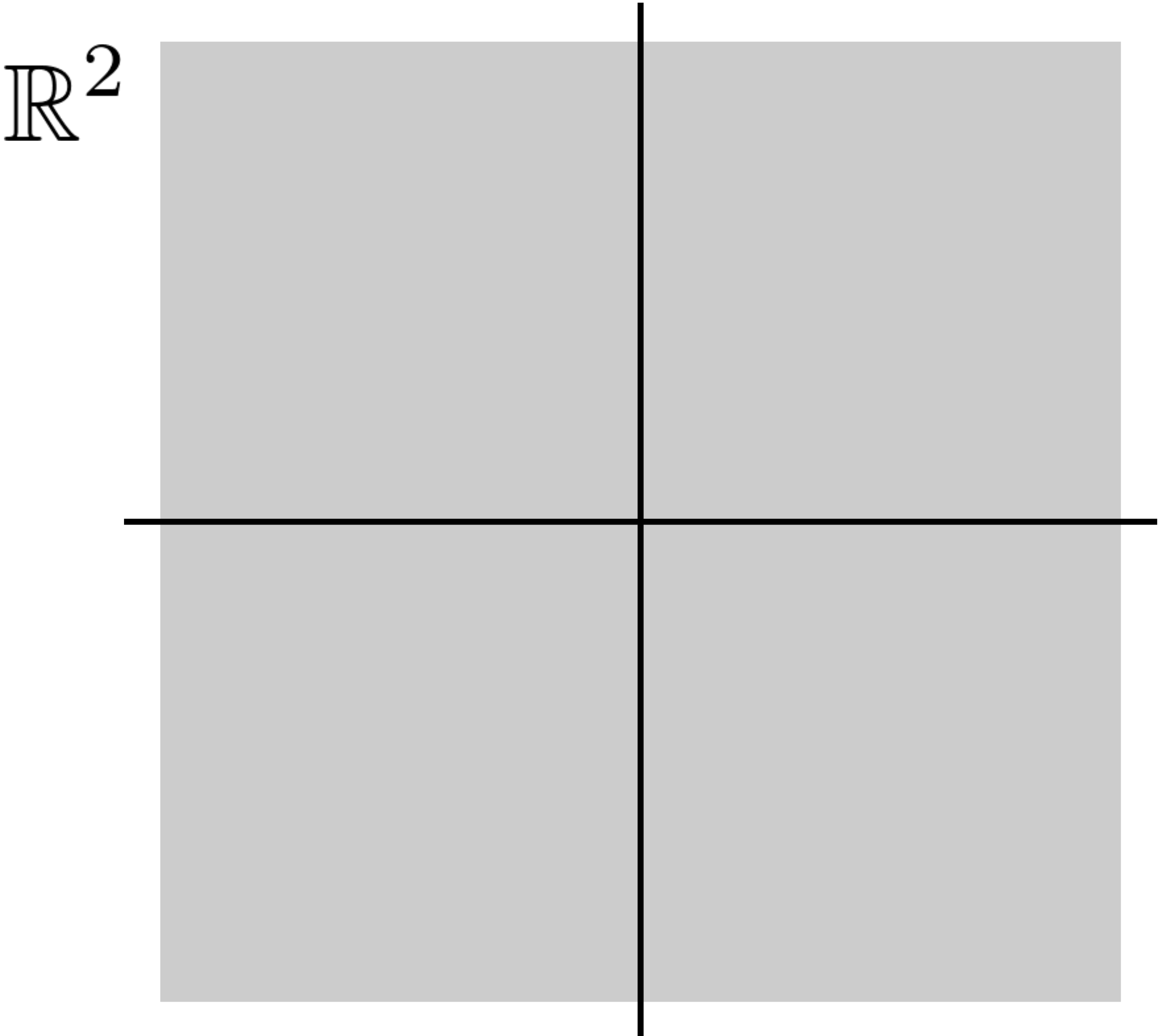
Lightning Talks II
Tech Topology Conference

