Lightning Talks II
Tech Topology Conference

December 11, 2016

The Milnor Fiber of the Braid Arrangement

Michael Dougherty

December 10, 2016
Tech Topology Conference

University of California, Santa Barbara

$\textsc{Sym}_n$ acts naturally on $\mathbb{C}^n$ by permuting coordinates.

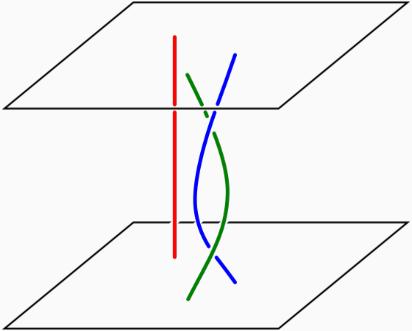
Each transposition $(i\ j)$ fixes the hyperplane $z_i = z_j$;
Each product of transpositions fixes an intersection of hyperplanes.

These hyperplanes form the *braid arrangement*

$$\mathcal{A}_n = \{\vec{z} \in \mathbb{C}^n \mid z_i = z_j \text{ for some } i, j\}$$

Goal: Understand the topology of $\mathbb{C}^n - \mathcal{A}_n$

Motivation: $\pi_1(\mathbb{C}^n - \mathcal{A}_n)$ is the *pure braid group*.



This is a (pure) braid!

How about the homology?

The homology of $\mathbb{C}^n - \mathcal{A}_n$ is well known (Arnol'd '69):

1. Betti numbers: coefficients of $(1+t)(1+2t)\cdots(1+(n-1)t)$
2. $H_*(\mathbb{C}^n - \mathcal{A}_n)$ is torsion-free.

What about other hyperplane arrangements?

Orlik-Solomon ('80): For any complex hyperplane arrangement $\mathcal{A}$,

1. We can compute $H_*(\mathbb{C}^n - \mathcal{A})$
2. $H_*(\mathbb{C}^n - \mathcal{A})$ is torsion-free.

But this is not the whole story...

Milnor ('68): For any complex hyperplane arrangement $\mathcal{A}$, $\mathbb{C}^n - \mathcal{A}$ is a fiber bundle over S^1.

1. Can we compute the homology of the Milnor fiber?
2. Is the homology of the Milnor fiber torsion-free?

Problem: Orlik-Solomon doesn't help!

Homology is too hard...unknown even for the braid arrangement.

Conjecture (Randell '11): Every Milnor fiber has torsion-free homology.

$\longrightarrow$ counterexample by Denham-Suciu in '14

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Theorem (D-McCammond '16): The Milnor fiber of the braid arrangement $\mathcal{A}_6$ has torsion in its homology.

The prism manifold realization problem

Faramarz Vafaee
joint with W. Ballinger, C. Hsu, W. Mackey, Y. Ni, and T. Ochse
(Summer Undergraduate Research Fellowship (SURF) program)

California Institute of Technology

December 2016

The spherical manifold realization problem

- Every closed three-manifold can be obtained by performing surgery on a link in S^3. (Lickorish–Wallace)
- Focus: Which closed 3–manifolds with finite fundamental groups can be realized by surgeries on nontrivial knots in S^3?

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  Spherical manifolds
- A spherical 3-manifold Y, with $G = \pi_1(Y)$, falls into one of the following five types, depending on the structure of $G/Z(G)$:
  C or cyclic, **D** or dihedral, **T** or tetrahedral, **O** or octahedral, **I** or icosahedral.

What is known?

- The spherical manifold realization problem: Which spherical manifolds can be realized by integral surgeries on nontrivial knots in S^3?
- Solution for **C**–type manifolds (lens spaces) by Greene
- Solution for **T**, **O**, and **I**–type manifolds by Gu

What is known?

- The spherical manifold realization problem: Which spherical manifolds can be realized by integral surgeries on nontrivial knots in S^3?
- Solution for **C**–type manifolds (lens spaces) by Greene
- Solution for **T**, **O**, and **I**–type manifolds by Gu
- This leaves the **D**–type manifolds (also known as the **prism manifolds**) as the only remaining case.

