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Left orderable groups that don't act on $\mathbb{R}$

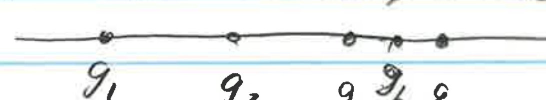
Def: G is left orderable if there is a total order $<$ s.t. $g<h \iff ag<ah$

example:
- $\mathbb{R}$, $>$
- torsion-free and nilpotent (Malcev '60s)
- Surface groups (Magnus)
- Braid group (Dehornoy)
- some, not all 3-mfd groups
  - if has $\mathbb{R}$-covered fol^{n} $\Rightarrow$ L.O.
  - Knot complements have left orderable π_1, but not all link complements.
- $\text{Homeo}_+ \mathbb{R}$ is L.O.
  - *pf:* $f<g$ if $f(0)<g(0)$
    if $=$ look at x, ... (check at dense countable set)

Useful tool: G countable

G is L.O. $\iff G \hookrightarrow \text{Homeo}_+ \mathbb{R}$

Proof. ($\Leftarrow$) inherit order above
($\Rightarrow$) enumerate $g_1, g_2, \ldots$
place g_i's on $\mathbb{R}$ using order



if $g_2 > g_1$

$g_i < g_n < g_j$

G acts on {labeled pts}

extend to closure
extend to $\mathbb{R}$

Question: What if not countable?

many nice examples of L.O. uncountable groups:

- $\mathrm{Homeo}_+ \mathbb{R}$
- $\mathrm{Diff}^{\infty}_+ \mathbb{R}$
- $\mathrm{Diff}^1(\mathbb{D}^2, \partial\mathbb{D}^2)$
- germs at $+\infty$ of homeos of $\mathbb{R}$ (Navas)

Unanswered Questions:

eg. Calegari: $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D})$ L.O.? (does not act on line)

Do uncountable L.O. groups act on $\mathbb{R}$? embed in $\mathrm{Homeo}_+ \mathbb{R}$?

silly counterexamples $\mathbb{Z}^{\kappa}$ κ large cardinal

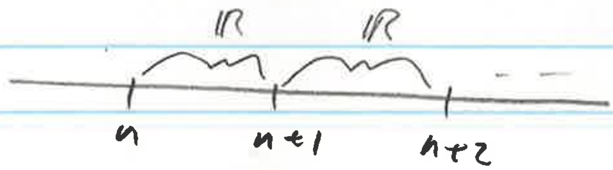
Thm (M–): let $\mathcal{G}_\infty$ be the group of germs at $+\infty$ of $\mathrm{Homeo}_+ \mathbb{R}$
any $\phi: \mathcal{G}_\infty \to \mathrm{Homeo}_+ \mathbb{R}$ is trivial

Remarks: 1) $\mathrm{Homeo}_+ \mathbb{R} \xrightarrow{\pi} \mathcal{G}_\infty$

Question: Navas: $\exists$ section?

Cor: No in fact $\exists$ finitely gen Γ in $\mathcal{G}_\infty$ st no section of π over Γ

2) $\exists$ $\mathrm{Homeo}_+\mathbb{R} \xhookrightarrow[\text{diag}]{i} \mathcal{G}_\infty$



$i(f)\big|_{(n,n+1)}$ "=" f ← under ident $(n,n+1) \cong \mathbb{R}$

$\mathrm{Homeo}_{\mathbb{Z}}\mathbb{R} = \{f(x)+1 = f(x+1)\}$

$\xhookrightarrow{i} \mathcal{G}_\infty$

Proof of th$^{\underline{m}}$:

techniques → hands on low dim'l dynamics.

→ understand or constrain actions of G on X via H_b^2 ← bounded cohomology.

