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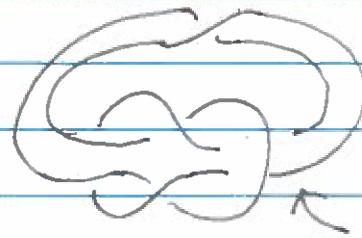
TTC '16

Annular Khovanov-Homology, braids & cobordisms

(Joint with Licata & Wehrli)

Slice-Ribbon Conjecture (Fox '60s)

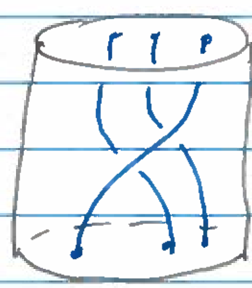
if $K \subseteq S^3$ bounds a disk B^4
then K bounds a ribbon-immersed disk in B^4 ($\Leftrightarrow$ K bounds a disk in B^4 with no maxima)



Stevedore's knot

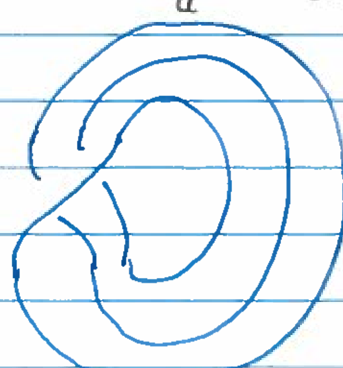
just ribbon singularities

Question: Get insight into SRC by studying links as closed braids?



$\beta \subseteq D \times I$

closure



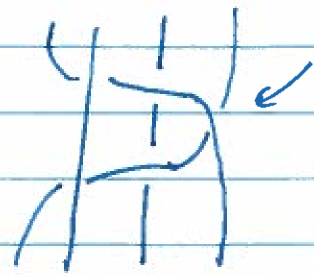
$\hat{\beta} \subseteq$ solid torus $\subseteq S^3$

Alexander/Markov represent all knots via closed braids

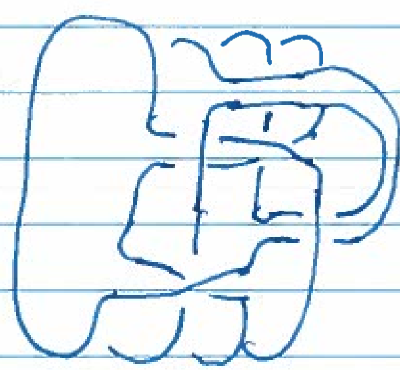
Theorem (L. Rudolph)

let S be an oriented ribbon-immersed surface (not nec. a disk) in S^3, bounded by a link L. Then S is isotopic to a braided banded surface

i.e. L is realized as $\hat{\beta}$ for $\beta = \prod_{j=1}^{c} \underbrace{w_j \sigma_{i_j}^{\pm 1} w_j^{-1}}_{\text{band.}}$



band



stevedore's knot as banded surface.

taking
- 0-handle (disk) for each strand of braid β.
- 1-handle (band) for each term in band presentation

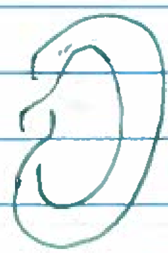
$$\Rightarrow \chi(C) = n - c$$

<u>Corollary</u>: If K is ribbon then $K = \hat{\beta}$ for a braid β of index n, band rank $(n-1)$

<u>idea</u>: find a ribbon obstruction by finding a braid conjugacy class invariant bounding band rank with nice properties under Markov de/stabilization

don't have all of this yet.

(Annular) Khovanov-Lee homology ($\mathbb{F} = \mathbb{Q}$)



$\xrightarrow{\text{Khovanov}} (\mathcal{C}, \partial)$ graded: i = hom. grading, j = quantum grading

chain complex

$L \subseteq S^3$

$\longrightarrow Kh(L) = H_*(\mathcal{C}, \partial)$

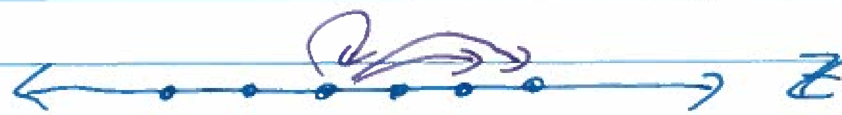
$\xrightarrow{\text{Lee}} (\mathcal{C}, \partial + \Phi)$ deformation of Khovanov differential

$\longrightarrow Lee(L) = H_*(\mathcal{C}, \partial + \Phi)$

$\cong \mathbb{Q}^{2^\ell}$ ℓ = # of components of L.