Prism manifolds

- Given a pair of relatively prime integers $p > 1$ and q, let $P(p, q)$ be the oriented prism manifold with Seifert invariants

$$(-1; (2, 1), (2, 1), (p, q)).$$

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- We provide the solution of the realization problem for prism manifolds $P(p, q)$ with $q < 0$.

Realizable $P(p, q)$, $q < 0$

Type	$P(p, q)$
1A	$P\left(p, -\frac{1}{2}(p^2 - 3p + 4)\right)$
1B	$P\left(p, -\frac{1}{22}(p^2 - 3p + 4)\right)$
2	$P\left(p, -\frac{1}{\lvert 4r+2 \rvert}(r^2 p + 1)\right)$
3	$P\left(p, -\frac{1}{2r}(p + 1)(p + 4)\right)$
4	$P\left(p, -\frac{1}{2r^2}\left((2r + 1)^2 p + 1\right)\right)$
5	$P\left(p, -\frac{1}{r^2-2r-1}(r^2 p + 1)\right)$
Sporadic	$P(11, -30)$, $P(17, -31)$, $P(13, -47)$, $P(23, -64)$

Conjectural list of realizable $P(p,q)$, $q>0$

Type	$P(p,q)$
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3	$P\left(p, \frac{1}{2r}(p-1)(p-4)\right)$
4	$P\left(p, \frac{1}{2r^2}\left((2r+1)^2p-1\right)\right)$
5	$P\left(p, \frac{1}{r^2-2r-1}(r^2p-1)\right)$
Sporadic	$P(11,19)$, $P(11,30)$, $P(13,34)$


Thank you

When is a Knot Diagram Legendrian?

Mark Lowell

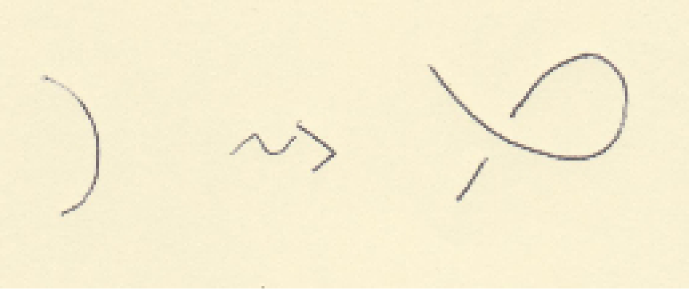
December 1, 2016

Introduction

- A **knot diagram** is an immersion $S^1 \to T^*\mathbb{R}$ equipped with crossing data:
- Two knot diagrams are **combinatorially equivalent** if they are isotopic without Reidemeister moves.
- When is a knot diagram combinatorially equivalent to the Lagrangian projection of a Legendrian knot?

Why We Care

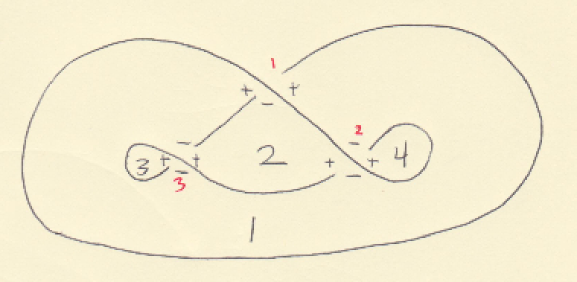
- My immediate goal is to use this in detecting if a Legendrian knot is stabilized.



- We can always stabilize a knot. There is no easy way to tell if it can be de-stabilized, that works in all circumstances.

Definitions

- Define a **minimal polygon** to be a bounded component of $T^*\mathbb{R} - K$. Index them P_1 through P_N.
- Index the crossings of K as c_1 through c_M. Put signs on each crossing:



- If K is Legendrian, then by Stokes' Theorem, the area of the minimal polygon P_i is the signed sum of the actions of the adjacent Reeb chords.

