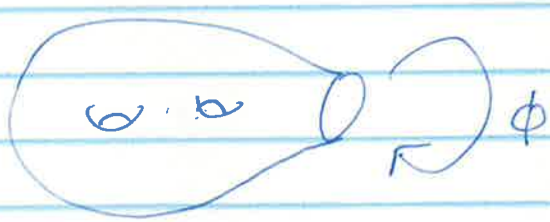
joint work: M. Hedden

fractional Dehn twist coefficients

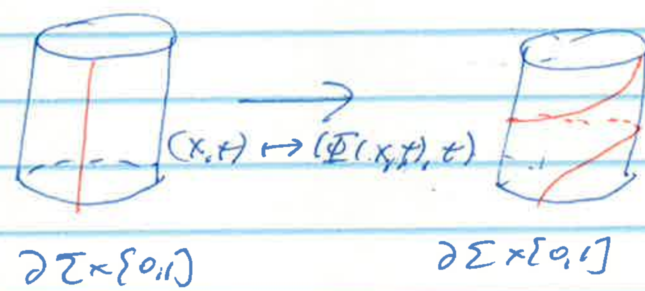
Σ surface with $\partial\Sigma = S^1$

$\phi : \Sigma \to \Sigma$ diffeo $\phi|_{\partial\Sigma} = id$

$\exists$ isotopy of ϕ (not fixing ∂) to a Nielsen - Thurston representative ϕ_{NT}



say Φ = isotopy restrict to $\partial\Sigma$ and consider



$\tau(\phi)$ = total rotation of image of pt $\times [0,1]$

<u>Properties:</u> • $\tau(\phi)$ depends only on isotopy class of ϕ rel $\partial\Sigma$

• $\tau(\phi) \in \mathbb{Q}$ satisfies

$$\tau(\phi^n) = n\,\tau(\phi)$$

$$\tau(\phi \circ t_\partial) = \tau(\phi) + 1$$

right Dehn twist around ∂-parallel curve

- $\tau(\phi)$ invariant under conjugation

Ex: if $K \subset Y$ a fibered knot
$\rightsquigarrow$ fibration described by monodromy $\Sigma \circlearrowleft_{\phi_K}$ fixing $\partial\Sigma$
$\rightsquigarrow$ twist coeff $\tau(K) = \tau(\phi_K)$

Remark: $\exists\, \phi$ with arbitrarily large $\tau(\phi)$ (pos. or neg.)

Th$^{\underline{m}}$ (Hedden-M): fix closed 3-mfd Y then there $\exists$ integer $N_Y \geq 0$ with the property: for any fibered knot $K \subset Y$, the twist coeff $\tau(K)$ satisfies

$$|\tau(K)| \leq N_Y + 1$$

Remark: Previously known for $Y = S^3$ by Gabai and Kazez-Roberts
also for L-spaces by Ozsváth-Szabó

Strategy: exploit connection with contact topology

Recall: fibered knot K $\rightsquigarrow$ open book decomp $\rightsquigarrow$ contact structure (T-W) ξ_K

Moreover any contact structure is supported by an open book like this

<u>Th$^{\underline{m}}$ (Honda-Kazez-Matić)</u>

if ξ_K is <u>tight</u> then $\tau(K) \geq 0$

so main result follows from

<u>Th$^{\underline{m}}$</u>: for Y there is an integer $N_Y \geq 0$ such that given any contact str ξ supported by an open book (Σ, ϕ) $\partial\Sigma = S^1$, then for any $n \geq N_Y$ the contact str determined by $(\Sigma, \phi \circ t_\partial^{\,n})$ is tight

<u>Indeed</u>: in this case $0 \leq \tau(\phi \circ \tau_\partial^{\,n}) = \tau(\phi) + n$

for upper bound replace ϕ by ϕ^{-1} (reverse or$^{\underline{n}}$ on Y)

<u>Remark</u>: Replace ϕ by $\phi \circ t_\partial^{\,n}$ corresponds to $Y \rightsquigarrow Y_{-1/n}(K)$

(surgery on K)

How to detect tightness use HF

Ozsváth - Szabó: contact str ξ on Y

$\rightsquigarrow$ invariant $c(\xi) \in HF^+(-Y)$

Th$^{\underline{m}}$ (OS) if $c(\xi) \neq 0$ then ξ is tight

Define: $c(\xi)$ O.-S. $\Rightarrow$ if M_0 is a fibered 3-mfd then $\exists$ canonical spin^c str $\mathfrak{t}$ s.t.

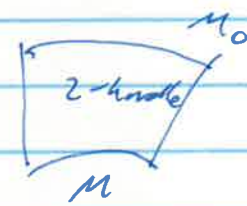
$$HF^+(M_0;\mathfrak{t}) = \mathbb{F}$$

$\mathbb{Z}/2\mathbb{Z}$

If $K \subset M$ is a fibered knot then $\exists$ cobord.

$M \to M_0$

induces a map



$$F: HF^+(-M_0,\mathfrak{t}) \to HF^+(-M)$$

image of generator is $c(\xi_K)$

Surgery exact sequence:

$$\begin{tikzcd} HF^+(-Y_0;\mathbb{F}[G_n]) \arrow[rr, "F"] & & HF^+(-Y_{1/n}(K)) \arrow[dl] \\ & HF^+(-Y) \arrow[ul, "G"] & \end{tikzcd}$$

$\mathbb{F}[G_n]$ = gp ring of $G_n = \mathbb{Z}/n\mathbb{Z}$

Theorem follows from:

- $HF^+(-Y_0, t; \mathbb{F}[c_n]) \cong \mathbb{F}[c_n]$

  hop dim n

- if F is nontrivial (restricting to t)
  then $c(\xi_n) \neq 0$
  ct str on $Y_{-1/n}(K)$

- Maps F, G are U equivariant
  action on $-Y_0$ is trivial
  $\Rightarrow$ G factors through coker U
  which is finite dimensional

$\Longrightarrow$ choose $n >$ dim coker U
so F nontrivial

Refinement using gradings:

Plane fields on 3-mfd classified up to
homotopy by 2 pieces of data:

- Spin^c structure s_ξ
- 3-dimensional invt
  if $c_1(s_\xi)$ is torsion
  this corresponds to
  Hopf invariant $h(\xi) \in \mathbb{Q}$

Fact: if $c_1(s)$ is torsion then HF^+ graded

Th^{m} (Hedden - M) let ξ be a contact str. on Y with $c_1(\xi)$ torsion, and let (Σ, ϕ) be a supporting open book with connected $\partial\Sigma$. Then $\tau(\phi)$ satisfies

$$-1 - \dim_{\mathbb{F}} HF^{red}_{2g-1-h(\xi)}(-Y, s_\xi) \le \tau(\phi) \le \dim HF^{red}_{-h(\xi)}(-Y, \xi) + 1$$

most of the time $\tau \in [-1, 1]$ (reduced homology usually 0)

Cor: take ξ, with $c_1(\xi)$ torsion then for all suff large g then adding a single right handed twist to boundary of supp. open book of genus g gives a tight contact str.

Cor: Fix Y then $\exists\, r_Y \in \mathbb{Q}$ with following property, let ξ be a contact str with $c_1(\xi)$ torsion and $h(\xi) \le r_Y$ then ξ becomes tight after adding 1 twist to any open book w/ connected boundary