Spectral Order Contact Invariant

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Contact structure: (M, ξ) ← 2 plane field

$\xi = \ker \alpha \quad \alpha \wedge d\alpha \neq 0$

Lutz, Martinet 70's contact structures exist on all oriented manifold

↑

"Lutz twist" give $D^2 \hookrightarrow (M, \xi)$ overtwisted disk

s.t. $\xi_p = T_p D \quad \forall p \in \partial D$

tight if no such disk

Dichotomy

↙ ↘

tight overtwisted

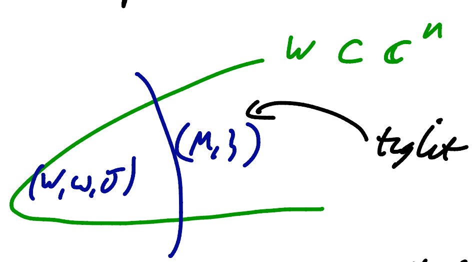
Bennequin: (S^3, ξ_{st}) is tight

↓

$S^3 = \partial B^4 \subset \mathbb{C}^2$ ← complex str.

$\xi_p = T_p S^3 \cap J T_p S^3$

Eliashberg, Gromov:



such a ξ called fillable

(W, ω, J) Stein domain

also have strong and weakly fillable

Eliashberg: overtwisted contact structures are classified upto isotopy by alg. topology tight contact structures: · only in finitely many homotopy classes

· ∄ on all manifolds

Question: existence, uniqueness, classification

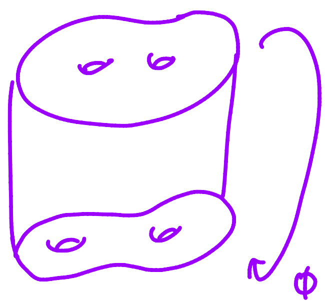
How to recognize if a contact str. is tight

Giroux: every contact structure comes from Thurston Winkelnkemper construction given by some open book

$$\begin{tikzcd} TW: \{\text{open books}\} \arrow[r] & \{\text{contact str}\} \arrow[l, bend left, "\text{Giroux}"] \end{tikzcd}$$

two open books for same ξ related by pos stabilization

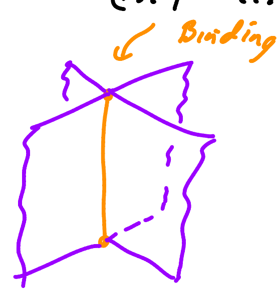
Open Books: (S, ϕ) $\phi \in Map(S, \partial S)$



$S \times I / \sim_\phi$

$(x,1) \sim (\phi(x), 0)$

$(x,t) \sim (x,s)$ $x \in \partial S$



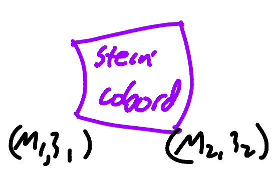
contact invariant in Heegaard-Floer homology

$c(\xi) \in \widehat{HF}(-M)$

1) $c(\xi) = 0$ if ξ is overtwisted

2) $c(\xi) \neq 0$ if ξ is Stein fillable

3)



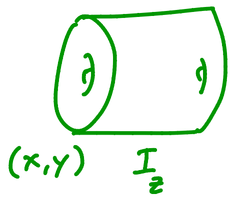
$$\widehat{HF}(-M_1) \longleftarrow \widehat{HF}(-M_2)$$
$$c(\xi_1) \longmapsfrom c(\xi_2)$$

Question: $c(\xi)=0 \Rightarrow OT$?

No: Gay, Ghiggini...

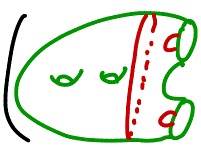
if (M,ξ) contains Giroux torsion, then $c(\xi)=0$

$T^2 \times I \hookrightarrow (M,\xi)$



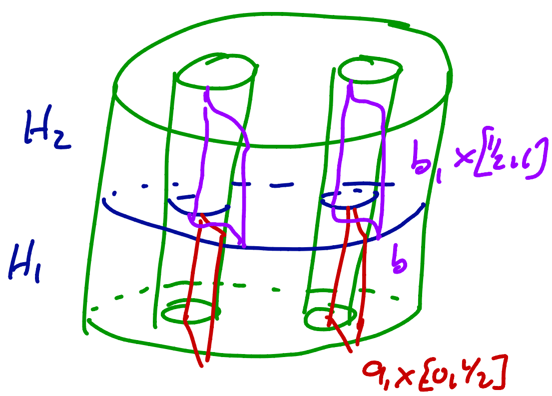
$$\xi_n = \ker(\cos(2\pi n z)\,dx + \sin(2\pi n z)\,dy)$$

Question: $c(\xi)=0 \Rightarrow$ Giroux torsion?