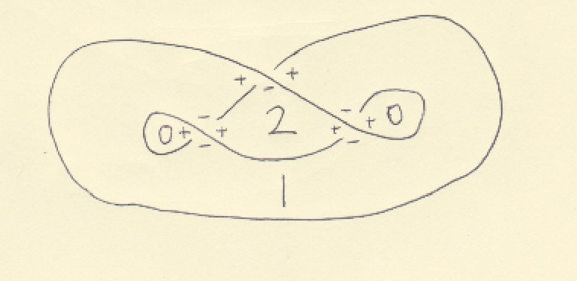
An If Condition

- Let A_K be the $M \times N$ matrix whose (i,j) entry is the signed sum of the number of times crossing i is adjacent to minimal polygon P_j.
- Let x be an N-dimensional vector whose entries are non-negative integers. Then, if K is Legendrian:

$$\sum_{i=1}^{N} x_i \text{Area}(P_i) = \sum_{j=1}^{M} (A_K^T x)_j \text{Action}(c_j)$$

- So, if we can find $x \neq 0$ so that $(A_K^T x)_j \leq 0$ for all j, K cannot be Legendrian. Because if it was, we would have a sum of areas with non-positive area.

An Example



An If and Only If Condition

This goes both ways:

Theorem: K is not Legendrian *if and only if* there exists a vector x such that $(A_K^T x)_j \leq 0$ for all j.

We can determine if a Legendrian knot can be de-stabilized using this approach, by undoing the Reidemeister Type I move and checking if the resulting knot is Legendrian.

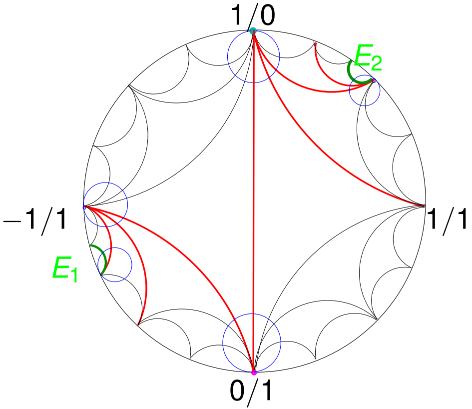
Calculating distance by twisting and projecting

Funda GÜLTEPE

University of Illinois at Urbana-Champaign
MSRI Postdoc

Tech Topology Conference 2016

Motivating Example: Distance estimate on Farey graph



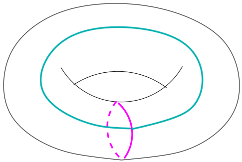


Figure: Walking on the Farey graph with 1 right, two left, 1 right, two left and two right turns, from E_1 to E_2.

Generalizing the picture

So we have: Start with a group ($SL_2(\mathbb{Z})$) and an associated object (Torus).

- Find a nice graph on which group acts *nicely*(Farey Graph $\mathcal{F}$), (i.e, find a way to express group elements (matrices) in terms of vertices/edges of the graph)
- Define projections to smaller components (into annuli)
- Count big enough ones to get an estimate for the word distance.

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So, lets do the following:

1. Torus $\rightarrow$ Surface $S = S_g^s$ with genus $g \geq 2$ and s boundary components
2. Farey Graph $\rightarrow$ Curve complex $\mathcal{C}(S)$
3. Edge in Farey graph $\rightarrow$ self homeo. of S (mapping class)

TOOL:

- Subsurface projection: Project curves to subsurfaces

Main Theorem

- $\mathbb{F}_n$: Free group of rank n
- $\text{Out}(F_n) = \text{Aut}(\mathbb{F}_n)/\,\text{Inn}(\mathbb{F}_n)$
- The *Mapping Class Group of S*, denoted $\text{Mod}(S)$ is,

$$\text{Mod}(S) = \text{Homeo}^+(S)/\text{isotopy}$$

Let $\iota : \text{Mod}(S_g^s) \to \text{Out}(\mathbb{F}_n)$ be the map induced from $\pi_1(S_g^s) \simeq \mathbb{F}_n$. we obtain the following.

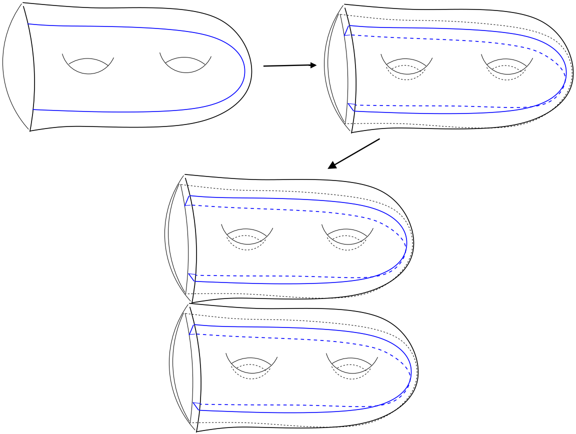
Theorem in Progress (with Rafi and Qing)

There is a distance formula for geometric outer automorphisms.

