Lightning Talks II
Tech Topology Conference

December 6, 2015

Penner's conjecture

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December 5, 2015

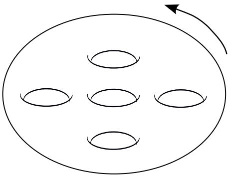
Mapping class groups

S_g – closed orientable surface of genus g

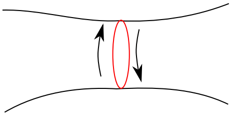
$\mathrm{Mod}(S_g) = \mathrm{Homeo}^+(S_g)/\text{isotopy}$

Theorem (Nielsen–Thurston classification)

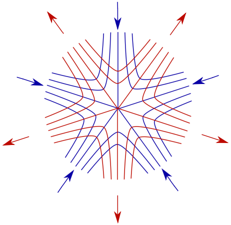
Every $f \in \mathrm{Mod}(S_g)$ is either finite order, reducible or pseudo-Anosov.



Finite order



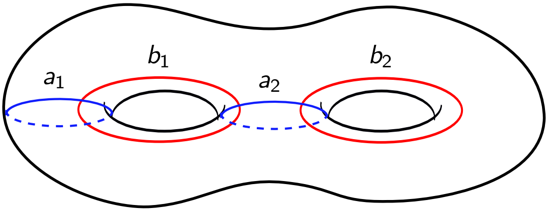
Dehn twist



Pseudo-Anosov

Penner's construction

$A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ filling multicurves. Any product of T_{a_i} and $T_{b_j}^{-1}$ containing each of these Dehn twists at least once is pA.



Conjecture (Penner, 1988)

Every pseudo-Anosov mapping class has a power arising from Penner's construction.

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Theorem (Shin-S.)

Penner's conjecture is false for $S_{g,n}$ when $3g + n \geq 5$.

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Sketch of the proof.

1. Every Dehn twist in Penner's construction can be described by a matrix. $\Longrightarrow$ Every Penner pA can be described by a matrix.

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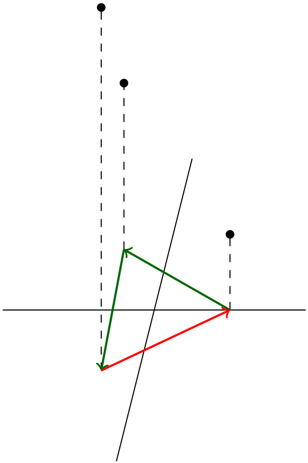
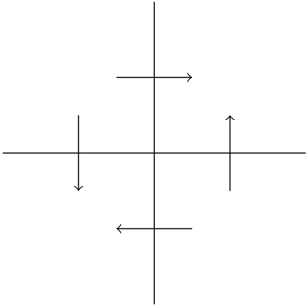
1. Every Dehn twist in Penner's construction can be described by a matrix. $\implies$ Every Penner pA can be described by a matrix.
2. Need to show that such matrices cannot have eigenvalues on the unit circle.
3. I.e., they cannot act on 2-dimensional invariant subspaces by rotations.
4. Construct a height function that is increasing after every iteration.

□



Example

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



An increasing height function: $h(x, y) = xy$.

Thank you!

Optimal cobordisms between knots

David Krcatovich

Rice University

Joint with Peter Feller (Boston College)

6th December 2015

The "cobordism distance" between two knots K and J is

$$d(K,J) = \min\left\{g(\Sigma) \mid \Sigma \text{ sm. emb. in } S^3 \times [0,1], \partial\Sigma = K \sqcup rJ\right\}$$

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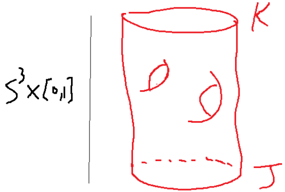

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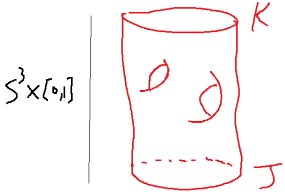

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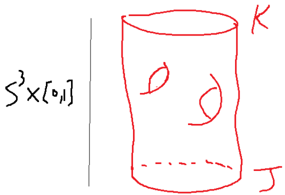

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triangle inequality: $d(K,J) \geq |g_4(K) - g_4(J)|$

