# Discrete Measured Foliations

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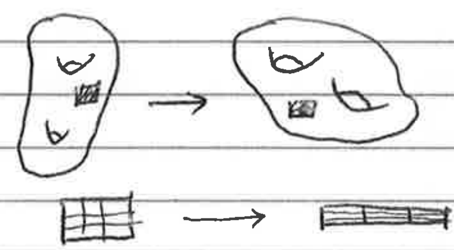
(1)

Goal: Finding "nice" measured foliations
e.g. harmonic discrete measured foliations

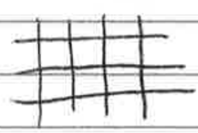
Will give a definition, but with applications coming!

Motivation: Compute Teichmüller maps in given homotopy class
(minimal conformal constant)



First, understand measured foliations.

I. Harmonic discrete functions

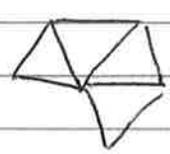


function defined on vertices.

harmonic means $f(v) =$ avg of $f$ on neighboring vertices.

Too rigid, so:

instead, <u>weighted</u> average.



$f(v) =$ weighted average of $f$ on neighboring vertices

Can extend to whole manifold (say, $\mathbb{R}^2$) linearly on the triangles.

Harmonic functions minimize

$$\int_{\mathbb{R}^2} |\nabla f|^2 = \sum_e K(e)\left(f(v_1) - f(v_2)\right)^2$$

$$K(e) = \tfrac{1}{2}\left(\cot(\alpha) + \cot(\beta)\right)$$

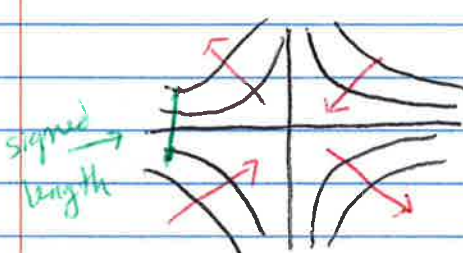
positive if $\alpha + \beta < \pi$

earliest reference: (MacNeal, 1949), regarding building resistor networks.

Use these to approximate the differential equations.

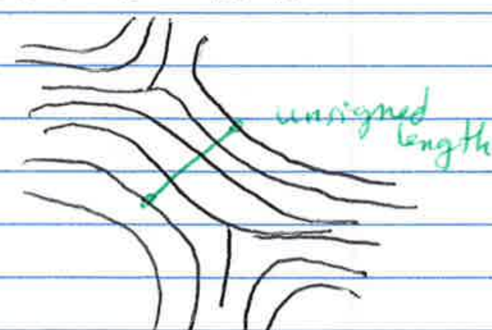
## II. Measured foliations

For a function, its level set might look like:




"know which way is uphill."
- signed length of transverse arcs.
- captures homology info.
- like electrical network.

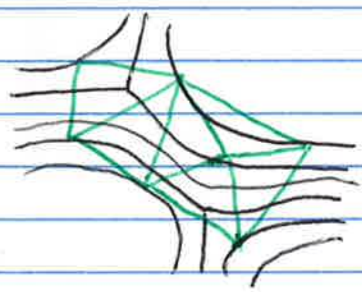
For a measured foliation
- unsigned length of transverse arcs
- captures homotopy info
- like a springs network




## III. Discrete measured foliation

Overlay a polygonal decomposition.
Edges have an unsigned length.

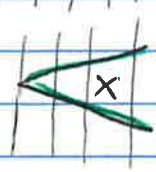
ℓ **Def.** For $\Sigma$ a closed surface, with $\Gamma \subset \Sigma$, filling ~~[illegible]~~, a <u>length structure</u> on $\Gamma$ is a choice of length for each edge of $\Gamma$



going uphill or downhill doesn't make sense, but "opposite" directions does — going back over leaves you've just crossed.

So at each corner, we can record whether we've turned around.

o: leaf running into corner



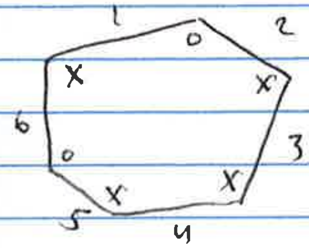
x: no leaf into corner

Notice: $k$-pronged singularities, $k \geq 3$

C **Def:** A <u>corner structure</u> on $\Gamma$ is a choice of X or O for each corner of $\Gamma$ so that
1) $\geq 2$ O's around each vertex
2) $\geq 2$ X's around each face

**Def.** A <u>discrete measured foliation</u> is ~~$\ell, c$~~ a measured foliation along with a length and a corner structure.

