Symplectic 4-manifolds and Heegaard Floer Homology – Jen Hom – Dec 6, 2015

(joint with Tye Lidman)

always: closed, smooth, simply connected 4-mfds.

Geography Problem: Which signatures and Euler characteristics can be realised by symplectic 4-mfds?

perhaps interested in exotic 4-mfds.

eg. X = K3 Surface blown up at a point
simply connected, symplectic, odd intersection form. $b_2^+ = 3$, $b_2^- = 20$.

Freedman $\Rightarrow$ $K3 \# \overline{\mathbb{CP}}^2 \underset{\text{homeom}}{\cong} \#_2 \mathbb{CP}^2 \#_{20} \overline{\mathbb{CP}}^2$

Taubes $\Rightarrow$ mfd w/ $b_2^+ \geq 2$ and $\mathbb{CP}^2$ summand cannot be symplectic

$\Rightarrow$ X is an exotic $\#_2 \mathbb{CP}^2 \#_{20} \overline{\mathbb{CP}}^2$ (ie. homeom but not diffeom.)

What about when X is positive definite, $b_2^+ \geq 2$?

Donaldson + Freedman $\Rightarrow$ $X \underset{\text{homeom}}{\cong} \#_n \mathbb{CP}^2$

Taubes $\Rightarrow$ $\#_n \mathbb{CP}^2$, $n \geq 2$ is not symplectic.

$\Rightarrow$ Simply connected, symplectic, positive definite, 4-mfd with $b_2^+ \geq 2$ is an exotic $\#_n \mathbb{CP}^2$

Def^n: A manifold is geometrically simply connected if it admits a handlebody decomposition with no 1-handles

Remark: Geometrically simply connected $\Rightarrow$ simply connected. ($\Leftarrow$ is unknown for closed 4-mfds)

Thm (H-Lidman) Let X be a closed, geometrically simply connected 4-mfd with $b_2^+ \geq 2$. If X is positive definite, then X is not symplectic.

Remark: (1) More generally, does every simply connected 4-mfd admit a perfect Morse function?
(ie. #crit points $= \sum b_i$)

(2) A simply connected, positive definite, symplectic 4-mfd with $b_2^+ \geq 2$ would provide a counterexample.

Cor : (TJ Li) If a simply connected closed 4-mfd has a positive definite intersection form, a perfect Morse function, and a symplectic structure, then it is diffeomorphic to $\mathbb{CP}^2$.

Pf of Cor : Thm $\Rightarrow b_2^+ = 1$.

X = 0-handle $\cup$ +1 framed 2-handle attached along K $\cup$ 4-handle.

No 1- or 3-handles $\Rightarrow S^3_{+1}(K) \cong S^3$.

Gordon-Luecke $\Rightarrow K$ = unknot.

standard handlebody decomposition for $\mathbb{CP}^2$

$\square$

Pf of Thm relies on Heegaard Floer closed 4-mfd invt $\Phi_{X,s}$.

Thm : (Ozsváth - Szabó) If X is symplectic and $b_2^+ \geq 2$, then $\Phi_{X,s} \neq 0$ for some spin^c structure s.

What is $\Phi_{X,s}$?

Heegaard Floer is a (3+1) TQFT.

closed 3-mfd $Y \longrightarrow HF^{\pm}(Y) = \bigoplus_{t \in Spin^c(Y)} HF^{\pm}(Y,t)$

graded $\mathbb{F}[U]$-module, $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, U = formal variable.

Spin^c cobordism (W,s) between (Y_1,t_1) and (Y_2,t_2) $\rightsquigarrow F^{\pm}_{W,s} : HF^{\pm}(Y_1,t_1) \to HF^{\pm}(Y_2,t_2)$ homomorphism

for Y = rational homology sphere

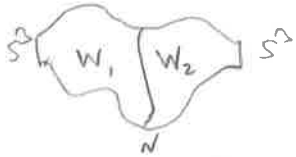
U-torsion

$HF^-(Y,t) \cong \mathbb{F}[U] \oplus HF_{red}(Y,t)$; $HF^+(Y,t) \cong \mathbb{F}[U^{-1}] \oplus HF_{red}(Y,t)$.

