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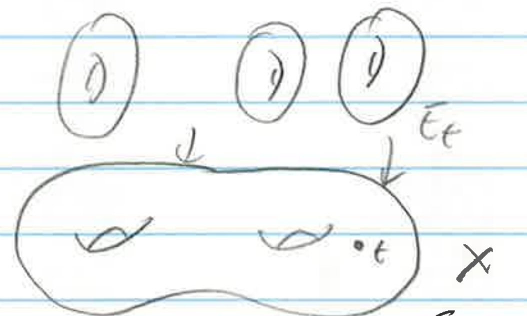
Elliptic Curves, Complex dynamics, and topology

X = compact Riemann surface
E = elliptic curve / function field $\mathbb{C}(X) = k$
$E = \{y^2 = x^3 + Ax + B\}$ $A, B \in k$



$E \subset \mathbb{P}^2(\bar{k})$
$\downarrow$
$E_t \subset \mathbb{P}^2(\mathbb{C})$
eval. A, B at t.

E is said to be isotrivial if all E_t are isomorphic$/\mathbb{C}$

Legendre Elliptic curves

$$E_t = \{y^2 = x(x-1)(x-t)\}$$

$t \in \mathbb{C} \setminus \{0,1\}$

$$X = \mathbb{P}^1 \qquad k = \mathbb{C}(t)$$

Classical problem: Understand structure of $E(k)$ = set of rational points of E (coords are in k)

Mordel-Weil Th^{m} (1920's)
if k = number field $E(k)$ is finitely gen.
In particular {torsion points} is finite

$k = \mathbb{C}(X)$ { c. 1960 $\rightsquigarrow$ if E is not isotrivial then $E(k)$ is finitely generated
$|\{P \in E(k) : \exists n \in \mathbb{N} \ n \cdot P = 0\}| < \infty$

Lang-Néron
Néron-Tate

rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

Endomorphism of E descends to a rational map on $\mathbb{P}^1$

$$\begin{array}{ccccc} (x,y) & & E & \xrightarrow{\phi} & E_{/k} \\ \downarrow & & p \downarrow \deg 2 & & \downarrow \\ x & & \mathbb{P}^1 & \xrightarrow[f_\phi]{} & \mathbb{P}^1_{/k} \end{array}$$

$\phi \in End(E)$

eg $\phi(P) = P + P = 2P$

$\deg \phi = 4$

$\deg f_\phi = \deg \phi$

$p \in E(\bar{k})$ is torsion $\Leftrightarrow$ proj $p(P)$ has finite orbit for f_ϕ

$\exists n \neq m$ st, $f^n(x) = f^m(x)$

any ϕ, $\deg \phi > 1$

back to Legendre Elliptic curves

$$E_t = \{ y^2 = x(x-1)(x-t) \}/_{\mathbb{C}}$$

$$\phi(p) = 2 \cdot p$$

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)}$$

deg 4 rational f$\frac{n}{s}$.

Th$\underline{^m}$ (M. Baker, 2008):

assume f is a ration function with coefficients in field $k = \mathbb{C}(X)$

X cpt Riemann Surface

(so $f = \{ f_t : t \in X \}$)

assume f is not isotrivial: not all f_t are conjugate by Möbius transf.
(eg $f_t(z) = z^2 - t$ $t \in \mathbb{C}$
$X = \hat{\mathbb{C}}$)

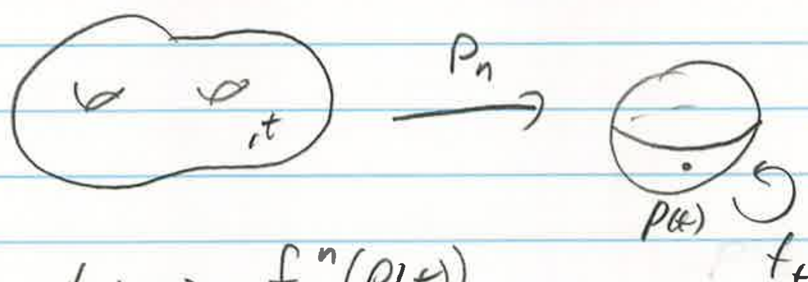
then $\{p \in \mathbb{P}^1(k) : p$ has finite orbit for $f\}$ is finite.