December 8, 2018

Adding points to configurations in closed balls

Justin Lanier, Georgia Tech
(with Lei Chen & Nir Gadish)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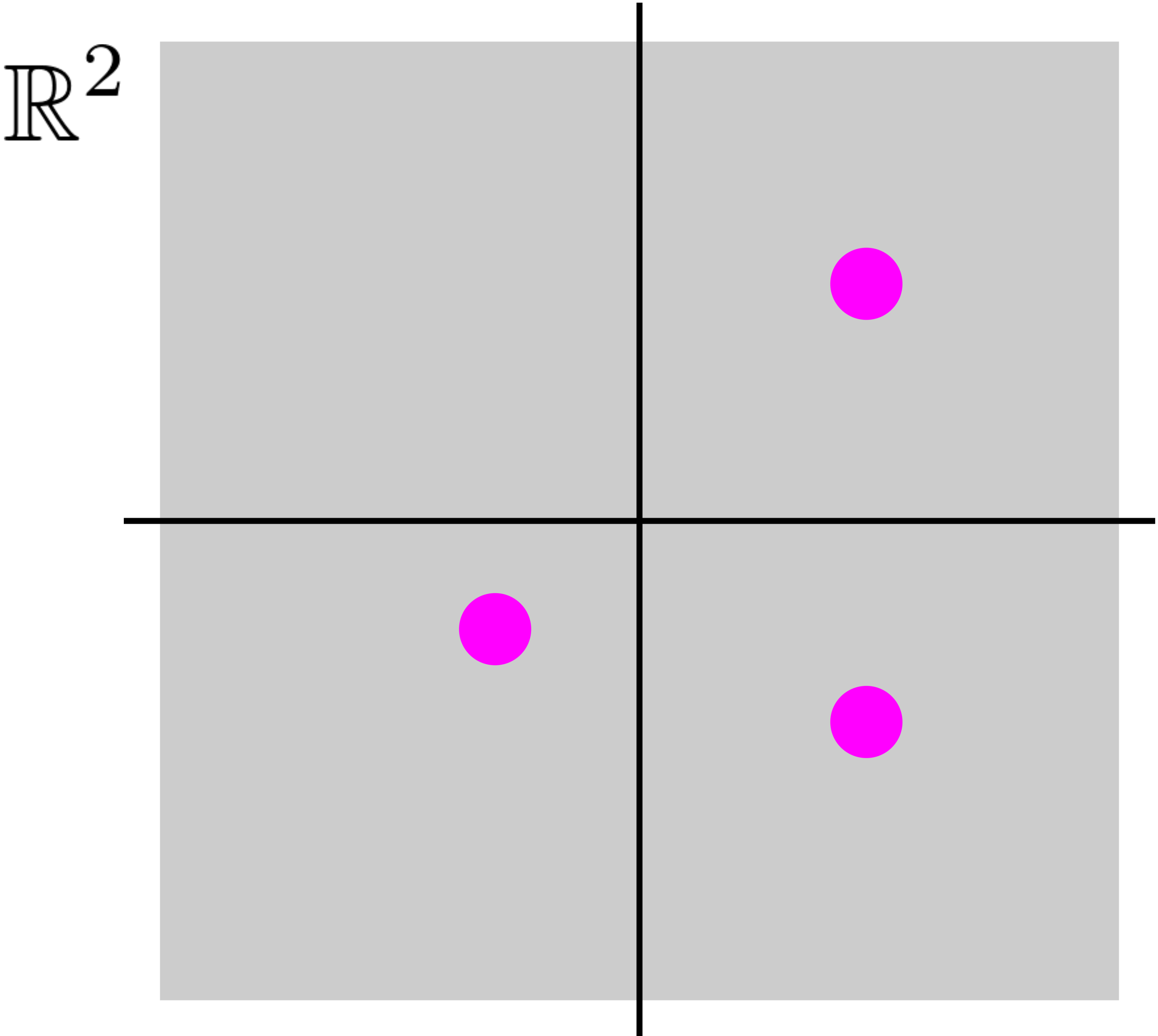
