Satellites and Concordance

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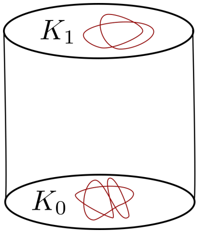
Concordance of knots

Definition

Knots $K_0, K_1 \hookrightarrow S^3$ are concordant if



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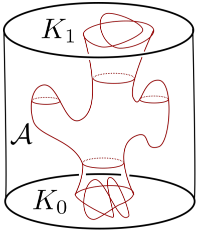

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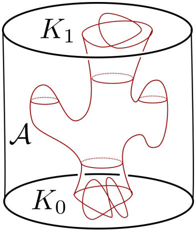
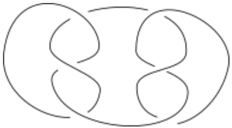


Warning! Whether we require a smooth or just topological(ly flat) embedding of $\mathcal{A}$ matters!

A concordance from $T_{2,3} \# - T_{2,3}$ to...



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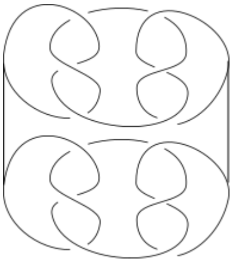

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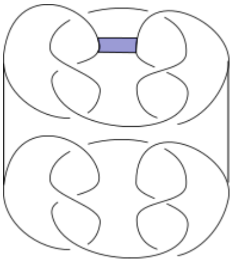

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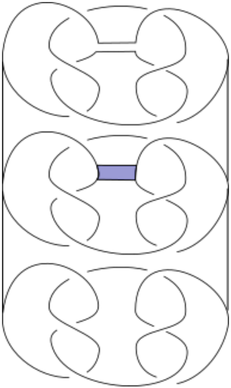

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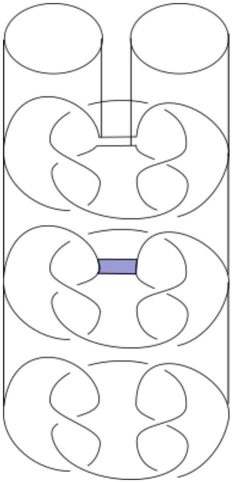
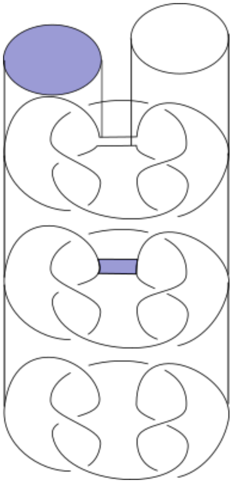


A concordance from $T_{2,3} \# - T_{2,3}$ to the unknot!



The concordance set

Definition

$$\mathcal{C}_* := \{\text{knots in } S^3\}/\sim_*, \;\; \text{where } * = \text{sm}, \text{top}$$

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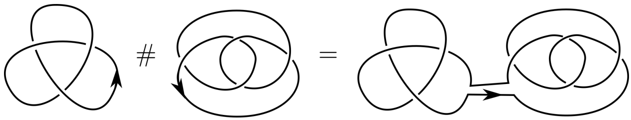

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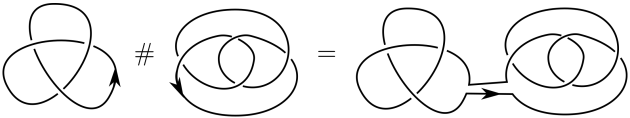

Theorem (Fox-Milnor)

The map $[K] + [J] := [K \# J]$ *is well-defined on* $\mathcal{C}_*$.

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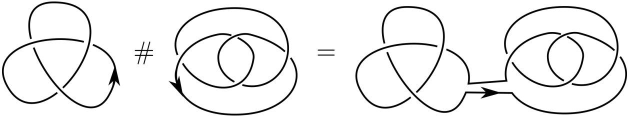

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The map $[K] + [J] := [K \# J]$ *is well-defined on* $\mathcal{C}_*$. *Moreover, it induces the structure of an abelian group!*

Known: $\mathcal{C}^*$ contains a $\mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty$-summand.