In fact, $\exists$ constant $C > 0$ so that
$\{P \in \mathbb{P}^1(k) : \hat{h}_f(P) < C\}$ is finite.

$P \in \mathbb{P}^1(k) \leftrightarrow P : X \to \mathbb{P}^1 = \hat{\mathbb{C}}$ holomorphic
constant or branched cover
and has well-def. topological degree.

$h(P)$ = Naive height = Weil height of P = $\deg P$

Canonical height $= \hat{h}_f(P) = \lim_{n \to \infty} \dfrac{\deg f^n(P)}{(\deg f)^n}$



$$\underbrace{t \longmapsto f_t^n(P(t))}_{P_n}$$

$(\exists\, C' \quad |\hat{h}_f(P) - h(P)| \le C')$

If P has finite orbit $\Rightarrow \hat{h}_f(P) = 0$
$\Leftarrow$ see below

to prove $\{ P \in \mathbb{P}^1(k) : \hat{h}_f(P) = 0 \}$ is finite

this implies $\{ p \in \mathbb{P}^1(k) : \hat{h}_f(P) = 0 \}$

$\shortparallel$

$\{P \text{ preperiodic}\}$

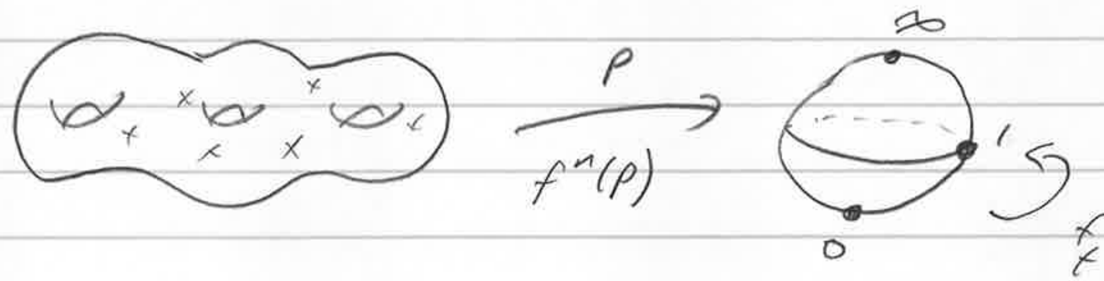
$(\hat{h}_f(f(P)) = (\deg f) \cdot \hat{h}_f(P))$

$\uparrow$

this & def^n $\hat{h}_f(P)$

$\Rightarrow$ reverse implication above)

Height theory : $\underbrace{\hat{h}_f(P) = 0 \iff \{\deg f^n(P)\} \text{ is bounded}}_{\text{assume this.}}$



1) change coords so f_t fixes $\{0,1,\infty\}$ $\forall t$

2) look at $S_{p,n} = (f^n(p))^{-1}(\{0,1,\infty\})$

$|S_{p,n}| < D_p$ ← uniform bound

$S_{p,n} \subset S_{p,n+1}$ because $\{0,1,\infty\}$ are fixed

so $S = \bigcup_{n \geq 0} S_{p,n}$ finite let $Y = X \setminus S$

Riemann-Hurwitz : only finitely many
(fact) non constant holomorphic
maps $Y \to \mathbb{P}^1 \setminus \{0,1,\infty\}$

fix $k = \mathbb{C}(X)$, fix $f \in h(\mathbb{C})$ $\deg f \geq 1$

It turns out $\hat{h}_f(P) > 0 \Leftrightarrow P$ undergoes bifurcation
(sometimes $P(t) \in J(f_t)$
sometimes $P(t) \notin J(f_t)$)

Julia set.