$\cong \mathrm{Span}\{[\mathfrak{s}_o] \mid o \text{ an or}^n \text{ on } L\}$

Rasmussen: $\mathcal{C}^{Lee}$ is $\mathbb{Z}$-filtered



$\partial + \Phi$ is monotonic with respect to j-grading.

$\Rightarrow$ Can define j-grading of homology classes of $\mathcal{C}^{Lee}$.

K a knot $\Rightarrow s(K) = j[\mathfrak{s}_o] + 1$
$\leq 2g_4(K)$
$\uparrow$ 4-genus.

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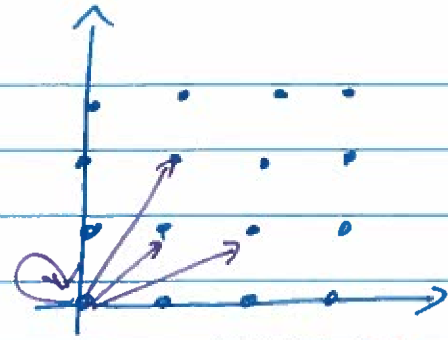
consider $L \subseteq A \times I$ thickened annulus
L a braid closure

Asaeda-Przytycki-Sikora, L. Roberts
get extra grading k — $sl(2)$-weight space grading

We can: $\mathcal{C}^{Lee}(\hat{\beta} \subseteq A \times I)$
is $(\mathbb{Z} \oplus \mathbb{Z})$-filtered.



get family of $\mathbb{R}$-filtered complexes (by projecting to lines of given slope)

$\Rightarrow$ family of annular Rasmussen invariants, one for each $t \in [0,2]$

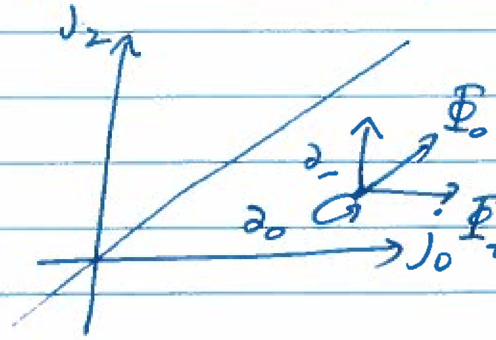
there are braid conjugacy class invariants except at $t = 0, 2$ where agree with Rasmussen

Lemma: annular Lee differential $\partial + \Phi$ splits into 4 (j,h)-homogeneous terms: $(\partial_0 + \partial_1) + (\Phi_0 + \Phi_1)$

$(j,k) = (0,0)\ (0,-2)\ (4,0)\ (4,2)$

note: if $j_t = j - tk$ $\quad t \in [0,2]$

then $(\partial + \Phi)$ is non-decreasing with respect to j_t-grading def^n: let $\beta \in B_n$, $d_t(\hat{\beta} \subseteq A \times I) := j_t[\mathfrak{s}_0]$



Lee complex symmetric w.r.t. j_0, j_2 grading

Theorem (G-Licata-Wehrli)

let $\beta \in B_n$ have writhe w, $t \in [0,2]$

(1) $d_t(\hat{\beta})$ is braid conjugacy invariant

(2) $d_{1-t}(\hat{\beta}) = d_{1+t}(\hat{\beta}) \quad t \in [0,1]$

(3) $d_0(\hat{\beta}) = d_2(\hat{\beta}) = s(\hat{\beta}) - 1$

(4) $d_1(\hat{\beta}) = w$

(5) $d_t(\hat{\beta})$ is piecewise linear with finitely many slopes

$m_t(\hat{\beta}) \in \{-n, -n+2, \dots, n-2, n\}$

right hand slope at t

(6) let F be a braid-orientable braid cobordism from $\hat{\beta}_0$ to $\hat{\beta}_1$ for which every component of F has a boundary component in $\hat{\beta}_0$

if F has $\begin{cases} a_1 & \text{odd index critical points} \\ b_0 & \text{even index critical points} \end{cases}$

then $d_t(\hat{\beta}_0) - d_t(\hat{\beta}_1) \le a_1 - b_0 |1-t|$

(Motivated by Ozsváth-Stipsicz-Szabó's Upsilon invariant as reinterpreted by Livingston)

<u>Corollary of (6):</u>

let $\beta \in B_n$

$$\max_{t \in [0,2]} \left| d_t(\hat{\beta}) - d_t(\hat{\mathbb{1}}_n) \right| \le r(\hat{\beta})$$

band rank.

<u>Example:</u> 10_{132} (Ng-Ozsváth-Thurston, Khandhawit-Ng, Hubbard-Saltz)

$$\sigma_3 \sigma_2^{-2} \sigma_3^{2} \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{2}$$

$w = 3$

$\tau(K) = 1$

$s(K) = 2$