(from exact triangle
$$\begin{tikzcd} HF^-(Y,t) \arrow[r] & HF^{\infty}(Y,t) = \mathbb{F}[U,U^{-1}] \arrow[dl] \\ HF^+(Y,t) \arrow[u] & \end{tikzcd}$$
)

The d-invariant is the grading of $1 \in \mathbb{F}[U]$. This is an invariant of homology cobordism.

Let X be a closed 4-mfd, $b_2^+ \geq 2$. Remove 2 4-balls, cut along N^3:



$(W = X \setminus (D^4 \sqcup D^4))$

N is an admissible cut if

- $b_2^+(W_i) \geq 1$
- $\delta H^1(N;\mathbb{Z}) = 0 \in H^1(W, \partial W; \mathbb{Z})$

δ is coboundary map in relative M-V.

$$\begin{tikzcd}
F^-_{W_1, s|_{W_1}} : HF^-(S^3) \arrow[r] \arrow[dr] & HF^-(N, s|_N) \\
& HF_{red}(N, s|_N) \arrow[u] \arrow[dr] \\
F^+_{W_2, s|_{W_2}} : HF^+(N, s|_N) \arrow[ur] \arrow[rr] & & HF^+(S^3)
\end{tikzcd}$$

So have $F^{mix}_{W,s} : HF^-(S^3) \to HF^+(S^3)$.

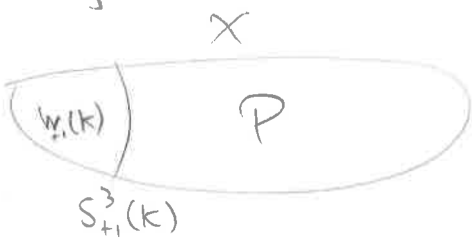
So we say $\Phi_{X,s} = 0 \iff F^{mix}_{W,s} = 0$.

trace of surgery: $W_n(K) = B^4 \cup$ n-framed 2-handle along K.
$\partial W_n(K) = S^3_n(K)$.

Prop^n (H.-Lidman) If $d(S^3_1(K)) = 0$, then $F^-_{W_n(K)} : HF^-(S^3) \to HF^-(S^3_n(K))$ is zero for all spin^c structures, any $n > 0$.

Sketch of Pf of Thm

(1) X as in thm has a handlebody decomposition with no 1-handles, and linking matrix for 2-handles $\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$.

(2) 

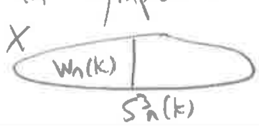
$b_2^+(W_{+1}(k)) = 1$, $b_2^+(P) = b_2^+(X) - 1 \geq 1$.
+ check other condition $\Rightarrow S^3_1(k)$ is an admissible cut.

$S^3_{+1}(k)$ bounds $W_{+1}(k)$, positive definite $\Rightarrow d(S^3_{+1}(k)) \leq 0$.
$-S^3_{+1}(k)$ bounds P, positive definite $\Rightarrow d(-S^3_{+1}(k)) = -d(S^3_{+1}(k)) \leq 0$
$\Rightarrow d(S^3_{+1}(k)) = 0$.

(3) Prop^n $\Rightarrow \Phi_{X,s} = 0$ for all spin^c structures s, so not symplectic. $\square$

Thm (H-Lidman) Let $K \subseteq S^3$ be a knot with $d(S^3_{+1}(K)) = 0$. Then for $n > 0$ and any contact structure on $S^3_n(K)$, the trace $W_n(K)$ cannot be a symplectic filling for $S^3_n(K)$.

Pf Sketch: By contradiction: if $W_n(K)$ symplectically fills $S^3_n(K)$, then $W_n(K)$ embeds in symplectic X with $b_2^+ \geq 2$. (Etnyre, Eliashberg).



Prop^n $\Rightarrow \Phi_{X,s} = 0$, contradiction. $\square$