A Quantitative Look at Lagrangian Cobordisms

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Bryn Mawr College

Joint work with Joshua M. Sabloff, *Haverford College*

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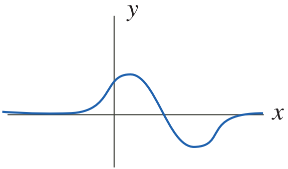
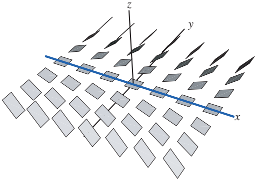
Lagrangians and Legendrians





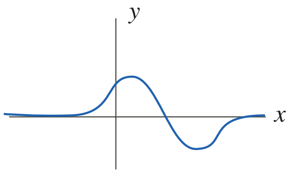
Symplectic Manifold (X^{2n}, ω)

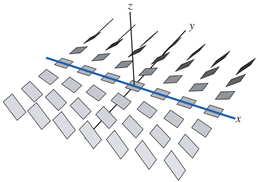
Lagrangian Submanifold
$L^n : \omega|_{TL} \equiv 0$

Contact Manifold (Y^{2n+1}, ξ)

Legendrian Submanifold
$\Lambda^n : T\Lambda \subset \xi$

Lagrangians and Legendrians





Symplectic Manifold (X^{2n}, ω)

Exact Symplectic : $\omega = d\lambda$

Lagrangian Submanifold

$L^n : \omega|_{TL} \equiv 0$

Exact Lagrangian: $\lambda = df$

Contact Manifold (Y^{2n+1}, ξ)

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The Symplectization of a Contact Manifold

Standard Contact Manifold: $(\mathbb{R}^{2n+1}, \ker\alpha)$

$$J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i$$

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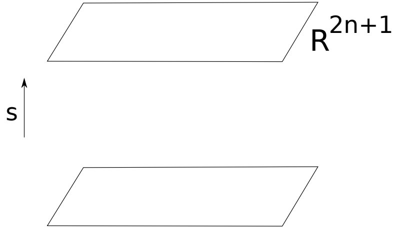

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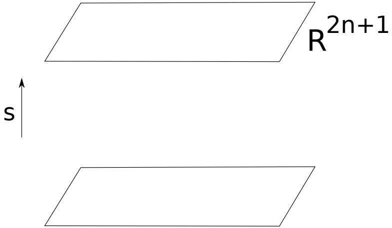

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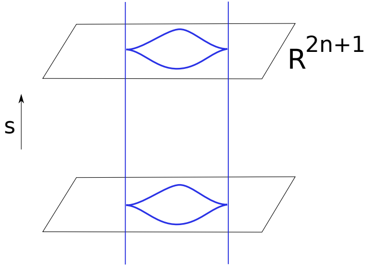
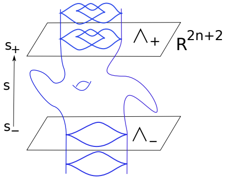


- There are no closed, exact Lagrangians (Gromov);
- For a Legendrian Λ, the cylinder $\mathbb{R} \times \Lambda$ is an exact Lagrangian.

Lagrangian Cobordisms between Legendrians

A Lagrangian cobordism from Λ_- to Λ_+ means:



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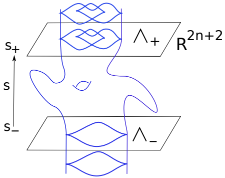

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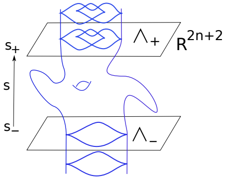

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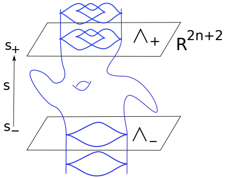

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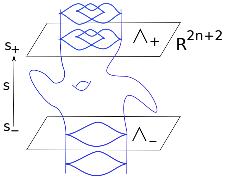

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Arise in relative SFT (Eliashberg-Givental-Hofer)

Qualitative Questions

- Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, does there exist a Lagrangian cobordism between them?