$\ell_1 + \ell_2 = \ell_3 \qquad \ell_4 \leq \ell_2 + \ell_5$ ← triangle inequality




groupings

$$\left.\begin{array}{c} l_1 + l_2 \\ l_3 \\ l_4 \\ l_5 + l_6 \end{array}\right\}$$ these satisfy a triangle inequality.

~~Def f face of $\Gamma$~~

Lemma about index:

$$\sum_f \mathrm{ind}(f) + \sum_v \mathrm{ind}(v) = -2\chi(\Sigma)$$

IV Harmonic discrete measured foliation

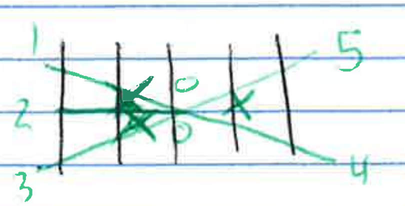
Fix constants $k(e)$ on each edge of $\Gamma$ (eg from cotan formula)

Want: discrete measured foliation $MF(l,c)$ representing a Whitehead equivalence class of a MF $F$ on $\Sigma$ minimizing

$$E(l,c) = \sum_e k(e)\, l(e)^2$$

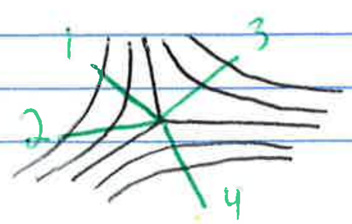
energy. think: stretched rubber bands

Ex.




$\rightsquigarrow$ $k_1 l_1 + k_2 l_2 + k_3 l_3 = k_4 l_4 + k_5 l_5$

net force on point should be zero.




$\rightsquigarrow$ $k_3 l_3 \le (k$

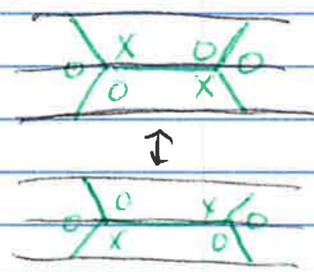
Whitehead equivalence



Def: A discrete MF $(\ell, c)$ is harmonic if $k_i \ell_i$ satisfy triangle inequalities when grouped between O's at each vertex

Thm (90%)

Given $\Gamma \subset \Sigma$ filling graph w/ elastic constants $k(e)$, then for any equivalence class of measured foliations $F$, there is a harmonic discrete MF $(\ell, c)$ representing $F$. $\ell$ is unique, $c$ is unique up to switching markings around zerO-length edges.



(Expected)

V Applications

1. for geometry, /computer graphics a fine triangulation
   - approximate harmonic MF
   - compute Teichmüller map
2. for topology, coarse

uniform set of coor

maybe better than other known coordinates

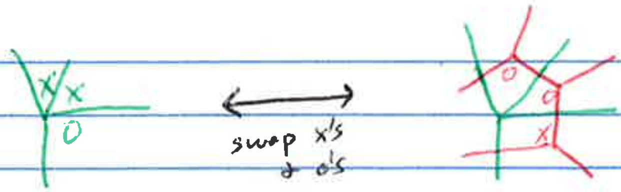
computations in $Mod(S_g)$.

Also: harmonic conjugate MF (holomorphic quadratic differentials)

Duality: Start with a graph $\Gamma$

$$\Gamma \longleftrightarrow \Gamma^*$$

$$k(e) \longleftrightarrow k(e^*) = \frac{1}{k(e)}$$




$$\ell(e) \longleftrightarrow \ell(p^*) = \kappa(e)\,\ell(e)$$

discrete MF condition $\longleftrightarrow$ harmonic condition

harmonic discrete MF $\longleftrightarrow$ harmonic discrete MF

Likely-to-be-difficult-to-prove convergence:
Stephenson's conjecture:
electrical networks
↳ extract Riemann maps
Proved by Hersonsky 2015