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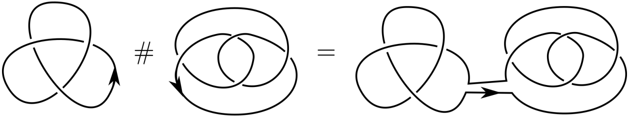

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Unknown: $\mathbb{Q} \hookrightarrow \mathcal{C}_*$? $\mathbb{Z}_n \hookrightarrow \mathcal{C}_*$, $n > 2$?

The satellite construction

Any $P\colon S^1 \hookrightarrow D^2 \times S^1$ defines a map on the set of knots in S^3:

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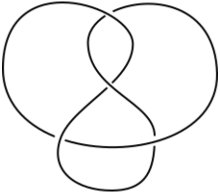

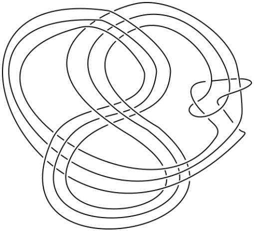

P:

$K \mapsto P(K)$

Satellites induce maps on concordance

Proposition

Let P be any pattern and K_0 and K_1 be concordant knots.
Then $P(K_0)$ and $P(K_1)$ are concordant.

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and observe that this is a concordance from $P(K_0)$ to $P(K_1)$! □

Satellites and concordance

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2. When does P induce a group homomorphism?

Winding number

Definition

Given a pattern P, we have $[P] = k[\{pt\} \times S^1] \in H_1(D^2 \times S^1)$ for some $k \in \mathbb{Z}$. We call $k =: w(P)$ the <u>winding number of P.</u>

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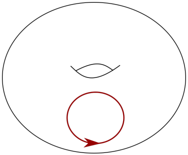	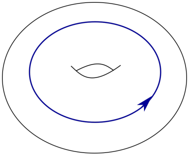	

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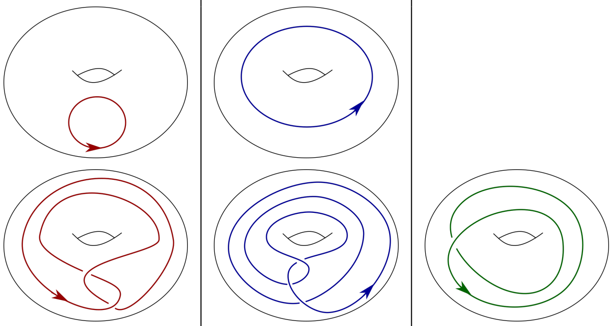

Satellite maps and surjectivity

Proposition (Folklore)

If P has $w(P) \neq \pm 1$, then P does not induce a surjection.

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"Easy": Uses classical invariants, e.g. Tristram-Levine signatures. □

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Theorem (Levine, 2014)

The Mazur pattern does not induce a surjection on $\mathcal{C}_{\mathrm{sm}}$.

Proof.

Difficult: Uses (bordered) Heegaard Floer theory! □

Satellite maps and injectivity

Proposition (M., 2018)

For each $n \in \mathbb{N}$, there exist winding number 0 patterns P which induce nonzero maps on $\mathcal{C}$ but for which there are at least n distinct concordance classes $K_1, \ldots, K_n$ such that $P(K_i)$ is slice for all i.

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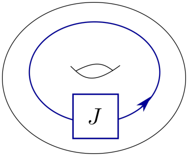

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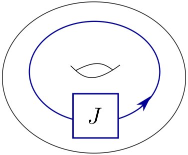
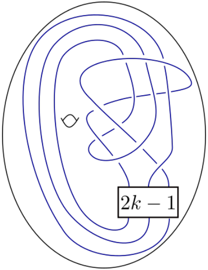


But boring:
$K \mapsto K \# J$.

Satellite maps and bijectivity



Theorem (M.-Piccirillo 2017)

There exist patterns P which induce bijective maps on $\mathcal{C}_{sm}$ and do not act by connected sum.