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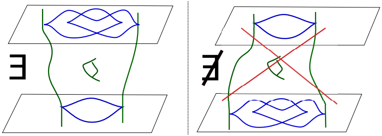

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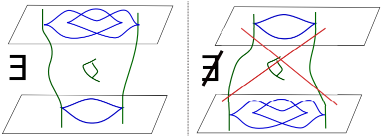

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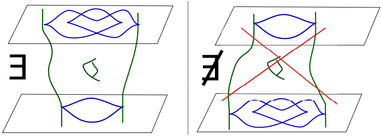

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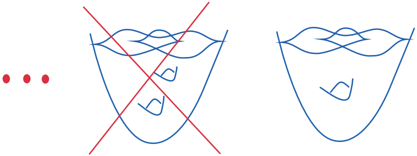

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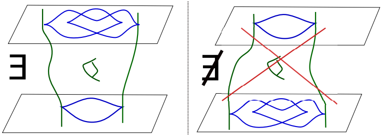

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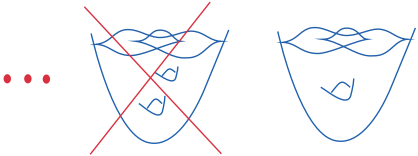
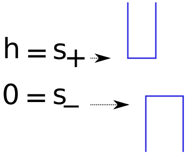


Fillings realize 4-ball genus!

A variety of qualitative questions have been studied by: Chantraine, Ekholm, Honda, Kálmán, Dimitroglou Rizell, Ghiggini, Golovko, Cornwell, Ng, Sivek, Bourgeois, Sabloff, Traynor, Capovilla-Searle, Hayden, Pan, . . .

Quantitative Questions

- **(Length)** Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, what is the minimal “length" of any cobordism between them?



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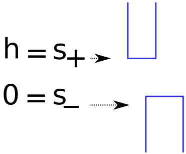
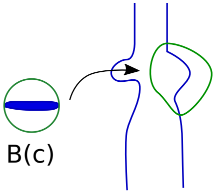


- **(Width)** Given a Lagrangian cobordism, what is its "width"?



Outline

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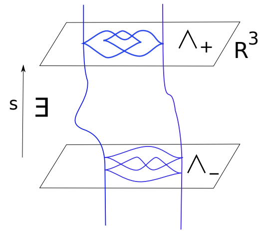
Constructions of Lagrangian Concordances

Isotopy Lemma (Eliashberg, Chantraine, Golovko, Ekholm-Honda-Kálmán, . . .)

Suppose Λ_- and Λ_+ are Legendrian isotopic. Then there exists a Lagrangian cobordism from Λ_- to Λ_+.



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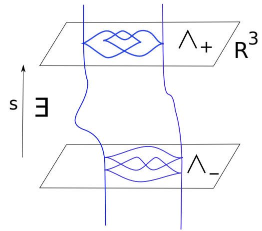
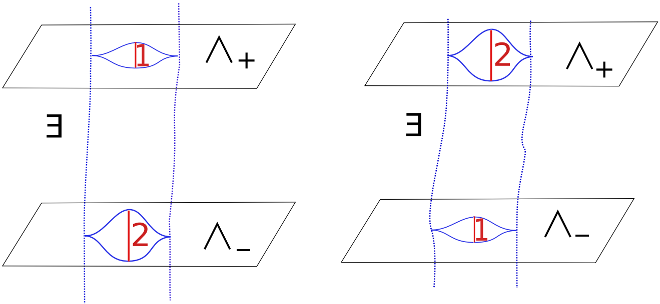


Remark: The Lagrangian is **not** the trace of the isotopy.

Most slices of the Lagrangian will **not** be Legendrian.