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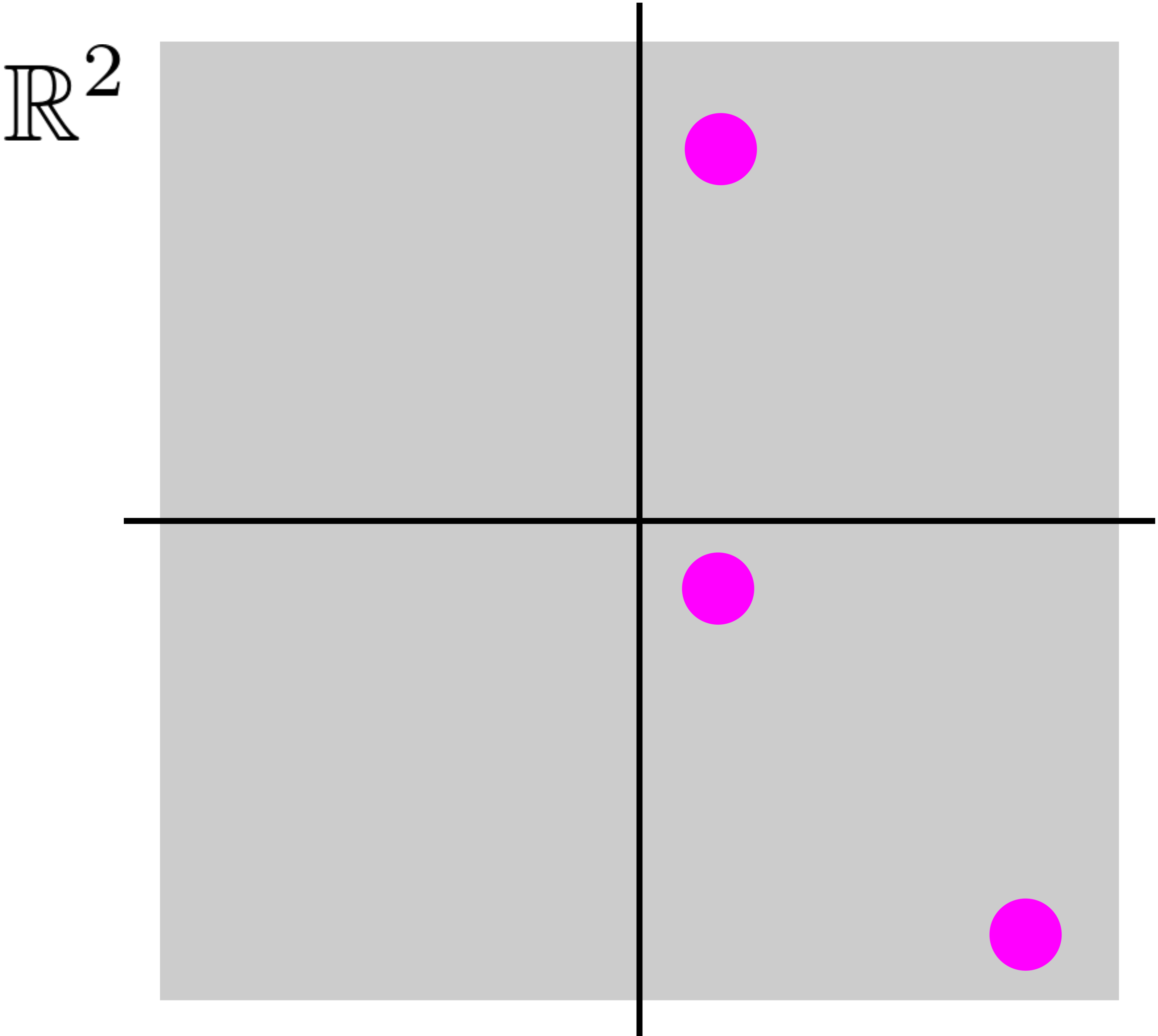
