Sucharit Sarkar Khovanov homotopy types

- Jones polynomial
- Khovanov homology
- Khovanov homotopy type.

① Jones polynomial.

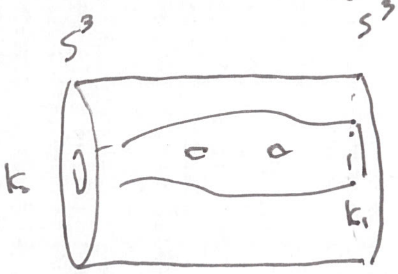
$V_K \in \mathbb{Z}[q, q^{-1}]$

[Tait conj (1880)
Alternating diagrams minimize crossing number.

by Kauffman, Murasugi.

$\sum_{i,j} (-1)^i q^j \, rk(Kh^{i,j}) = V_K$

② Khovanov homology.



- group : $Kh(K) = \bigoplus_{i,j} Kh^{i,j}(K)$.
- $\exists$ knots K_0 and K_1 with same Jones poly but different Khovanov homology.

- Rasmussen Lee perturbation of Khovanov homology (Bar-Natan) $\rightsquigarrow$
  - $s(K) \in 2\mathbb{Z}$.
  - $g_4(K) \geq \left| \frac{s(K)}{2} \right|$
  - $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.

Kronheimer - Mrowka. $Kh(K) \cong Kh(U) \iff K \cong U$.

Ng $\overline{\text{th}}$ comp from Kh } bounds.

$\overline{\text{sl}}$ comp from Kh

Plamenevskaya $\psi(K)$ transverse invariant.

Piccirillo Conway knot is not slice.

③ Khovanov homotopy type.

Robert Lipshitz] even
Tyler Lawson]

Construct a space

$(Kh^{i,j}) \leftarrow$ homology.

$H_*(C_{Kh}^{i,j}) = Kh^{i,j}.$

Khovanov chain complex. $\quad C_{Kh}^{i,j} \longrightarrow C_{Kh}^{i+1,j}.$

CW cplx X^j with $\tilde{C}^*_{cell}(X^j) = C_{Kh}^{*,j}.$

[chain homotopy type is a knot invt.
[stable homotopy type of X^j is a knot invt.

Thm. $\exists$ knots K_0 and K_1 with $Kh(K_0) = Kh(K_1)$

but $X(K_0) \not\simeq X(K_1)$ not stably homotopy equivalent.

- Extra structure

  stable homology operations.

  $$Sq^k : Kh^{i,j}(K;\mathbb{Z}_2) \longrightarrow Kh^{i+k,j}(K;\mathbb{Z}_2).$$

- Refinements of s-invt.

  $s_{Sq^2}(K)$ ← new invts. $\left(g_4(9_{42} \# T_{p,q}) = 1 + \frac{(p-1)(q-1)}{2}. \right)$

- $G \curvearrowright X$ localization theorem.

  $\mathbb{Z}_2 \curvearrowright X.$

  $$rk(H^*(X;\mathbb{Z}_2)) \geq rk(H^*(X^{fix};\mathbb{Z}_2))$$

  Smith.

<u>Stroffgen-Zheng.</u>

p-periodic knot.

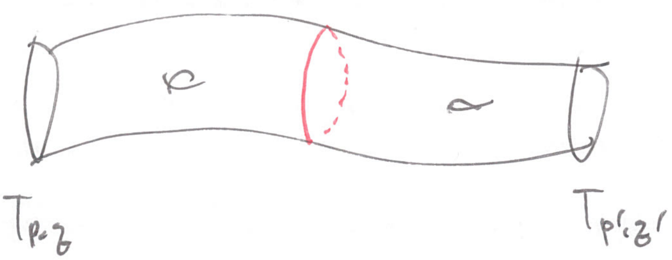
$\mathbb{Z}_p \curvearrowright X(K)$ fixed point $=$ AX(quotient knot).

In particular, $rk(Kh(K)) \geq rk(Kh(\bar{K}))$ (with $\mathbb{Z}_p$ coeff).

