The number of fiberings of a surface bundle over a surface

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Basic fact

A manifold E sometimes can be a manifold bundle over a manifold in many ways

- Thurston studied 3-manifold case (Thurston norm)

  if E^3 is a surface bundle over circle and $b_1(E^3) > 1$

  $\Rightarrow$ infinitely many ways to fiber over a circle

  related to subsets of $H^1(E, \mathbb{R})$

This talk 4-manifolds

$$SFIB(E^4) = \{ \underset{g>1}{S_g} \to E \to \underset{B \text{ sfc}}{B} \} / \pi_1{}^* E$$

π_1 Equivalent

$$1 \to \pi_1(F_1) \xrightarrow{P_{1*}} \pi_1(E) \to \pi_1(B_1) \to 1$$

$P_1 \sim P_2$ if $P_{1*}(\pi_1(F_1)) = P_{2*}(\pi_1(F_2))$

$\Rightarrow$ fiberwise diffeomorphism

2 What is known

① FEA Johnson

$SFib(E^4) < C(\chi(E)) < \infty$

$\cdot\ \chi(E) > 0$ $\quad \chi(E) = \chi(B) \times \chi(F)$

case $\chi(E) = 0$ $\quad E = S^1 \times E^3$

Proof idea: ⓐ $\chi(E) = \chi(B)\chi(F)$

only finitely many # for $\chi(B)$, $\chi(F)$

ⓑ $g(B) > g(F) \Rightarrow 1$ fibering (prove later)

ⓒ $\exists\ \tilde{B} \to B$

$g(\tilde{B}) > g(F)$ $\quad \tilde{E}$ has 1 fibering

finite # of covers

② Salter: Monodromy criterion for $SFib(E) = 1$

$$\begin{array}{c} S_g \to E \\ \quad\ \downarrow \\ \quad\ B \end{array}$$

$$\rho: \pi_1(B) \to Mod(S_g) \cong \pi_0(Diff(S_g))$$

$$\rho_H: \pi_1(B) \to Aut(H_1(S_g)) = Sp(2g;\mathbb{Z})$$

Salter: $H_1(S_g)^{\pi_1(B)} = \{0\}$

$\Rightarrow SFib(E) = 1$

Prop (Salter) $\Rightarrow$ ⓑ

$F_1 \to E \xrightarrow{p_i} B_1 \qquad p_1 \neq p_2$

$p_1^* H^1(B_1;\mathbb{Q}) \cap p_2^* H^1(B_2;\mathbb{Q}) = \{0\}$

$\Rightarrow$ ⓑ $\quad b_1(E) \leq b_1(B) + b_1(F)$

$b_1(E) > b_1(B_1) + b_1(B_2)$

Pf of Salter Prop:

$Hom(\pi_1(X);\mathbb{Q}) = H^1(X;\mathbb{Q})$

pf by contradiction

$\alpha \in p_1^* H^1(B_1) \cap p_2^* H^1(B_2)$

$\alpha \neq 0$

$$\begin{tikzcd} \pi_1(F_2) \arrow[r] & \pi_1(E) \arrow[r] \arrow[d] & \pi_1(B_2) \arrow[d, "\alpha_2"] \\ p_1(\pi_1(F_2)) \trianglelefteq & \pi_1(B_1) \arrow[r, "\alpha_1"] & \mathbb{Q} \end{tikzcd}$$

finite gen

Key property: nontrivial fin gen normal subgp of $\pi_1(S_g)$ or F_n $(g \geq 2)$ is finite index

so trivial $\Rightarrow \pi_1 E$

nontrivial fin. index $\Rightarrow$ contradiction

$SFib(S_g \times S_n) = 2$ Salter

<u>Problem:</u> Do you have E nontrivial $SFib(E) > 1$?

3. The Atiyah - Kodaira Example

Thm (Atiyah-Kodaira)

$\exists\, M_{AK}$ s.t. $\# \, SFib(M_{AK}) \geq 2$

$$S_{321} \to M_{AK} \to S_3$$
$$S_6 \to M_{AK} \xrightarrow{p} S_{129}$$

$\sigma(M_{AK}) \neq 0$

$\rho_H = 0 \Rightarrow \sigma(E) = 0$

Idea (construction)