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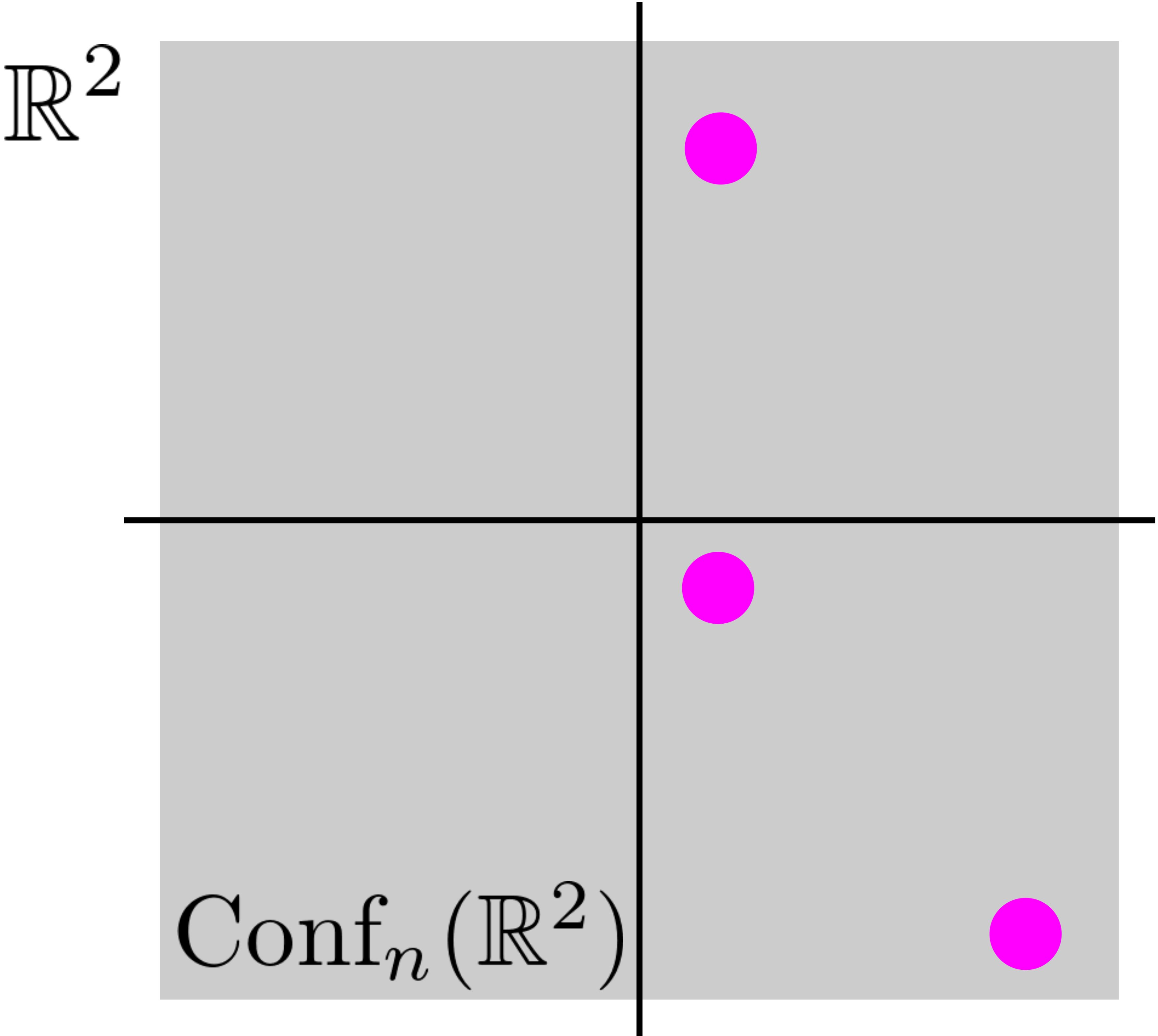
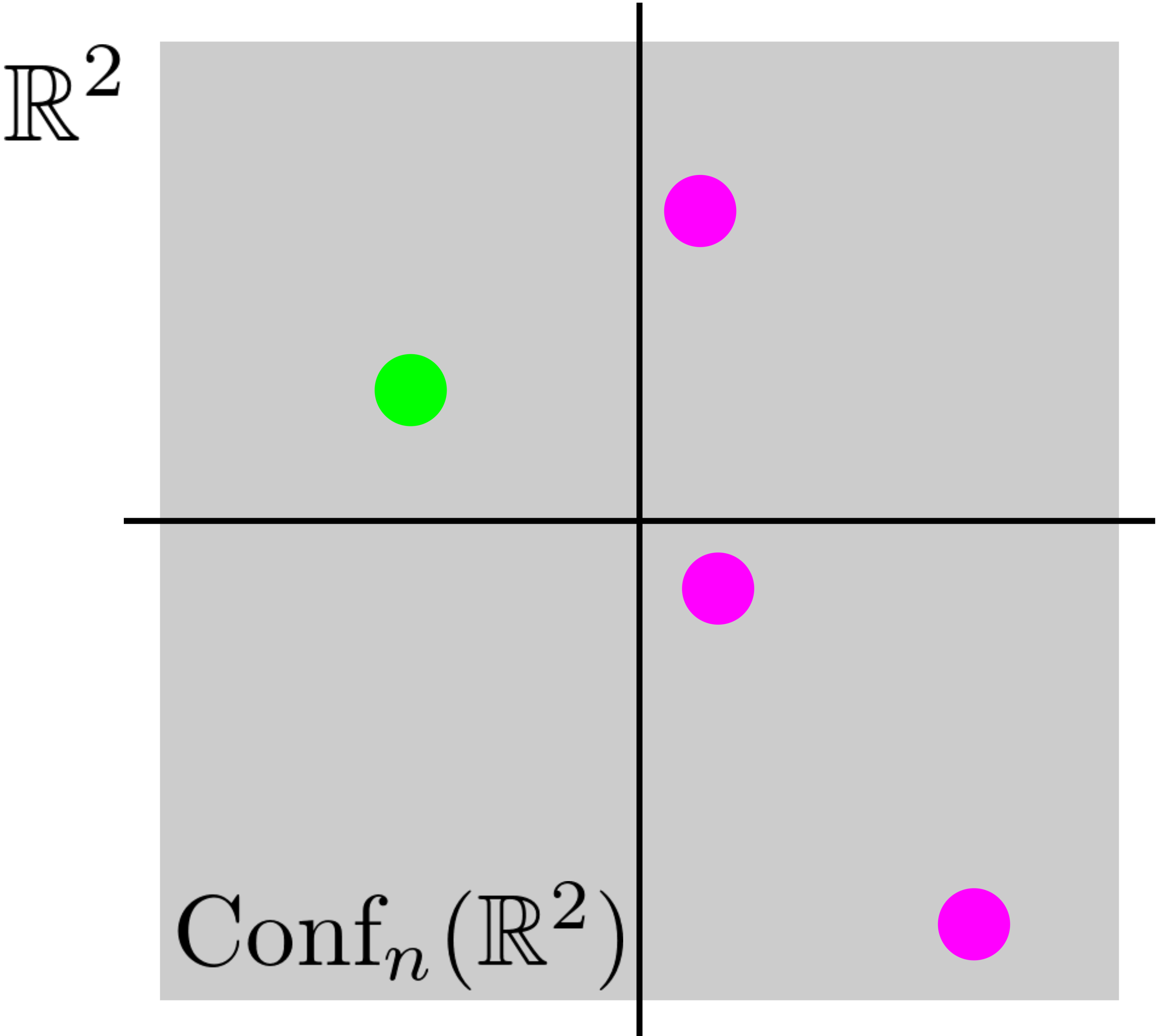