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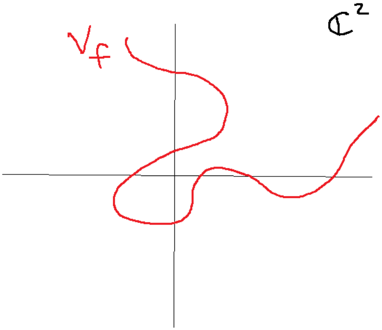
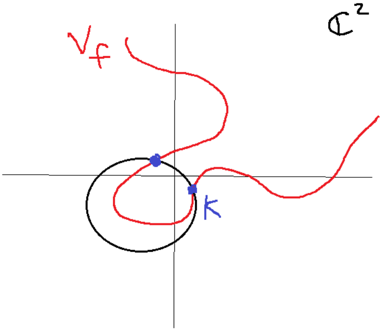
Q: When do optimal cobordisms exist?

Suppose V_f is the zero set of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$.



Suppose V_f is the zero set of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$.
Then $K = V_f \cap S^3_r$ is generically a knot or link in S^3.
And K bounds the surface $\Sigma_K = V_f \cap B^4_r$



(Rudolph, Boileau–Orevkov): K is a "quasipositive" knot

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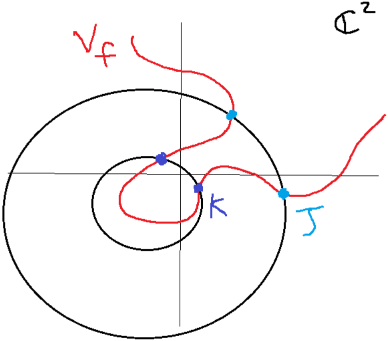

Theorem

Suppose K and J are quasipositive knots; K has braid index m, and J is the closure of a QP n-braid which contains k full twists. Then

$$d(K, J) \geq g_4(K) - g_4(J) + k(n - m).$$

Corollary

If an algebraic cobordism exists between two knots, the one with bigger genus cannot have smaller braid index.

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Corollary

If a knot K is the closure of a quasipositive n-braid with a full twist, then n is the braid index of K.

Proof of theorem uses Upsilon invariant from Heegaard Floer homology (Ozsváth - Stipsicz - Szabó), and the fact that for quasipositive knots, the slice–Bennequin inequality is sharp

Thank you!

Nontrivial examples of bridge trisection of knotted surfaces in S^4

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December 6, 2015

Definitions

Definition (by J.Gay and Kirby)

Let X be a closed, connected, oriented, smooth 4-manifold. A (g, k_1, k_2, k_3)-*trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$, such that

1. $X_i \equiv \natural^{k_i}(S^1 \times B^3)$,
2. $H_{ij} = X_i \cap X_j$ is a genus g handlebody, and
3. $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface of genus g

Definition

The 0-*trisection* of S^4 is a decomposition $S^4 = X_1 \cup X_2 \cup X_3$, such that

1. X_i is a 4-ball,
2. $B_{ij} = X_i \cap X_j = \partial X_i \cap \partial X_j$ is a 3-ball and
3. $\Sigma = X_1 \cap X_2 \cap X_3 = B_{12} \cap B_{23} \cap B_{31}$ is a 2-sphere.

Definitions

A <u>*trivial c-disk system*</u> is a pair $(X, \mathcal{D})$ where X is a 4-ball and $\mathcal{D} \subset X$ is a collection of c properly embedded disks $\mathcal{D}$ which are simultaneously isotopic into the boundary of X.

Definition (by J. Meier and A. Zupan)

A $(b; c_1, c_2, c_3)-$ *bridge trisection* $\mathcal{T}$ of a knotted surface $\mathcal{K} \subset S^4$ is a decomposition of the form $(S^4, \mathcal{K}) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_3, \mathcal{D}_3)$ such that

1. $S^4 = X_1 \cup X_2 \cup X_3$ is the standard genus zero trisection of S^4,
2. $(X_i, \mathcal{D}_i)$ is a trivial c_i-disk system, and
3. $(B_{ij}, \alpha_{ij}) = (X_i, \mathcal{D}_i) \cap (X_j, \mathcal{D}_j)$ is a $b-$strand trivial tangle.