Qualitatively Symmetric Concordances:



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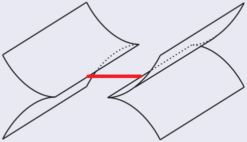
Theorem (Dimitroglou Rizell, Ekholm-Honda-Kálmán, Bourgeois-Sabloff-T)

If Λ_+ is obtained from Λ_- by a "cusp-surgery",



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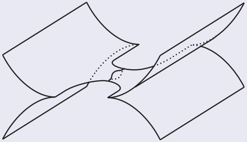

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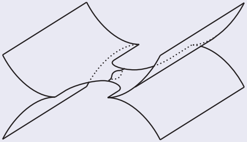
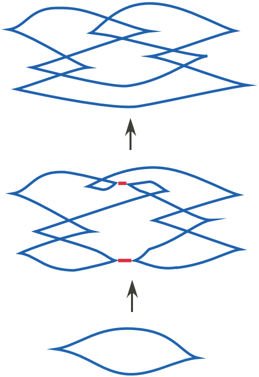


then there exists a Lagrangian cobordism from Λ_- to Λ_+.

Construction Example

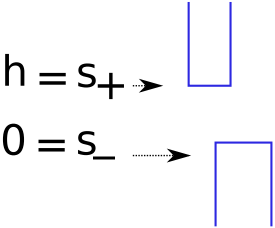
Lagrangian genus 1 filling of a Legendrian $m(5_2)$:



Legendrian isotopy and cusp pinches as you move up!

Outline

Question: Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, what is the "minimal length" of any cobordism between them?



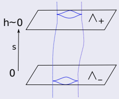
$$\text{minimal length} = \inf\{h : \exists \text{ Lagrangian cobordism from } \Lambda_- \text{to } \Lambda_+ \text{ that is cylindrical outside} [0, h]\}.$$

Flexibility

Theorem (Sabloff-T, '16: *Selecta Mathematica*)

There exists an arbitrarily short Lagrangian cobordism between

1. *a Legendrian and its vertical translate,*



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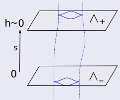

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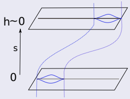

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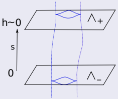

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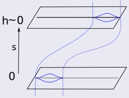
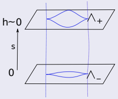


3. *a Legendrian and its vertical expansion.*



Rigidity

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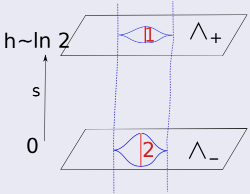

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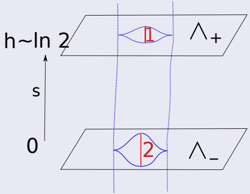
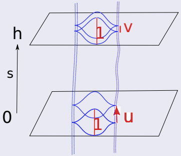


2. *vertically shifted Hopf links:*



$$h \sim \begin{cases} \ln\left(\frac{1-u}{1-v}\right), & \text{if } u \leq v, \\ \ln\left(\frac{u}{v}\right), & \text{if } u \geq v. \end{cases}$$

Lower Bound to Length

(Step 1) Assign "capacities" to a Legendrian

$$c(\Lambda, \varepsilon, \theta) \in \mathbb{R}_{>0} \cup \{\infty\},$$

ε is an augmentation of the DGA $\mathcal{A}(\Lambda)$, $\qquad \varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}_2, 0)$,
$\theta \in LCH^*(\Lambda, \varepsilon)$.

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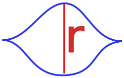
Example:



$\exists 0 \neq \lambda \in LCH^1(U(r), \varepsilon)$; $c(U(r), \varepsilon, \lambda) = r$.

Fundamental Class Fundamental Capacity

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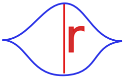

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Fundamental Class Fundamental Capacity

For $\theta \neq 0$, $c(\Lambda, \varepsilon, \theta)$ is *always* the height of a Reeb chord!

Lower Bound to Length

(Step 2) From ε_-, θ_- for Λ_- and Lagrangian cobordism *L* from Λ_- to Λ_+, get induced ε_+, θ_+ for Λ_+.

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(Step 2) From ε_-, θ_- for Λ_- and Lagrangian cobordism L from Λ_- to Λ_+, get induced ε_+, θ_+ for Λ_+.