$\mathbb{R}^2$
$\mathrm{Conf}_n(\mathbb{R}^2)$


$\mathbb{R}^2$
$\mathrm{Conf}_n(\mathbb{R}^2)$

add a point

$$\mathrm{Conf}_n(X) \to X$$

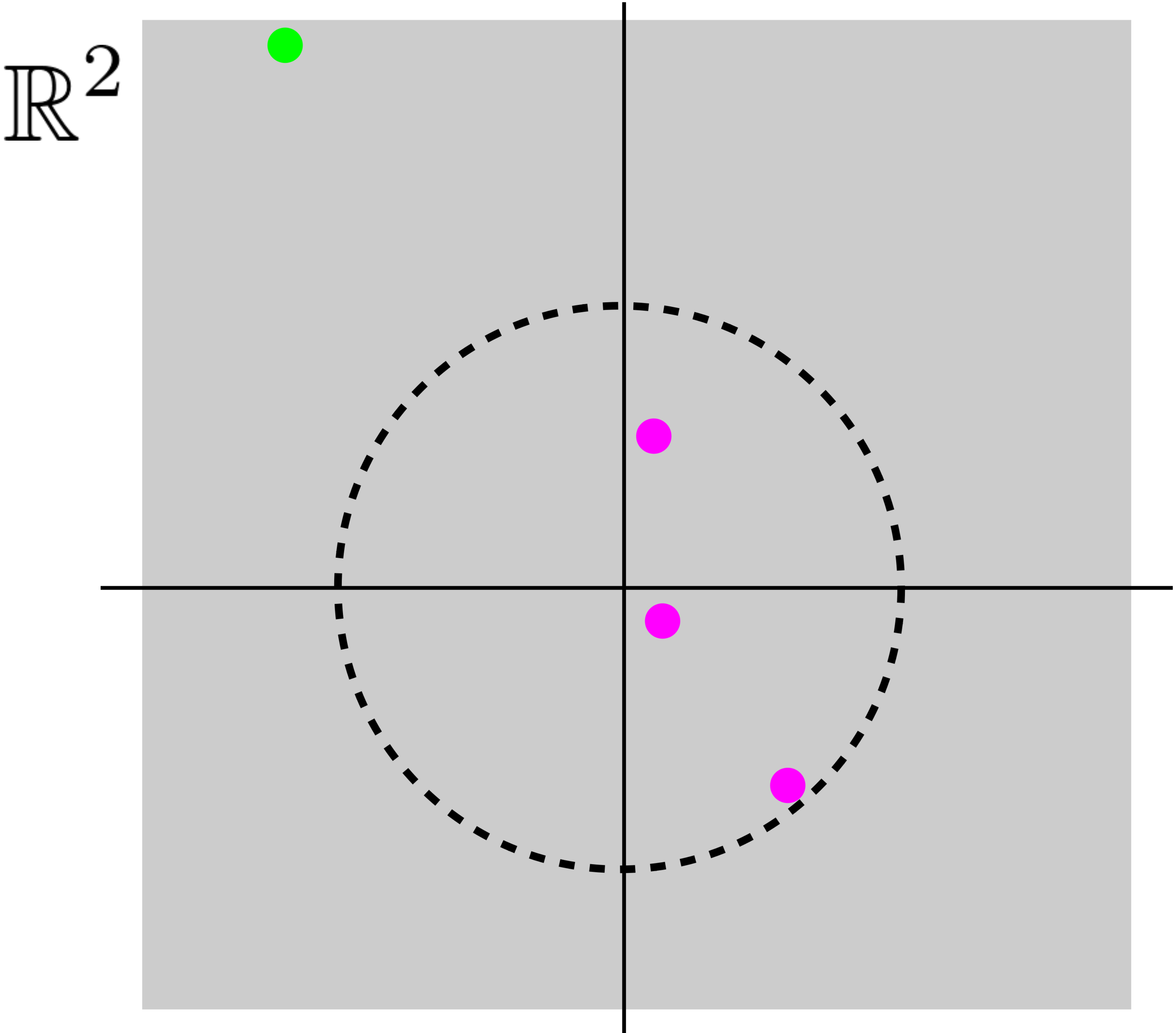
add a point

$$\mathrm{Conf}_n(X) \to X$$

