Band Surgeries on the trefoil

Tye Lidman (joint A. Moore & M. Vazquez)

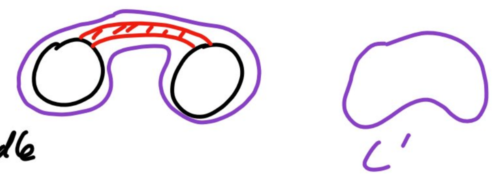
I Band Surgery

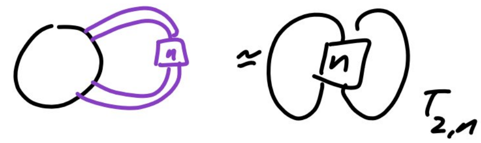
Simple construction

link in S^3

$L \subseteq S^3 \xrightarrow[\text{bands}]{\text{attach}}$

2 dim'l 1-handle

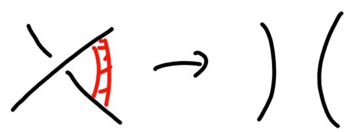




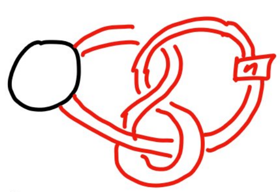
What does band surgery do to L?

⓪ Change or preserve # of components (depends on or$^{\underline{n}}$ and feet of handle)

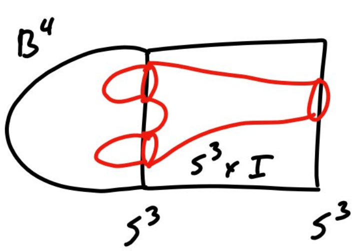
①



② Every (2,n)-cable of knot is band surgery on unknot



③ If ∃ "coherent" band surgery from knot K to unlink then K bounds smooth disk in B^4



__Question:__ How many bands does it take to go between L_1 and L_2?

__Simplest example__ All $T_{2,n}$ torus links are one band surgery away from unknot

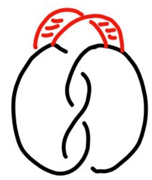
What is distance between two $T_{2,n}$'s?

$\underline{Th^m \text{ (L-Moore-Vazquez)}}$ $\exists$ band surgery from $T_{2,3}$ to $T_{2,n}$

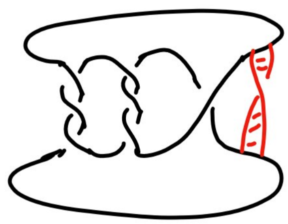
$$\Longleftrightarrow$$

$$n \in \{\pm 1, \pm 2, 3, 4, -6, 7\}$$

<u>Ex:</u>



$\rightsquigarrow T_{2,4}$



$T_{2,-6}$

<u>Remark:</u> Case of n even was originally due to Darcy–Ishihara–Medikonduri–Shimokawa

II. <u>Dehn Surgery</u>

<u>Def</u>: Let $K \subseteq S^3$ $\quad p, q$ rel prime

$$S^3_{p/q}(K) = S^3 \setminus \nu(K) \cup D^2 \times S^1$$
$$p\mu + q\lambda \longleftarrow \partial D^2$$

<u>Example:</u> $S^3_{p/q}(O) =$  $= L(p,q)$

$$p\mu + q\lambda \longleftarrow \partial D^2$$

We can do this for $K \subseteq Y$ ← arbitrary

if $[K] = 0 \in H_1(Y)$, then

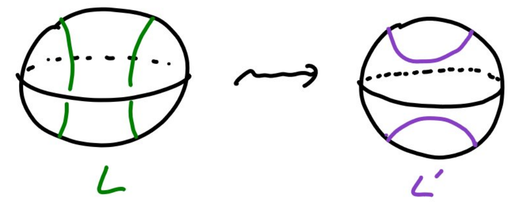
$$H_1(Y_{p/q}(K)) = H_1(Y) \oplus \mathbb{Z}/p$$

a surgery is <u>integral</u> if image of ∂D^2 intersects μ once (if null-homologous $q = \pm 1$)