Topological model for Out($\mathbb{F}_n$)

Take a surface with $\pi_1(S) = \mathbb{F}_n$, Thicken it, then double it to $\sharp_n(S^2 \times S^1)$.

Arcs on S $\rightarrow$ Disks in H $\rightarrow$ Spheres in $\sharp_n(S^2 \times S^1)$



- Realize mapping classes as arc systems and project them into subsurfaces. (including annuli)

$$d_{\mathrm{Mod}(S)}(f_1, f_2) \asymp \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_k$$

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- Thicken arc systems to sphere systems (ADD twisting to replace annulus proj.):

$$\sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_k \prec \sum_{Y \subseteq S} [d_{\mathcal{S}(Y)}(\sigma_1, \sigma_2)]_k + \sum_{\alpha \in \mathbb{F}_n} [twist_\alpha(\sigma_1, \sigma_2)]_k.$$

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- Using Bestvina-Feighn subfactor projections:

$$\sum_{Y \subseteq S} [d_{\mathcal{S}(Y)}(\sigma_1, \sigma_2)]_k \prec d_{\mathrm{Out}(\mathbb{F}_n)}(\iota f_1, \iota f_2)$$

- Finally using our twisting number distribution along folding lines:

$$\sum_{\alpha \in \mathbb{F}_n} [twist_\alpha(\sigma_1, \sigma_2)]_k \prec d_{\mathrm{Out}(\mathbb{F}_n)}(\iota f_1, \iota f_2)$$

THANK YOU!

Random Groups and Cubulations

Yen Duong, University of Illinois at Chicago

Tech Topology Conference 2016

December 2, 2016

What is a Random Group?

How many words of length l in an alphabet of m letters?

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Definition

Choose $0 < d < 1$. Fix m generators $a_1, \ldots, a_m$. Choose $l > 0$, and with uniform probability choose $(2m-1)^{dl}$ many words of length l to form a relator set R. Then $\langle a_1, \ldots, a_m | R \rangle$ is a *random group at density d*.

Random Groups can have properties

Definition

Now let $l \to \infty$. If $\frac{|\text{Groups with P}|}{|\text{All random groups}|} \to 1$ as $l \to \infty$, we say that *random groups at density d have property P* (asymptotically almost surely).

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- If $d < 1/6$, random groups act freely and cocompactly on a $CAT(0)$ cube complex (Ollivier-Wise).

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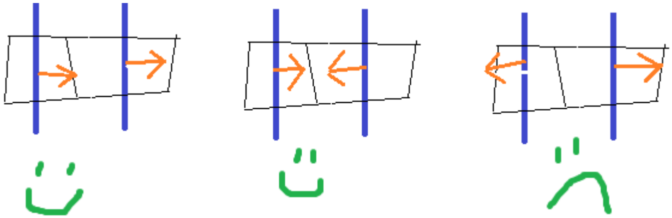
- If $d < 1/6$, random groups act freely and cocompactly on a $CAT(0)$ cube complex (Ollivier-Wise).
- If $d > 1/3$, any action on a $CAT(0)$ cube complex has a global fixed point [has Property (T)](Zuk).

Cube Complexes

Cube Complexes

Good thing you went to Dani Wise's talk!

Making a cube complex from a Cayley graph



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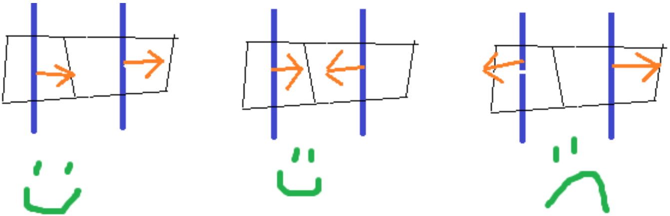

