3 Mfld Mutation and Heegaard Floer Hom

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Q: Does Heegaard Floer detect 3 mfld mutation?

Def: (Ozsváth-Szabó) Take M^3's closed, connected, orient.

$$M^3 \longrightarrow \underset{\text{abelian group}}{\widehat{HF}(M)} \longrightarrow \underset{\in \mathbb{N}}{rk\, \widehat{HF}(M)}$$

mutation ≡ cut along embedded surf, glue back together.

Four choices: mfld M, surf Σ, embedding $f: \Sigma \hookrightarrow M$,
gluing $\varphi: \Sigma \circlearrowleft$

take Σ closed, conn., orient., genus ≥ 2 and let gluing maps preserve orient.

Def: genus 2 hyperelliptic involution

 τ induces $-id\ H_1(\Sigma_2)$

Thm: Given $\varphi: \Sigma \circlearrowleft$ s.t. $\varphi \not\simeq id$ and $\varphi \not\simeq \tau$
$\exists\ M^3$ and $f: \Sigma \hookrightarrow M^3$ s.t. $rk\, \widehat{HF}(M) \neq rk\, \widehat{HF}(M_f^\varphi)$.

Remark: Mutation by elem of Torelli group pres. $H_*(M)$.

Pf steps: ① reformulate as MCG question
② reduce to pseudo-Anosov elem of Torelli group.
③ show $\exists$ elem can change rank.

MCG formulation

$$\mathrm{Mod}(\Sigma) = \mathrm{Diff}'(\Sigma) / \text{isotp.}$$

<u>Def</u>: $N = \{ [\varphi] \in \mathrm{Mod}(\Sigma) \mid$ for all M, f $\mathrm{rk}\, \widehat{HF}(M) = \mathrm{rk}\, \widehat{HF}(M_f^\varphi) \}$

Goal: $N = \{id\}$ or $\{id, \gamma\}$.

<u>Lemma</u>. N is a normal subgroup.

<u>Def</u>: Torelli group $\mathcal{I}(\Sigma) \triangleleft \mathrm{Mod}(\Sigma)$

maps that induce id $H_1(\Sigma)$

pseudo-Anosov elem. $[\varphi] \in \mathrm{Mod}(\Sigma)$

s.t. $\exists$ rep. $\varphi : \Sigma \circlearrowright$, pair of $\pitchfork$ measured foliations, const.

- ∞ order
- violent, understood

Facts: • $\mathcal{I}(\Sigma)$ is torsion free

- ∞ normal subgroups of $\mathrm{Mod}(\Sigma)$ contain pA elem.

<u>Thm</u> (Long) normal, non-trivial, non-central subgroups intersect non-trivially.

$$Z(\mathrm{Mod}(\Sigma)) = \begin{cases} \langle \gamma \rangle & g = 2 \\ \mathbb{1} & g > 2 \end{cases}.$$

Prop: If $N \cap \mathcal{I}(\Sigma)$ has no pA elem., then $N = \mathbb{1}$ or $\langle \iota \rangle$.

pf:

$N \cap \mathcal{I}(\Sigma)$

finite
$\Downarrow$
trivial
$\Downarrow$
N trivial or cent.
$\Downarrow$
$N = \mathbb{1}$ or $\langle \iota \rangle$

infinite + normal
$\Downarrow$
pA
$\Rightarrow\Leftarrow$

$\boxtimes$

pA elem. of $\mathcal{I}(\Sigma)$

idea: look mnfld w/ small rk $\widehat{HF}$

e.g. rk $\widehat{HF}(S^1 \times S^2) = 2$

Thm (Hedden-Ni): If M irred., $b_1 > 0$ and rk $\widehat{HF}(M) = 2$ then $M \cong \pm S_0^3(3_1)$

Prop: Let $\varphi : \Sigma \circlearrowleft$ be a pA and elem Torelli. $\exists\ f : \Sigma \hookrightarrow S^1 \times S^2$ s.t. rk $\widehat{HF}((S^1\times S^2)_f^{\varphi}) \neq 2$.

pf idea:

$f : \Sigma \hookrightarrow S^1 \times S^2$ Heegaard Splitting

$\exists\ M$ st. $(S^1 \times S^2)_f^{\varphi^n}$ mutant irred w/ no essential tori. $S_0^3(3_1)$ genus 1 knot implies $S_0^3(3_1)$ has essential torus $\therefore$ rk $\widehat{HF}((S^1\times S^2)_f^{\varphi^n}) \neq 2$ $\boxtimes$

Cor: $N = \mathbb{1}$ or $\langle \tau \rangle$.

Mutation by τ:

Thm: Mutation by τ is detected by $\widehat{HF}(M, s)$

spin^c str.

idea: $\widehat{HF}$ detect Thurston norm mutant knots
w/ diff. genus.

Thm: Mutation along non-sep. surface can be detected
by rk $\widehat{HF}$.

Thm: Mutation by elliptic inv. $-$ π
sep. torus w/ one side bounding a knot complement.
$\widehat{HF}(M, s)$ is pres.

Thm: (Moore - Starkston) infinite family of mutant
knot pairs where rk $\widehat{HFK}$ agree but
$\widehat{HFK}$ distinguish.