Chris Leininger
TTC 2014



# Dynamics & free-by-cyclic groups

jt w/ Dowdall & I. Kapovich

Group

$$G = \langle x_1, x_2, x_3, t \mid t^{-1}x_1 t = x_2,\ t^{-1}x_3 t = x_1,\ t^{-1}x_2 t = [x_2^{-1}, x_1^{-1}]x_3 \rangle$$

<u>note</u>: mapping cylinder of free group on 3-gen

How many other ways can you write as such?

$u_0 : G \to \mathbb{Z}$, $t \mapsto 1$, $x_i \mapsto 0$

$H^1(G) = \mathrm{Hom}(G; \mathbb{R}) \ni u$ s.t. $u(G) = \mathbb{Z}$ is called primitive integral.

<u>Question</u>: When can I find $Q_u < G$ & some homomorphism $\phi_u : Q_u \to Q_u$ st.

(1) $Q_u$ finitely gen, $\phi_u$ injective

(2) $G = \langle Q_u, \tau \mid \tau^{-1} x \tau = \phi_u(x)\ \forall x \in Q_u \rangle$

(3) $u(Q_u) = 0$, $u(\tau) = 1$

$Q_u$ free $\Leftarrow$ (1)–(3)

i.e. $G = Q_u *_{\phi_u}$

Special Case: $Q_u = \ker(u) \Rightarrow \phi_u$ automorphism

and $Q_u *_{\phi_u} = Q_u \rtimes_{\phi_u} \mathbb{Z}$

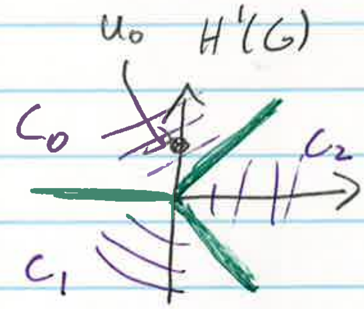
<u>Answer:</u> BNS-cone $\Sigma \subset H^1(G)$

open and scale inv't

and $\forall u \in H^1(G)$ with

$u(G) = \mathbb{Z}$ we have

$G = Q_u *_{\phi_u} \iff u \in \Sigma$




cone is complement of green

Moreover $G = Q_u \rtimes_{\phi_u} \mathbb{Z} \iff u \in \Sigma \cap -\Sigma$

$\forall u \in \Sigma$ w/ $u(G) = \mathbb{Z} \rightsquigarrow \phi_u : Q_u \to Q_u$

- if $\ker u = Q_u$ (so $\phi_u$ auto)
  $\Rightarrow \phi_u$ is well-defined upto inner auto.
  i.e. $\phi_u \in \mathrm{Out}(Q_u)$
- otherwise $Q_u$ isn't unique, so neither is $\phi_u$
- $\lambda(\phi_u)$ depends only on $u$
  ↳ growth rate of word length under $\phi_u$

$$\lambda(\phi_u) = \sup_{x \in Q_u} \limsup_{n \to \infty} \sqrt[n]{\|\phi_u^n(x)\|}$$

word length w/r.t. genset

## Theorem (D-K-L)

$\exists$ functions $h_i : C_i \to \mathbb{R}$ $\quad i = 0, 1, 2$

$h_1$ convex

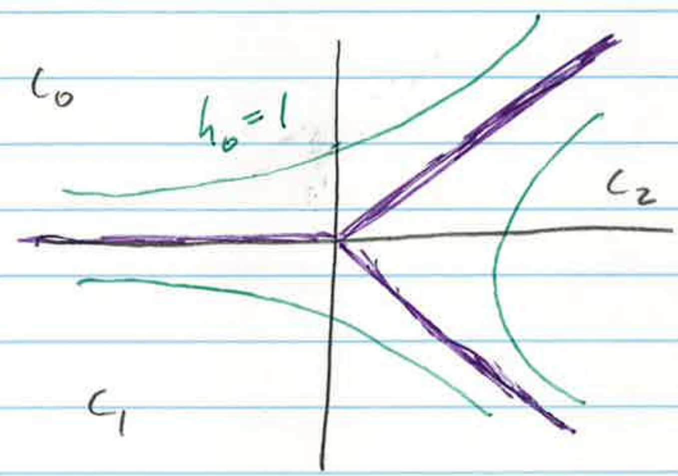
$h_1$ real analytic

$h_1(tu) = \frac{1}{t} h_1(u)$

$h_1(u) \to \infty$ as $u \to \partial C_1$

$\forall u$ s.t. $u(G) = \mathbb{Z}$ we have

$h_1(u) = \log(\lambda(\phi_u))$




**Thrm (DKL):** $\forall u \in \Sigma$ w/ $\ker(u) = Q_u$

$\phi_u$ fully irreducible

(i.e. no periodic conjugacy classes of free factors)

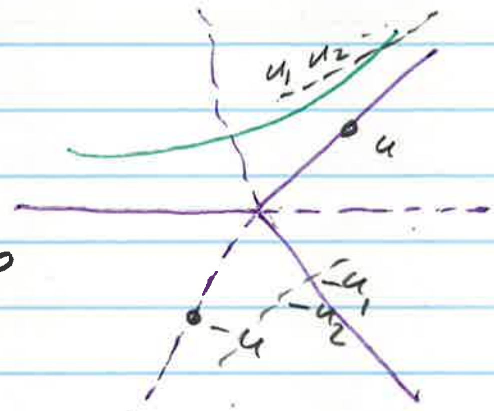
<u>(Handel-Mosher)</u>: $\exists$ fully irred auto.
$\phi : F_N \to F_N$ st.
$\lambda(\phi) \neq \lambda(\phi^{-1})$

$\exists\, C > 0$ st. $\dfrac{1}{C} \leq \dfrac{\log(\lambda(\phi))}{\log(\lambda(\phi^{-1}))} < C$

independent of $\phi$
(depends on $N$)

$u \in \partial C_0 \quad \{u_n\} \subset C_0 \quad \exists \{t_n\} \subset \mathbb{R}_+$ st.

$$\frac{u_n}{t_n} \to u$$




then $\phi_u \in Out(Q_{u_n})$

and $\lim\limits_{n\to\infty} \dfrac{\log(\lambda(\phi_{u_n}))}{\log(\lambda(\phi_{u_n}^{-1}))} = \infty$

<u>proof</u>: $\phi_{u_n}^{-1} = \phi_{-u_n}$

$$\lim_{n\to\infty} \frac{\log(\lambda(\phi_n))}{\log(\lambda(\phi_n^{-1}))} = \lim \frac{t_n \log(\lambda(\phi_n))}{t_n \log(\lambda(\phi_{-u_n}))}$$

$$= \lim_{n\to\infty} \frac{t_n h_0(u_n)}{t_n h_1(t u_n)}$$

$$= \lim_{n\to\infty} \frac{\overbrace{h_0(u_n/t_n)}^{\nearrow\, \infty}}{\underbrace{h_1(-u_n/t_n)}_{\to\, h_1(u)}} = \infty$$