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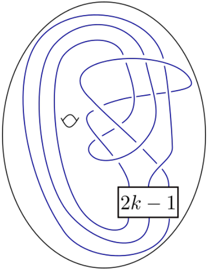

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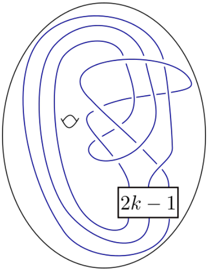

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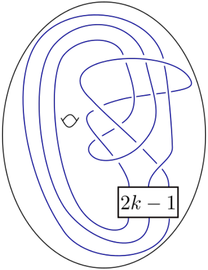

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Hard problem:
Do any winding number 1 patterns not act by connected sum on $\mathcal{C}_{\text{top}}$?

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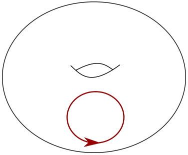

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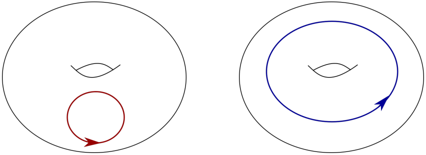

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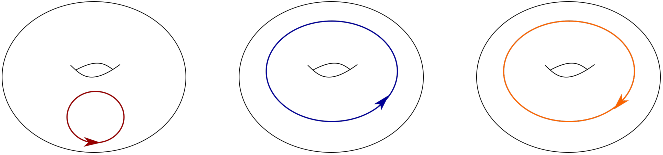

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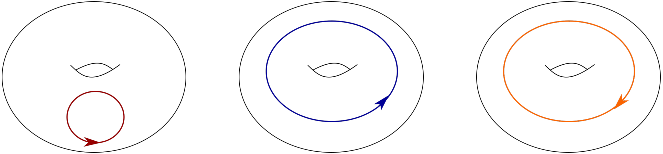

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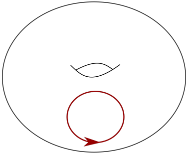

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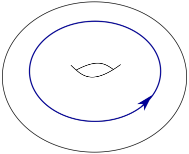

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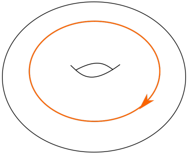

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Conjecture (Hedden)

If P induces a homomorphism on $\mathcal{C}$, then the induced map must be $K \mapsto K$, $K \mapsto U$, or $K \mapsto K^{rev}$.

Initial observations

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(i.e., the easily computed invariants– $\Delta_K(t)$, $\sigma_K(\omega)$– can't help!)

Some results

Theorem (Gompf; Levine; Hedden)

None of the Whitehead pattern, the Mazur pattern, or the (m,1) cable $C_{m,1}$ for $m > 1$ induce homomorphisms on $\mathcal{C}_{\mathrm{sm}}$.

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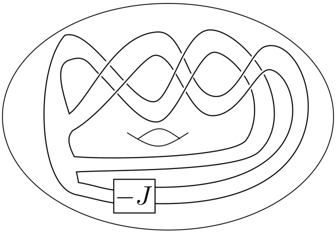
Problem

Given a pattern P with $P(U)$ slice, find an obstruction to P inducing a homomorphism on $\mathcal{C}_{top}$.

Winding number 0 case

Proposition (M.–Pinzón-Caicedo)

For any knot J, let P_J be the winding number 0 pattern shown. Then $P_J(U) \sim U$. Also, if $\sigma_J(e^{2\pi i/3}) \neq 0$, then P_J does not induce a homomorphism on $\mathcal{C}_{\text{top}}$.