No: $c\Big($  $\times S^1\Big) = 0$

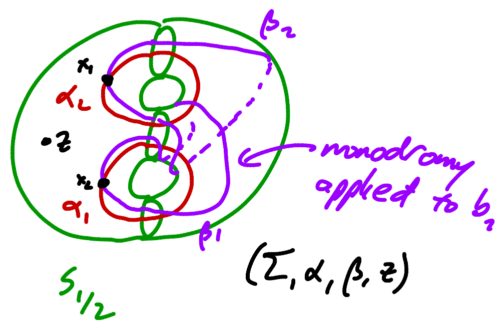
so if (M,ξ) contains this then $c(\xi)=0$

note open book gives Heegaard decomp



$$\partial H_1 = \partial H_2 = \Sigma = S_{1/2} \cup (-S_0)$$

$$S_0 \overset{\phi}{=} S_1$$



$(\Sigma, \alpha, \beta, z)$

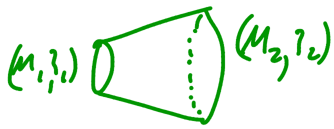
$x_\xi = (x_1, x_2) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$c(\xi) = [x_\xi] \in HF(-M)$

Th$^{\underline{m}}$ (Kutluhan - M - VHM - Wand) "spectral order" of ξ

there is an invariant $o(\xi) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ that has the following properties

1. $o(\xi) = 0$ if ξ is overtwisted
2. $o(\xi) = \infty$ if ξ is Stein fillable
3. $o(\xi)$ is non-decreasing under Stein cobordism



$\Rightarrow o(\xi_1) < o(\xi_2)$

spectral order can be detected in a single open book

Th$^{\underline{m}}$: If (M_1,ξ_1) and (M_2,ξ_2) are two ct strs, then

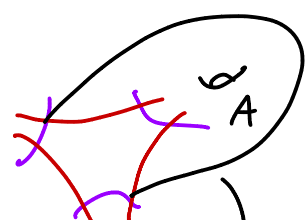
$$o(M_1 \# M_2, \xi_1 \# \xi_2) = \min(o(\xi_1), o(\xi_2))$$

There are families of examples (M_k, ξ_k) with $o(\xi_k)$ taking ∞ many values and $c(\xi_k) = 0$

Question: $o(\xi) = 0 \Rightarrow$ O.T. ?

$o(\xi) = \infty \Rightarrow$ fillable ?

$$HF(-M) = H(CF(\Sigma, \alpha, \beta, z))$$
$$= H(CF(S, \phi, \underline{a}, J))$$
$\circlearrowleft \partial$



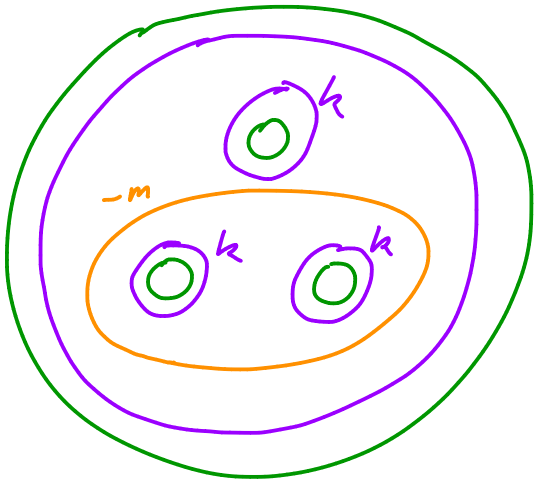
$$\partial x = \sum_{Y} \sum_{A \in \pi_2(x,y)} \#(\ )\, y$$

$J_+(A) = $ filtration $= 2\ell$

$$\partial = \partial_0 + \partial_1 + \ldots + \partial_\ell + \ldots$$

$$\partial_\ell = \sum_{Y} \sum_{\substack{A \text{ w/} \\ J_+(A) = 2\ell}} \#(\ )\, y$$

$o(\xi)$ is "first" ℓ st. $c(\xi)$ dies examples:



$k \geq 2 \quad m > k$

$(M_{k,m}, \xi_{k,m})$

$c(\xi) = 0$

$\lceil k/4 \rceil \leq o(\xi) \leq k$