Theorem (Meier, Zupan)

Every knotted surface $\mathcal{K}$ in S^4 admits a bridge trisection.

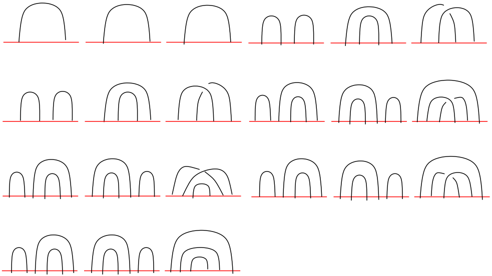

Figure: The seven standard bridge trisections:
$(1,1) : S^2, (2,1), (2,1) : \mathbb{RP}^2, (3,1), (3,1), (3,1), (3,1) : \mathbb{T}^2$

Any trisection obtained as the connected sum of some number of these standard trisections, or any stabilization thereof, will also be called *standard*.

Theorem (Meier, Zupan)

Every knotted surface $\mathcal{K}$ with $b(\mathcal{K}) \leq 3$ is unknotted and any bridge trisection of $\mathcal{K}$ is standard.

Theorem (Meier, Zupan)

Any two bridge trisections of a given pair $(S^4, \mathcal{K})$ become equivalent after a sequence of stabilizations and destabilizations.

Theorem (Meier, Zupan)

Any two tri-plane diagrams for a given knontted surface are related by a finite sequence of tri-plane moves. (Reidemeister move, mutual braid transpositions, stabilization/destabilization.)

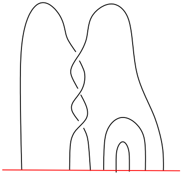

Figure: A (4, 2)-bridge trisection: Spun Trefoil

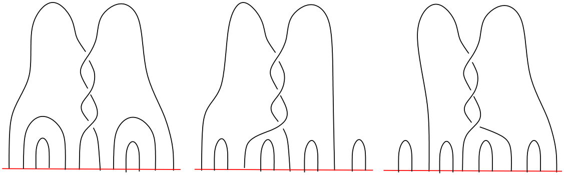

Figure: A (6, 2)-bridge trisection: Spun Torus from Trefoil Knot

Propostion[Meier, Zupan]
If $\mathcal{K}$ is orientable and admits a $(b; c_1, 1, c_3)$-bridge trisection, then $\mathcal{K}$ is topologically unknotted.

Question
Can a surface admitting a $(b; c_1, 1, c_3)$-bridge trisection be smoothly knotted?.

Interesting examples

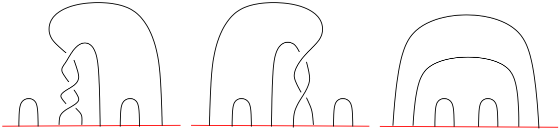

Figure: A $(4, 1)$-bridge trisection: $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

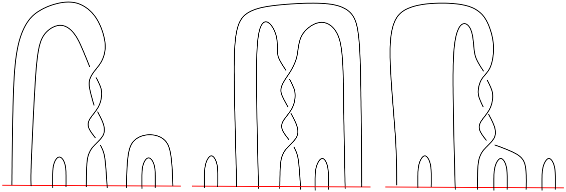

Figure: A $(5, 1, 2, 2)$-bridge trisection: $\mathbb{T}^2$ or $\mathbb{RP}^2 \# \mathbb{RP}^2$

Link maps in the 4-sphere

Tech Topology Conference, Georgia Tech 2015

Ash Lightfoot

Indiana University

December 6, 2015

Talk Outline / Result

1. Link maps, link homotopy

2. Kirk's σ invariant

3. Open problem: does $\sigma = 0 \Rightarrow$ link nullhomotopic?

4. Result: $\sigma = 0 \Rightarrow$ get "clean" Whitney discs

$\Rightarrow$ $\boxed{+\text{ve evidence to affirmative answer}}$

Classifying link maps

Link map:

$$f : S^p_+ \cup S^q_- \to S^n, \qquad f(S^p_+) \cap f(S^q_-) = \varnothing$$

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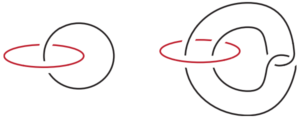
$LM^n_{p,q}$ = set of link maps $S^p_+ \cup S^q_- \to S^n$ mod link homotopy