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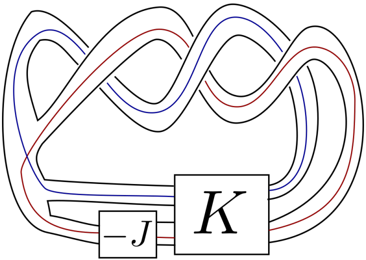

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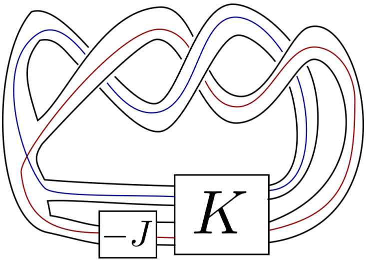

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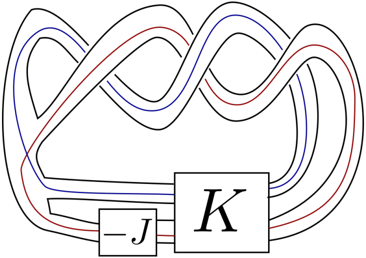

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3. $P_J(\#^n J) \not\sim U$ for $n >> 0$: Casson-Gordon signatures.

□

Casson-Gordon signatures

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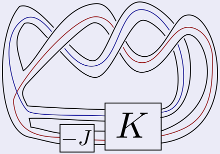
Proof of Step 3.



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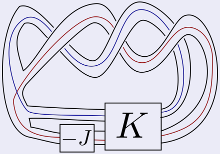

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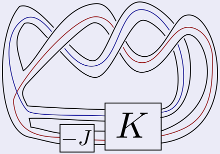

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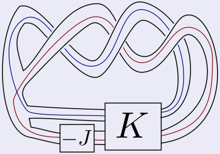

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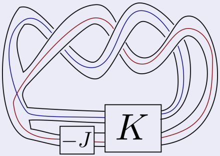

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So we can choose $n >> 0$ so that $\sigma(P_J(\#^n J),\chi) = 0$ only if $\chi(b) = 0$. But such characters do not vanish on a metabolizer for the torsion linking form. □

Nonzero winding number case

Theorem (M.–Pinzón-Caicedo)

For each $n \neq \pm 1$, there exist a pattern P_n of winding number n such that $P_n(U) \sim U$ and P_n does not induce a homomorphism on $\mathcal{C}_{\text{top}}$.

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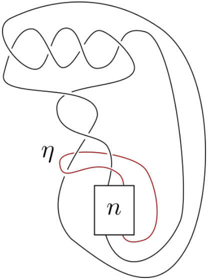

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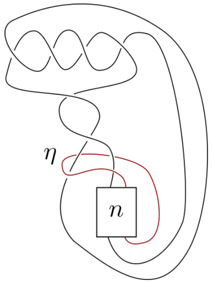

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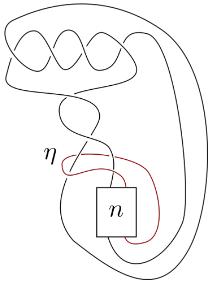

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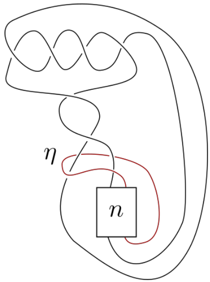

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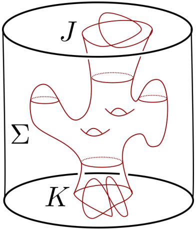
3. Analyse the linking form and show that $P(K \# K) \not\sim P(K) \# P(K)$ for some K.

□

The concordance ~~set~~ metric space



$$d([K],[J]) := \min\{g(\Sigma) : \Sigma \hookrightarrow S^3 \times I \text{ with } \partial\Sigma = -K \times \{0\} \sqcup J \times \{1\}\}.$$

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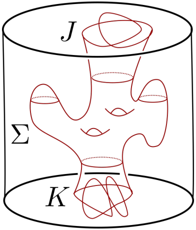

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Question

When do P and Q induce roughly the same action on $(\mathcal{C}, d)$?
i.e. When does there exist $C = C(P, Q)$ such that

$$d(P(K), Q(K)) \leq C \text{ for all } K \in \mathcal{C}.$$

When such a C exists, we say P and Q are 'bounded distance'.

Winding number and metric structure

Proposition (Cochran-Harvey, 2014)

If $w(P) = w(Q)$ *then* P *and* Q *are bounded distance.*

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Proof idea: Show that $d(P(\#^n T_{2,3}), Q(\#^n T_{2,3})) \to \infty$ via Tristram-Levine signatures.

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Idea: Casson-Gordon signatures again! □