[Ekholm-Honda-Kálmán]

$$
\begin{array}{ccccc}
\Lambda_+ & \rightsquigarrow & \mathcal{A}(\Lambda_+) & & \\
 & & \downarrow \Phi(L) & \searrow^{\varepsilon_+} & \\
\Lambda_- & \rightsquigarrow & \mathcal{A}(\Lambda_-) & \xrightarrow{\varepsilon_-} & \mathbb{F}_2
\end{array}
\qquad
\begin{array}{ccc}
LCH^*(\Lambda_+, \varepsilon_+) & \qquad & \theta_+ \\
\uparrow \Psi_{L,\varepsilon_-} & & \uparrow \\
LCH^*(\Lambda_-, \varepsilon_-) & & \theta_-
\end{array}
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[Ekholm-Honda-Kálmán]

$$\begin{array}{ccccc} \Lambda_+ & \rightsquigarrow & \mathcal{A}(\Lambda_+) & & \\ & & \downarrow \Phi(L) & \searrow^{\varepsilon_+} & \\ \Lambda_- & \rightsquigarrow & \mathcal{A}(\Lambda_-) & \xrightarrow{\varepsilon_-} & \mathbb{F}_2 \end{array} \qquad \begin{array}{c} LCH^*(\Lambda_+, \varepsilon_+) \\ \uparrow \Psi_{L,\varepsilon_-} \\ LCH^*(\Lambda_-, \varepsilon_-) \end{array} \qquad \begin{array}{c} \theta_+ \\ \uparrow \\ \theta_- \end{array}$$

Question: How do capacities $c(\Lambda_+, \varepsilon_+, \theta_+)$ and $c(\Lambda_-, \varepsilon_-, \theta_-)$ compare?

Lower Bound to Length

(Step 3) Relate capacities for ends of a Lagrangian cobordism.

Length-Capacity Inequality (Sabloff-T)

If L is a Lagrangian cobordism from Λ_- *to* Λ_+ *that is cylindrical outside* $[0, h]$*, then*

$$e^0 c(\Lambda_-, \varepsilon_-, \theta_-) \leq e^h c(\Lambda_+, \varepsilon_+, \theta_+).$$

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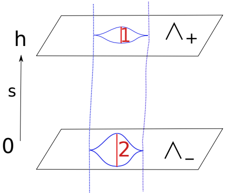
Get lower bounds to length of a cobordism!

$$\ln\left(\frac{c(\Lambda_-, \varepsilon_-, \theta_-)}{c(\Lambda_+, \varepsilon_+, \theta_+)}\right) \leq h.$$

Lower Bound to Length of a Contraction



By Length-Capacity Inequality:

$$\ln\left(\frac{2}{1}\right) = \ln\left(\frac{c(U(2)), \varepsilon_-, \lambda_-)}{c(U(1)), \varepsilon_+, \lambda_+)}\right) = \ln\left(\frac{c(\Lambda_-, \varepsilon_-, \lambda_-)}{c(\Lambda_+, \varepsilon_+, \lambda_+)}\right) \leq h.$$

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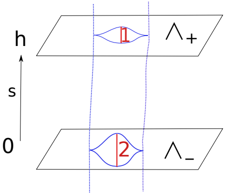

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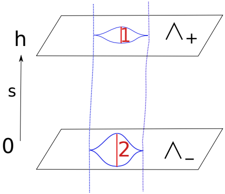

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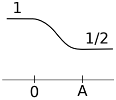

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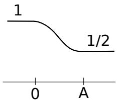

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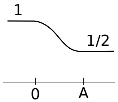

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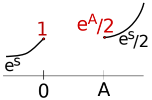

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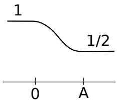

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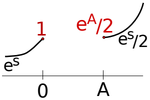

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So,

$\exists$ embedded Lagrangian cobordism when $1 < e^A/2 \iff \ln 2 < A$.