Classifying link maps

$$LM^3_{1,1} \xrightarrow[\cong]{\text{linking } \#} \mathbb{Z}$$



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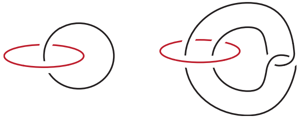

$$LM^4_{2,2} \xrightarrow[\text{(Kirk)}]{(\sigma_+, \sigma_-)} \mathbb{Z}[t] \oplus \mathbb{Z}[t]$$

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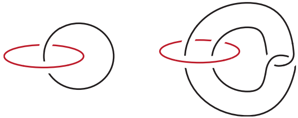

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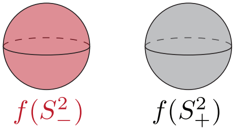
Q: Does $\sigma(f) = (0,0) \Rightarrow f$ link homotopically trivial?

(Trivial link map: two embedded 2-spheres bounding disjoint 3-balls)



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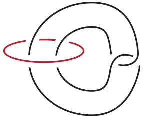

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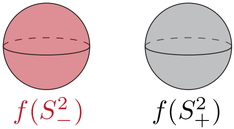

Does $\sigma(f) = (0,0) \Rightarrow f$ link homotopic to embedding?

(Bartels-Teichner '99)

Kirk's invariant $\sigma = (\sigma_+, \sigma_-)$

Given $f : S^2_+ \cup S^2_- \to S^4$,

$\sigma_\pm(f)$ obstructs homotoping $f|_{S^2_\pm} : S^2_\pm \to S^4 \setminus f(S^2_\mp)$ to embedding via the "Whitney trick":

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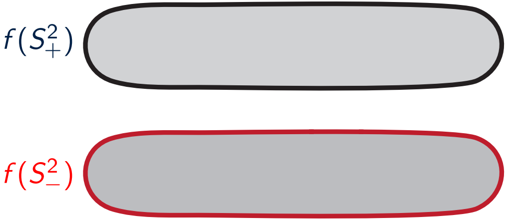

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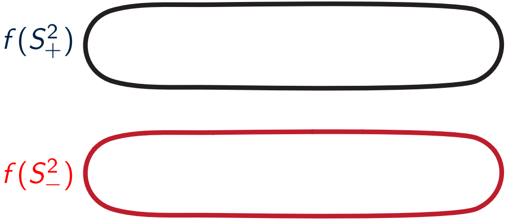

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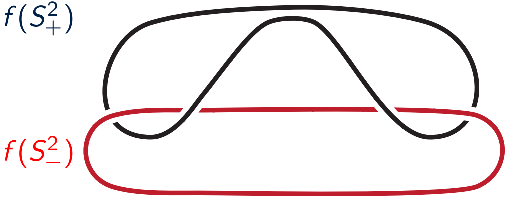

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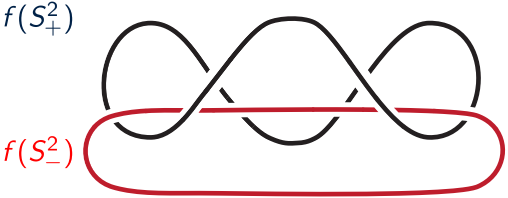

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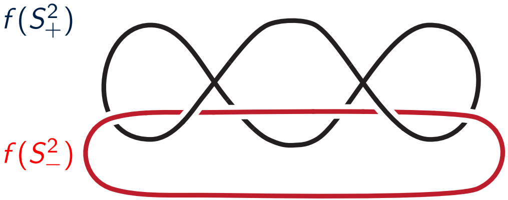

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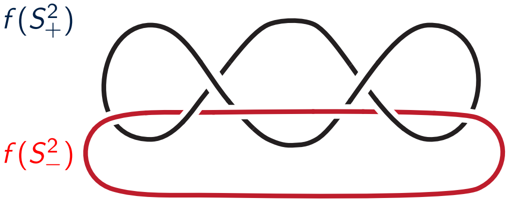