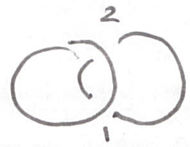
<u>Lewark - Lobb - Feller</u>

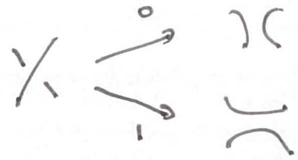
K squeezed if it is a slice of a minimal genus cobordism between torus knots

Obstruction being squeezed come from X(



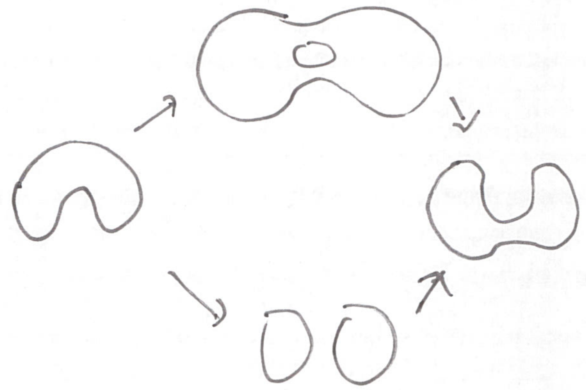
$$\underline{2} = \begin{tikzcd} 0 \arrow[r] \arrow[loop below, "id"] & 1 \arrow[loop below, "id"] \end{tikzcd}$$





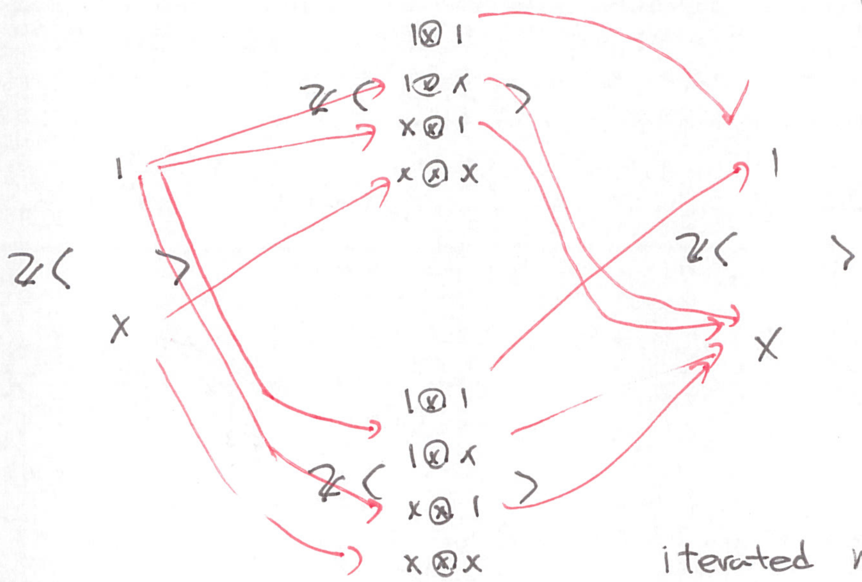
$$\boxed{\underline{2}^n \xrightarrow{Kh} Ab}$$

$$\begin{tikzcd} & 10 \arrow[dr] & \\ 00 \arrow[ur] \arrow[rr] \arrow[dr] & & 11 \\ & 01 \arrow[ur] & \end{tikzcd}$$



Kh

$$\begin{tikzcd} & \mathbb{Z}^4 \arrow[dr] & \\ \mathbb{Z}^2 \arrow[ur] \arrow[dr] & & \mathbb{Z}^2 \\ & \mathbb{Z}^4 \arrow[ur] & \end{tikzcd}$$



iterated mapping cone.

$$\begin{tikzcd} & Top \arrow[d, "H_*"] \\ \underline{2}^n \arrow[ur, dashed] \arrow[r] & Ab. \end{tikzcd}$$

Carlsson

No such section exists!

(Moore space is not functorial).

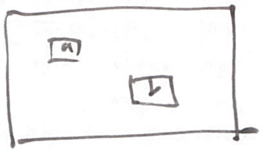
Problem.

$$\begin{tikzcd} S^t \arrow[r, "f"', "\deg 2."] \arrow[d] & S^t \arrow[d] \\ \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \end{tikzcd}$$

Choice $\pi_1 \cong \mathbb{Z}_2$.

Consider instead labeled boxes

S^k



space of labeled boxes in $[0,1]^k$.

$(k-2)$ connected.

As $k \to \infty$, contractible.

$$\begin{tikzcd} & \{a,b\} \arrow[dl] \arrow[dr] & \\ \mathbb{Z}\langle x\rangle & & \mathbb{Z}\langle y\rangle \end{tikzcd}$$

<u>Correspondence</u>

$$\begin{tikzcd} & \{a,b\} \arrow[dl, "t"'] \arrow[dr, "s"] & \\ \{x\} & & \{y\} \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "t"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$$

<u>Burnside category</u>

Object : finite sets X

morphism : $\mathrm{Hom}(X,Y) =$ $\begin{tikzcd} & A \arrow[dl, "t"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$

$$\begin{tikzcd} & \mathrm{Top} \arrow[d] \\ B \arrow[ur] \arrow[r] & \mathrm{Ab} \end{tikzcd}$$