- Vertices: choice of okay orientations

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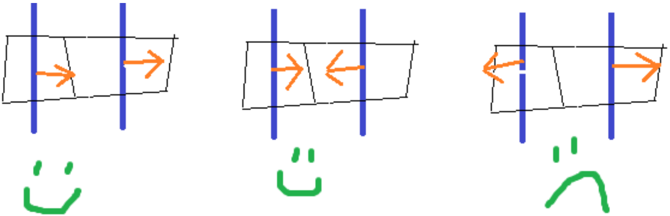

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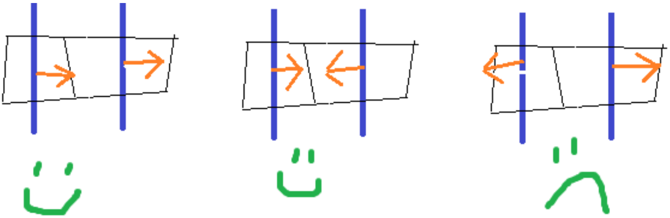

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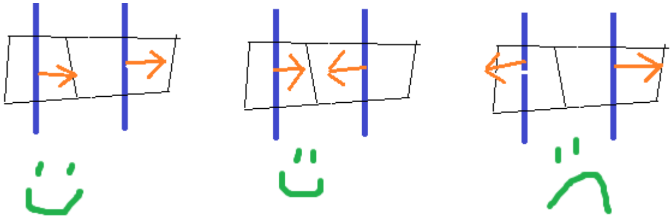

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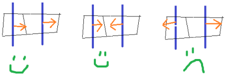

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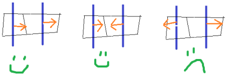

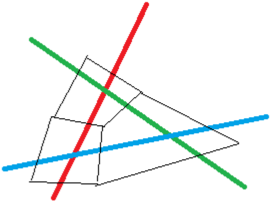

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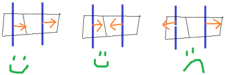

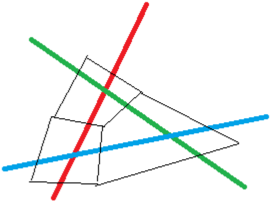

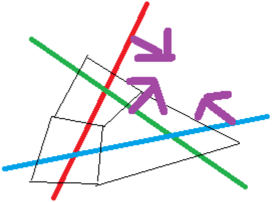

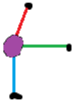

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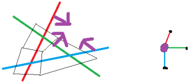

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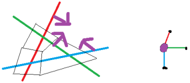

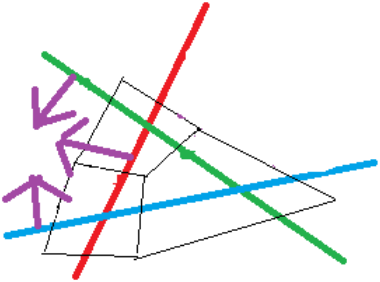

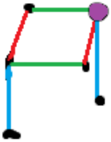

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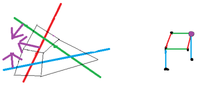

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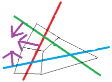

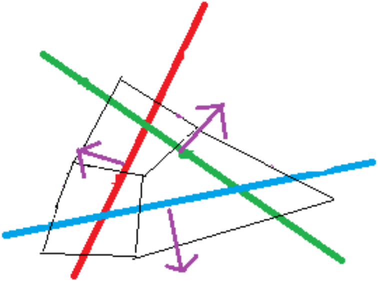

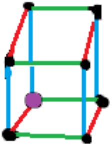

Divergence of CAT(0) Cube Complexes and Right-Angled Coxeter Groups

Ivan Levcovitz

CUNY Graduate Center

Right-Angled Coxeter Groups

Γ, graph

$S = \{s_1, s_2, ..., s_n\}$, vertices of Γ

E, edge set of Γ

Definition (Right-Angled Coxeter Group (RACG))

$$W_\Gamma =< S \mid s_i^2 = 1, s_i s_j = s_j s_i \ for \ (s_i, s_j) \in E >$$

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RACGs act geometrically on a natural CAT(0) cube complex.

Divergence

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Definition (Divergence)

$Div(G)$ is the supremum over all lengths of minimal paths in the Cayley graph of G, which avoid a ball of radius r, connecting two points that are distance about $2r$ apart.