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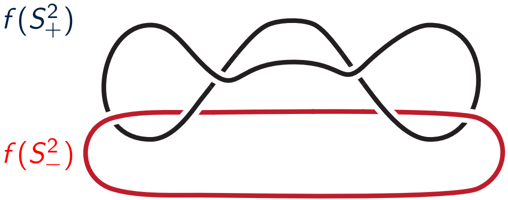

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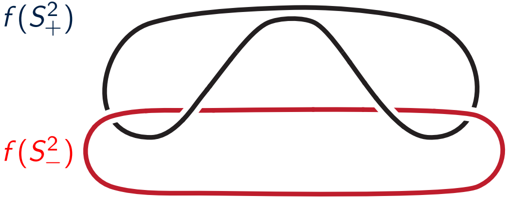

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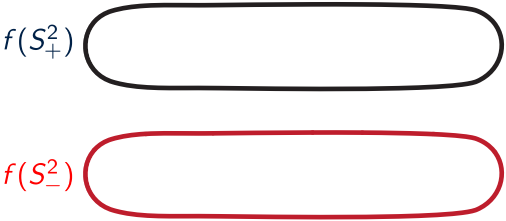

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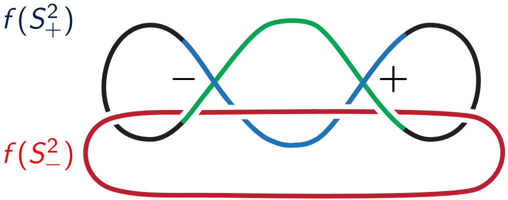

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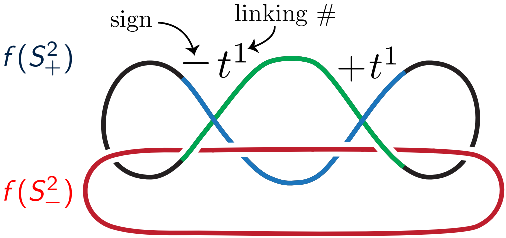

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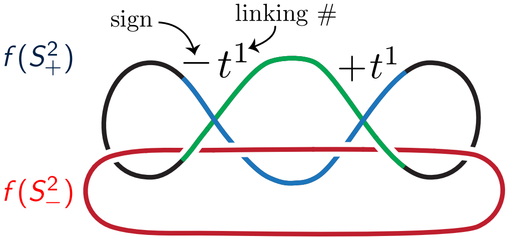

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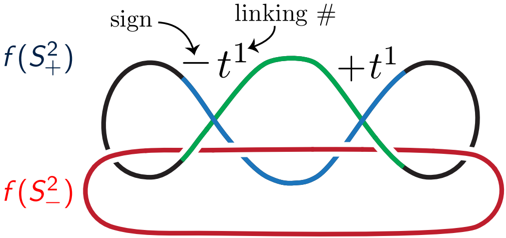

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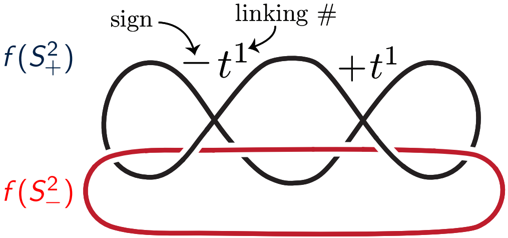

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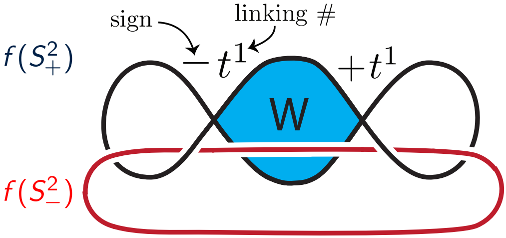

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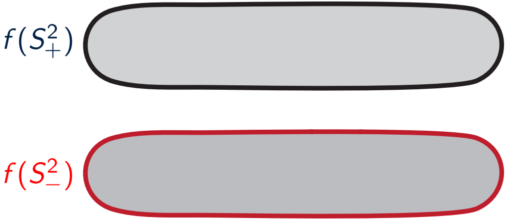

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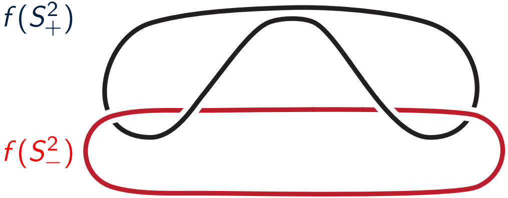