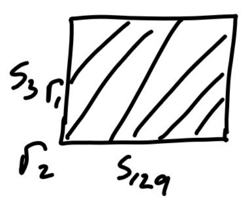
τ rot^n 180°

for each x take 2 fold branched cover over S_3 branched over $x, \tau(x)$

can't do globally so take cover of S_3 to ~~get~~ S_{129}

Global picture: $\pi_1(S_3) \to H_1(S_3; \mathbb{Z}/2)$

$\# \ \overset{\shortparallel}{2^6}$



$$\begin{tikzcd} S_{129} \arrow[r, "i"] \arrow[dr] & S_3 \arrow[d, "\tau"] \\ & S_3 \end{tikzcd}$$

$[\Gamma_1] + [\Gamma_2] = 0 \quad H_2(S_3 \times S_{129}; \mathbb{Z}/2)$

cover $M_{AK} \to S_3 \times S_{129}$

construction not unique

Question: Does there exist other fiberings?

4. Main Result

Thm (C.): $\# \, SFib(M_{AK}) = 2$

Prop (Criterion for $SFib(E) = 2$)

$$F_1 \to E \xrightarrow{p_1} B_1 \quad g(F_1) > 1$$
$$g(B_1) > 1$$

$$(p_1\ p_2)^* : H^1(E;\mathbb{Q}) \cong p_1^* H^1(B_1) \oplus p_2^* H^1(B_2)$$

$$H^2(B_1 \times B_2) \xrightarrow[inj]{} H^2(E;\mathbb{Q})$$

$$\Rightarrow SFib(E) = 2$$

Key lemma (C)

$$H^1(M_{AK};\mathbb{Q}) = p_1^* H^1(S_3,\mathbb{Q}) \oplus p_2^* H^1(S_{129};\mathbb{Q})$$

Proof of Main result using key lemma

$$M_{AK} \to S_3 \times S_{129} \quad 2\ \text{cover}$$

$$H^4(M_{AK}) \overset{\cong}{\Leftarrow} H^4(S_3 \times S_{129})$$

Poincaré Duality

$$H^*(S_3 \times S_{129};\mathbb{Q}) \longrightarrow H^*(M_{AK};\mathbb{Q}) \quad inj \quad \forall *$$

$$\alpha \longrightarrow 0$$

$$\alpha \cup \rho = [S_3 \times S_{129}]$$

$$H^1(M_{AK};\mathbb{Q}) = p_1^* H^1(S_3;\mathbb{Q}) \oplus p_2^* H^1(S_{129};\mathbb{Q})$$
$$a = b + c$$

if 3^{rd} fibering p_3

$$p_3^* H^1(B;\mathbb{Q}) \longrightarrow H^1(E;\mathbb{Q})$$
$$b+c \quad b \neq 0$$

$$(b+c) \cup (b'+c') \neq 0$$

$$g(B) \geq 2 \quad \pi_1(B) = a_1\ b_1\ a_2\ b_2 \ldots$$
$$a_1 \cup a_2 = 0$$

$$\exists\ a \cup a' = 0 \checkmark$$

Application of Criterion

$Th^{m}(c)$ $\quad g > 1 \quad k > 1$

$$E \to S_g \times S_k \quad \text{cover}$$
$$\Rightarrow SFib(E) = 2$$

Pf of Key lemma

$$0 \to H^1(S_{129};\mathbb{Q}) \to H^1(M_{AK};\mathbb{Q}) \to H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})} \to 0$$

Only need to show $H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})} \xleftarrow[inj]{} H^1(S_3;\mathbb{Q})$

Monodromy description M_{AK}

What do you mean by taking branched cover at sections?

Key: point-pushing

$$\pi_1(S_{129}) = \pi_1(S_3)^2 = \langle g^2 \mid g \in \pi_1(S_3) \rangle$$

Computation $\Rightarrow H^1(S_3;\mathbb{Q}) = H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}$

5. More ...

<u>Question:</u> $\exists E$ st. $SFib(E) > 2$?

Salter 2015: $\forall N > 0 \ \exists M^4$ st.

$$\# SFib(M^4) > N$$

$$M_S = (S_g \times S_g \setminus \Delta) \cup_\phi (S_g \times S_g \setminus \Delta)$$

$$\begin{array}{cccc} \swarrow p_1 & \searrow p_2 & \swarrow p_1 & \searrow p_2 \\ S_g & S_g & S_g & S_g \end{array}$$

get $SFib(M_S) \geq 4$

<u>Th^{m}(C):</u> $|SFib(M_S^4)| = 4$

Remark: Only values of $|SFib(M_S)|$ $0,1,2,4$

Open Question: Can you find M^4 s.t. $|SFib(M^4)| = 3$?