Outline

Width of a Symplectic Manifold

$$B^{2n}(c) := \left\{ (x_1, y_1, \dots, x_n, y_n) : \pi \sum_i (x_i^2 + y_i^2) \leq c \right\} \subset (\mathbb{R}^{2n}, \omega_0).$$

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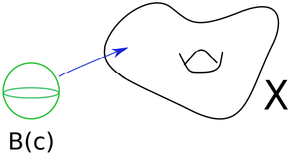
$$w(X) := \sup\{c : \exists \psi : B^{2n}(c) \to X, \psi^* \omega = \omega_0\}.$$



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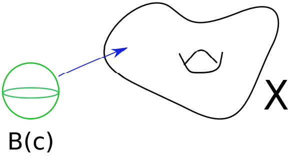
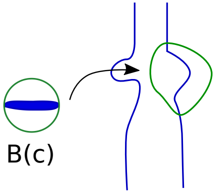


We are working in $X = \mathbb{R} \times J^1 M$: $w(\mathbb{R} \times J^1 M) = \infty$.

Width of a Lagrangian

Given a Lagrangian submanifold $L \subset (X, \omega)$, **relative width** is:

$$w(X, L) = \sup \left\{ c \mid \exists \psi : B^{2n}(c) \to X, \psi^* \omega = \omega_0, \psi^{-1}(L) = B^{2n}(c) \cap \mathbb{R}^n \right\}.$$



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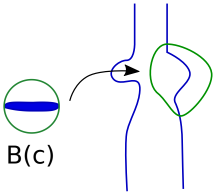
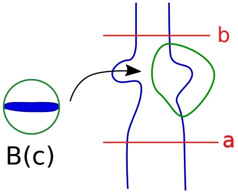


Introduced by Barraud and Cornea, '05.

Widths of Lagrangian Cobordisms

Given a Lagrangian cobordism L, for $-\infty \leq a < b \leq \infty$,

$$L_a^b := \{(s, x, y, z) \in L : a < s < b\} \subset (a, b) \times J^1 M.$$



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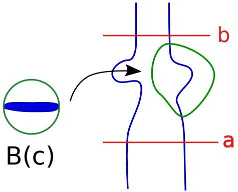

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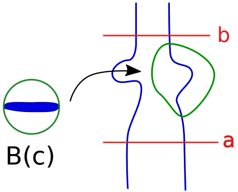

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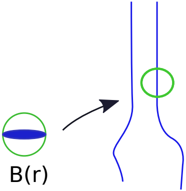

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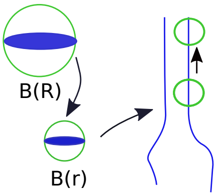

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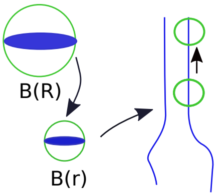

Chop off top! We will consider:

$$a = -\infty, \qquad s_+ \leq b < +\infty.$$

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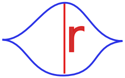

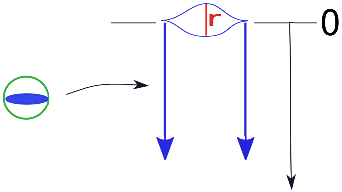

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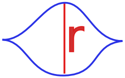

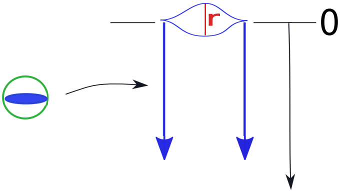

Theorem (Sabloff-T)

$$w((\mathbb{R} \times U(r))^0_{-\infty}) = 2r.$$

Upperbound to Width of a Legendrian

$w\left((\mathbb{R}\times U(r))^{0}_{-\infty}\right) \leq 2r$ follows from:

Theorem (Sabloff-T)

Suppose Λ is a Legendrian that admits an augmentation. Then

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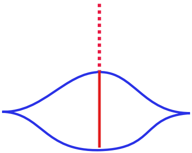
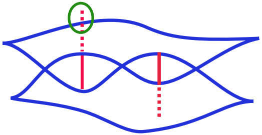


Vertically Extendable



No Yes

Constructing Embeddings

Proof Sketch:

$$\Psi : \mathbb{R} \times J^1 M \to T^*\mathbb{R}_+ \times T^*M$$
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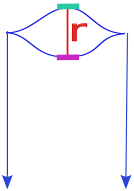