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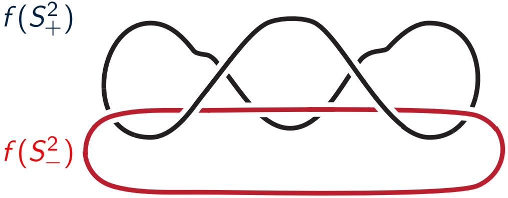

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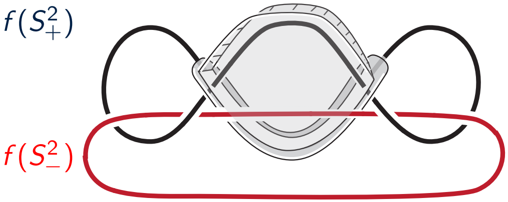

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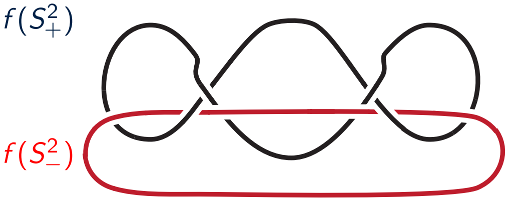

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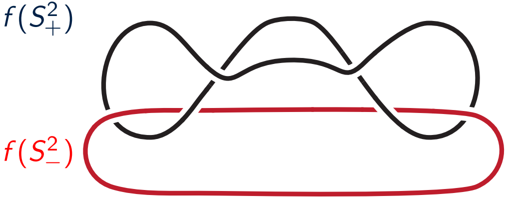

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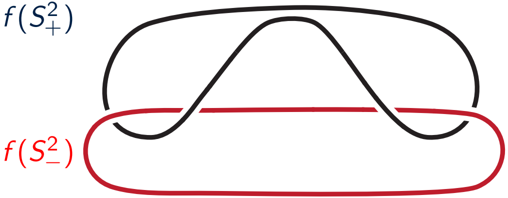

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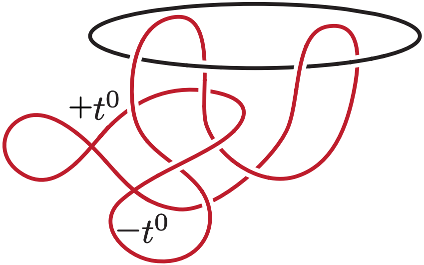
Is σ the complete obstruction to embedding?

That is, is the existence of Whitney discs alone enough to embed?

$f(S^2_+)$

$f(S^2_-)$



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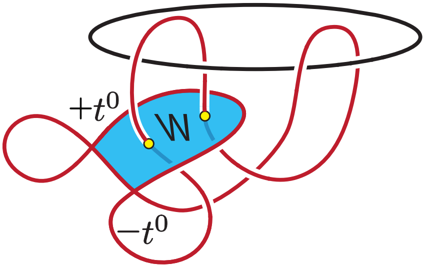

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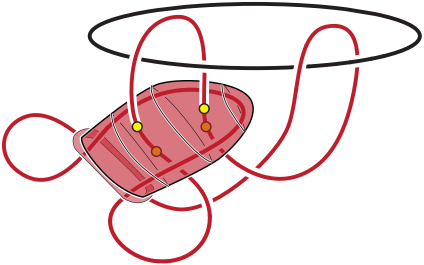

$f(S^2_-)$

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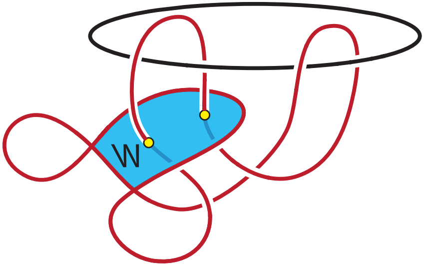

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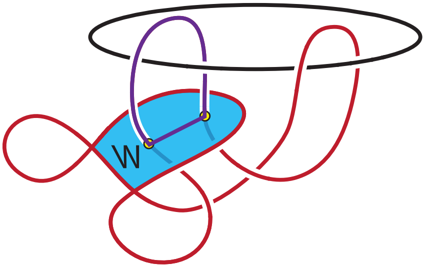