Divergence

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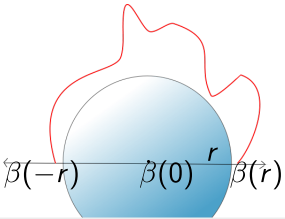
The divergence function roughly measures the fastest rate a pair of geodesic rays can stray apart from one another.



Divergence

- $Div(G)$ is a quasi-isometry invariant.
  (under a coarse equivalence of functions)

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  (under a coarse equivalence of functions)

- We say $Div(G)$ is linear, quadratic, $r^{1.5}$, exponential, etc.
  Multiplicative and additive constants are not very important.

From Graphs to Groups

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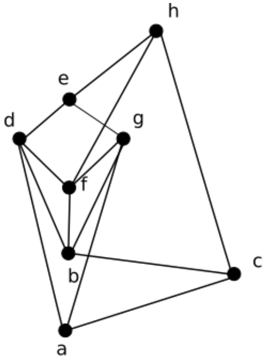
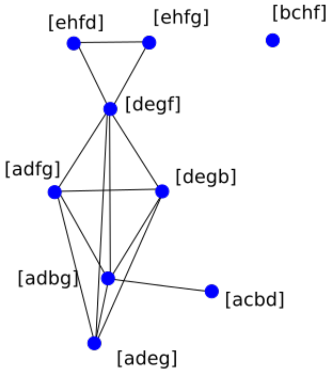
- Theorem follows from more general results on CAT(0) cube complexes.
- A different theorem also lets us to recognize infinite families of RACGs of polynomial divergence of any integer degree.

CFS Condition

Γ



□(Γ)



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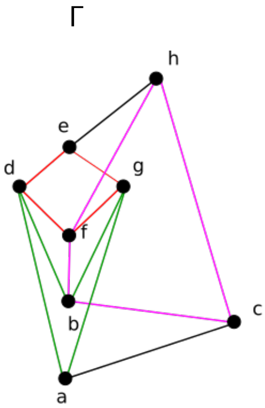

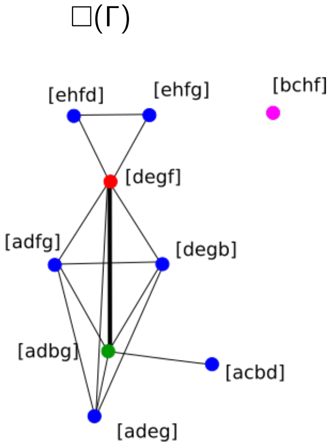

Application: Random RACGs

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Theorem (Behrstock–Falgas-Ravry–Hagen–Susse, L.)

$p(n)$ bounded away from 1, $\epsilon > 0$, $\Gamma = \Gamma(p(n), n)$ random graph.

If $p(n) > n^{-\frac{1}{2}+\epsilon}$, then W_Γ asymptotically almost surely exhibits quadratic divergence.

If $p(n) < n^{-\frac{1}{2}-\epsilon}$, then W_Γ asymptotically almost surely exhibits at least cubic divergence.

Thank You!

Divergence Definition

Fix constants $0 < \delta \leq 1$, $\lambda \geq 0$ and consider the linear function $\rho(r) = \delta r - \lambda$. Let $a, b, c \in X$ and set $k = d(c, \{a, b\})$.

$$div_\gamma(a, b, c, \delta)$$

is the length of the shortest path from a to b which avoids the ball of radius $\rho(k)$ about c.

$$Div_\gamma^X(r, \delta)$$

is the supremum of $div_\gamma(a, b, c, \delta)$ over all a, b, c with $d(a, b) \leq r$.

CFS Graphs

Given a Coxeter Diagram Γ, the *square graph*, $\square(\Gamma)$, is the graph with vertex set:

$$V(\square(\Gamma)) = \{\text{induced 4-cycles in } \Gamma\}$$

And edge set:

$$E(\square(\Gamma)) = \{(F_1, F_2) \mid F_1 \cap F_2 \subset \Gamma \text{ contains a pair of nonadjacent vertices}\}$$

Γ is **CFS** if $\square(\Gamma)$ contains a component, C, such that for every $v \in \Gamma$, there is a 4-cycle $F \in C$ with v a vertex of F.