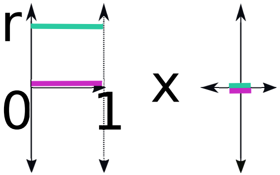

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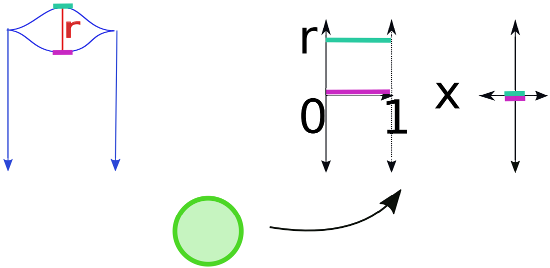

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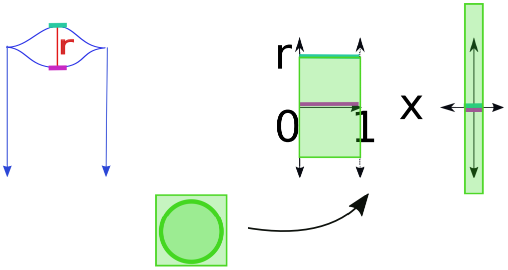

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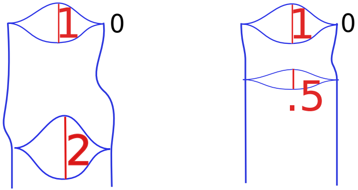

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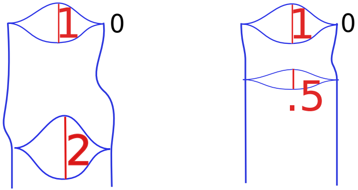

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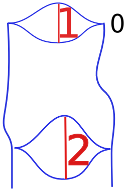

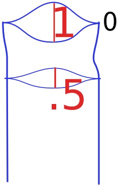

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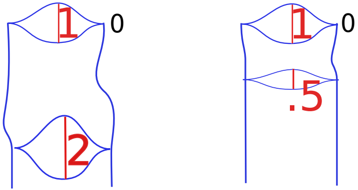

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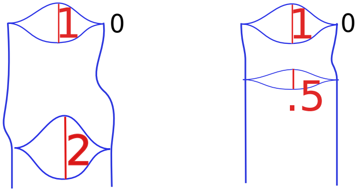

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Width does not see the negative end!

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If L is a Lagrangian cobordism from Λ_- to Λ_+ and Λ_- is fillable, then

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where $c(\Lambda_+)$ is the minimum fundamental capacity (for any augmentation).

Upper Bound for Width of Lagrangian Cobordisms

Theorem (Sabloff-T)

If L is a Lagrangian cobordism from Λ_- to Λ_+ and Λ_- is fillable, then

$$w\left(L^0_{-\infty}\right) \leq 2c(\Lambda_+),$$

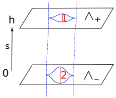
where $c(\Lambda_+)$ is the minimum fundamental capacity (for any augmentation).

Proof is similar in spirit to the proof when $L = \mathbb{R} \times \Lambda$:

Use Seidel Isomorphism to get the existence of a J-holomorphic disk through $\psi(0) \in \psi(B(\alpha))$.

Length-Width Connection

Can reprove our earlier length result between $\Lambda_- = U(2)$ and $\Lambda_+ = U(1)$: $\quad h \geq \ln 2$



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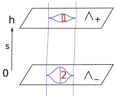

Corollary

Suppose L is a Lagrangian cobordism from $\Lambda_- = U(2)$ to $\Lambda_+ = U(1)$ that is cylindrical outside $[-h, 0]$. Then

$$\ln 2 = \ln\left(\frac{2}{1}\right) = \ln\left(\frac{c(U(2))}{c(U(1))}\right) \leq h.$$

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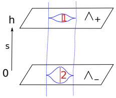

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Proof:

$$2e^{-h}c(U(2)) = e^{-h}w((\mathbb{R}\times\Lambda_-)^0_{-\infty}) = w((\mathbb{R}\times\Lambda_-)^{-h}_{-\infty}) \leq w(L^0_{-\infty}) \leq 2c(U(1))$$

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Thank you!