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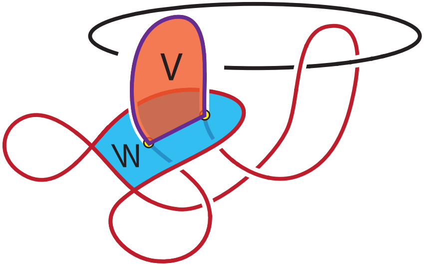

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Solution: try to form a "secondary" Whitney disc V

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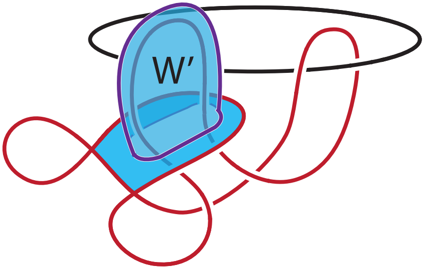

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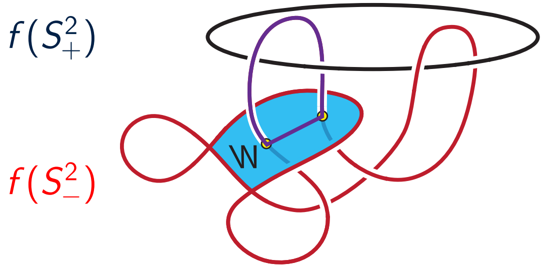

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Solution: try to form a "secondary" Whitney disc V

$\rightsquigarrow$ define a "secondary" invariant that obstructs this

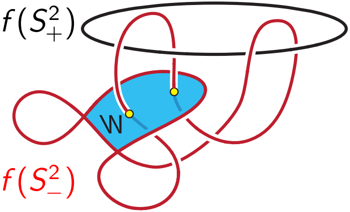
(Li '97) $\omega : \ker \sigma \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Result: $\sigma(f) = (0,0) \Rightarrow \omega(f) = (0,0)$

Theorem (L.)

If $f : S^2_+ \cup S^2_- \to S^4$ is a link map with **both** $\sigma_+(f) = 0$ *and* $\sigma_-(f) = 0$*, then:*

(after a link homotopy) each component $f_\pm$ can be equipped with framed, immersed Whitney discs whose interiors are disjoint from $f(S^2_\pm)$.



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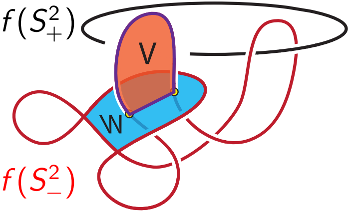

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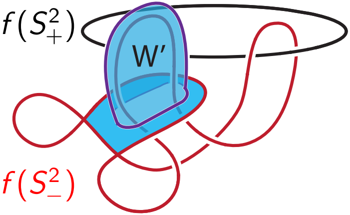

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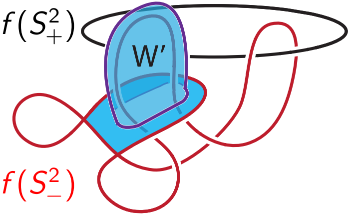
- **Still open**: Is $\sigma : LM^4_{2,2} \to \mathbb{Z}[t] \oplus \mathbb{Z}[t]$ the complete obstruction?



Other results and questions

- **Theorem:** Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map. After a link homotopy, the Schneiderman-Teichner τ-invariant applied to $f|S^2_+$ is $\mathbb{Z}_2$-valued and vanishes if $\sigma(f) = (0, 0)$.
- New proof of the image of σ.
- **Theorem:** There is a link map f with $\sigma_-(f) = 0$, $\omega_-(f) = 0$ but $\sigma_+(f) \neq 0$.
- **Question:** is $LM^4_{2,2}$ an abelian group with respect to connect sum?
- **Question:** Is σ injective?
- **Question:** Can a secondary invariant for $LM^4_{2,2,2}$ be defined? Is it stronger than σ?
- **Question:** Can ω be related to invariants of links?

Character Varieties of 2-Bridge Knot Complements

Leona Sparaco

Florida State University