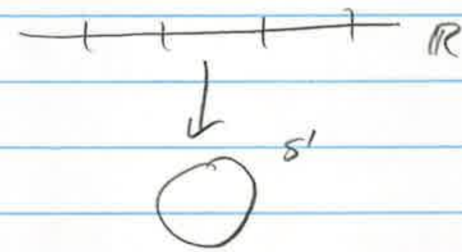
Step 1: $\mathcal{G}_\infty$ is a simple group

(hence any $\phi : \mathcal{G}_\infty \to \mathrm{Homeo}^+\mathbb{R}$ is faithful or trivial)

(elementary dynamical argt)

2. Actions of $\mathrm{Homeo}_{\mathbb{Z}}\mathbb{R}$ on $\mathbb{R}$ (Militon)

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}\mathbb{R} \to \mathrm{Homeo}_+ S^1 \to 1$$



__Thm__ (Matsumoto, Ghys '87 ish)

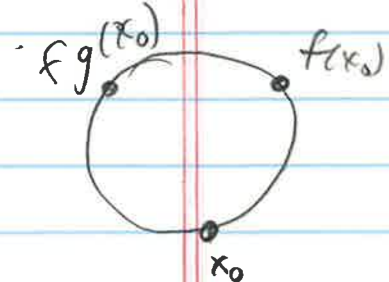
any nontrivial homomorphism $\mathrm{Homeo}_+ S^1 \to \mathrm{Homeo}_+ S^1$ is induced by conjugation

__idea of pf:__

1) __Ghys:__ $\exists\, c \in H^2_b(\mathrm{Homeo}_+ S^1; \mathbb{Z})$ s.t. for any G, any homomorphism $G \xrightarrow{\rho} \mathrm{Homeo}_+ S^1$ is completely determined by $\rho^*(c) \in H^2_b(G;\mathbb{Z})$

up to semi-conjugacy

eg $H^2_b(G;\mathbb{Z}) = 0 \Rightarrow$ any $G \curvearrowright S^1$ has a fixed pt



Representative cocycle "orientation"

$$c(f,g) = \begin{cases} \pm 1 & \text{orientation of } (x_0, f(x_0), fg(x_0)) \\ 0 & \text{if 2 agree} \end{cases}$$

11) <u>Matsumoto-Morita:</u> if G is uniformly perfect (eg. $Homeo_+ S^1$) then $H^2_b(G, \mathbb{R}) \to H^2(G^\delta; \mathbb{R})$ is injective (also for $\mathbb{Z}$)

group cohomology

if $\phi : Homeo_+ S^1 \to Homeo_+ S^1$

then get

$$\begin{array}{ccc} H^2(Homeo_+ S^1) & \xleftarrow{\phi^*} & H^2(Homeo_+ S^1) \cong \mathbb{Z} \\ \pm\varepsilon \text{ or } 0 & \longleftarrow & \varepsilon \text{ generator} \\ \uparrow & & \uparrow \\ H^2_b(Homeo\, S^1) & \xleftarrow{\phi^*} & H^2_b(Homeo_+ S^1) \\ \phi^*(c) & \longleftarrow & c \end{array}$$

if $\phi^*(c)$ goes to 0 then action trivial

so injectivity $\Rightarrow \phi^*(c) = \pm c$

$\therefore$ conjugate via orient pres/rev. homeo.

Suppose we have

$\phi : Homeo_{\mathbb{Z}} \mathbb{R} \to Homeo_+ \mathbb{R}$

(case) suppose T, generator of central $\mathbb{Z}$ has no fixed pts $\mathbb{R}/\phi(T) \cong S^1$

and $Homeo_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \cong Homeo_+ S^1$ acts on it so $\mathrm{Homeo}_{\mathbb{Z}}\mathbb{R}$ action conjugate to std

In general if you have fixed set

fixed

Step 3: now act on pieces in "std" way.

Now if we have $\phi: \mathcal{G}_\infty \to \mathrm{Homeo}_+ \mathbb{R}$

$$\begin{matrix} \mathcal{G}_\infty \\ \cup \\ \mathrm{Homeo}_{\mathbb{Z}}\mathbb{R} \end{matrix}$$

we know how $\circlearrowleft$ acts

<u>idea</u>: germ of $T: x \mapsto x+1$ is conjugate to germ of $x \mapsto ex$

$\log(e\exp(x)) = x+1$

in general $x \mapsto x+t$ conj (on germs at ∞) of $x \mapsto \eta_t x$

but <u>not</u> conjugate at homeo.

this gives contradiction

Questions:

1) $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$ is L.O.?
(Doesn't act on $\mathbb{R}$)

2) $\mathrm{Diff}(\mathbb{D}^2, \partial\mathbb{D})$ is L.O., does it act on $\mathbb{R}$?
(think can say no)

Big Question:

given G, what is the "simplest" space on which G acts?