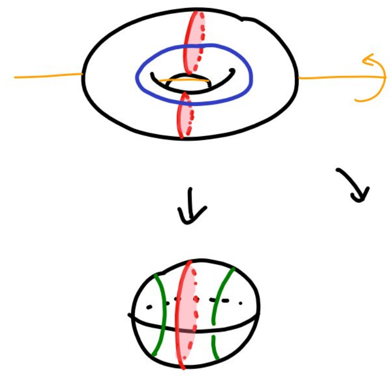
Who cares?

Montesinos Trick: if L and L' are related by band surgery then $\underbrace{\Sigma_2(L) \text{ and } \Sigma_2(L')}_{\text{branched double cover}}$ related by integral surgery

Proof: Band surgery is a local move



branched double cover



remove this solid torus in $\Sigma_2(L)$ and replace with "blue" torus

Th$^{\underline{m}}$ (LMV):

if $L(n,1)$ is integral surgery on a knot in $L(3,1)$, then $n \in \{\pm 1, \pm 2, 3, 4, -6, 7\}$

double cover of $T_{2,3}$

Key Invariant

(Y, s) $\quad b_1(Y) = 0$

$s \in Spin^c(Y) \cong H^2(Y)$

$\rightsquigarrow$

$d(Y,s) \in \mathbb{Q}$

d-invariant (from Heegaard Floer homology)

* $d(L(n,1), i) = -\frac{1}{4} + \frac{(n-2i)^2}{4n} \qquad 0 \le i \le n$
* $d(-Y,s) = -d(Y,s)$
* if Y' is obtained from ± 1 surgery from Y

  then $\{d(Y,s)\}_{s \in Spin^c(Y)} \equiv \{d(Y',s')\}_{s' \in Spin^c(Y')} \pmod 2$

Ex: $L(-3,1)$ is not integral surgery on knot in $L(3,1)$

Homological condition $\rightsquigarrow$ knot is nullhomologous

$\rightsquigarrow \frac{p}{q} = \pm 1$

so $\{d(-L(3,1), s)\} \equiv \{d(L(3,1), s)\} \pmod 2$

$\{d(-L(3,1), s)\} = \{-\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{6}\}$, $\quad \{d(L(3,1), s)\} = \{\tfrac{1}{2}, -\tfrac{1}{6}, -\tfrac{1}{6}\}$ $\quad \otimes$

*(Ni-Wu, Rasmussen)

let $K \subset L(3,1)$ w/ $[K] = 0$

$\exists$ non-negative integers $V_i(K)$ s.t.

① $d(L(3,1)_m(K), s_i) = d(L(m,1), i) - 2V_i(K) + d(L(3,1), 0)$

for $0 \le i \le \frac{m}{2}$

② $V_i \ge V_{i+1} \ge V_i - 1$

<u>Example:</u> $L(-42,1)$

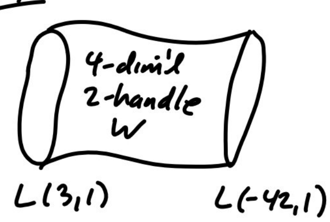
* $[K] = 0$, $\frac{p}{q} = \pm 14$

~~$m = +14$~~ Only one spin^c structure satisfies ① for $i = 0$

$\Rightarrow V_0 = 7$

① for $i = 1 \Rightarrow V_1 = 3$ $\quad \otimes$ condition ②

~~$m = -14$~~



W spin
$b_2^- = 1$
$b_2^+ = 0$

<u>Technical Lemma</u> (Pin(2) Seiberg-Witten Theory)

if $(W, s): (Y_1, s_1) \to (Y_2, s_2)$ is spin cobordism between spin L-spaces

then $d(Y_1, s_1) - d(Y_2, s_2) = -\frac{1}{4}$

nothing for $L(3,1)/L(-42,1)$ satisfies this ✓