or

$$\mathrm{Conf}_n(X) \xleftarrow{\text{forget}} \mathrm{Conf}_{n,1}(X)$$

$$\mathrm{Conf}_n(X) \xrightarrow[\text{section}]{} \mathrm{Conf}_{n,1}(X)$$

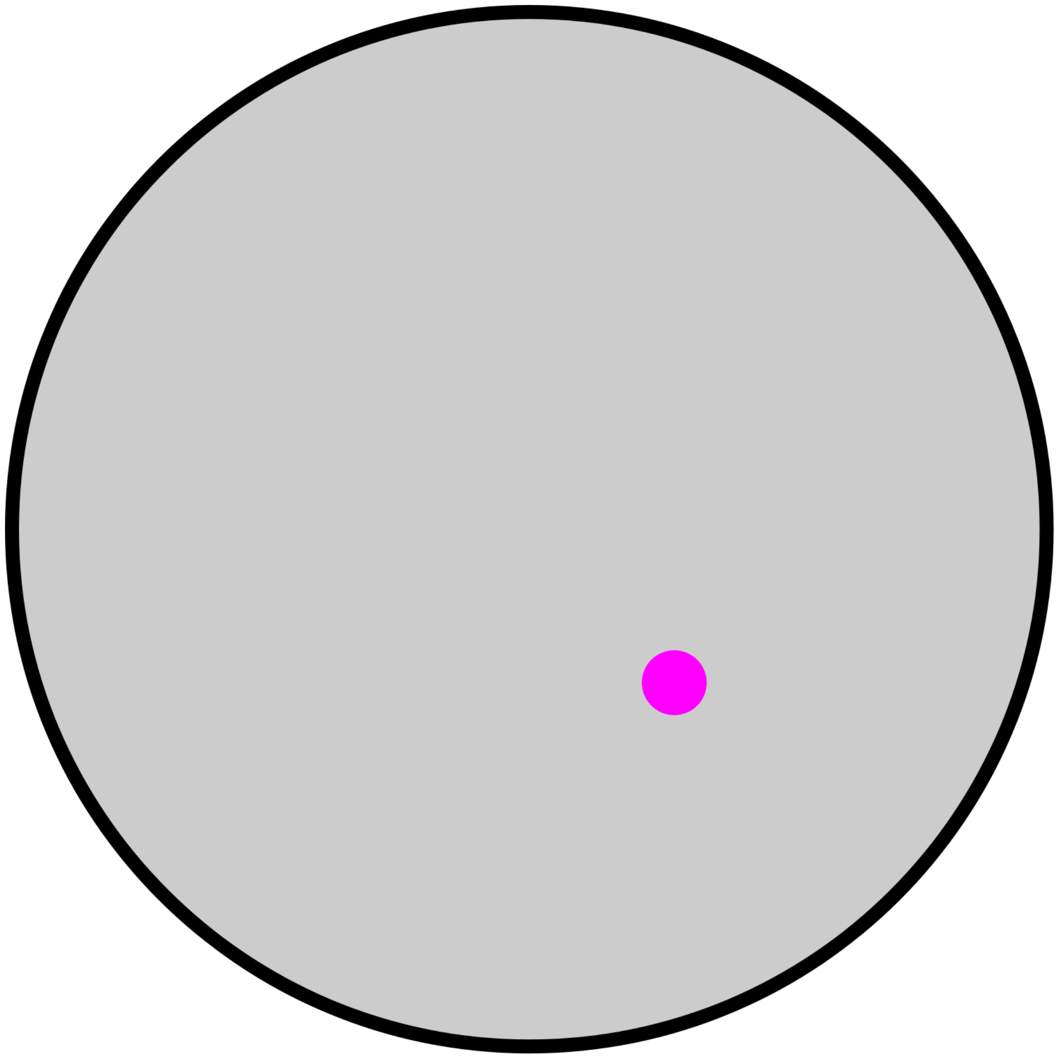

$\mathbb{R}^2$

Prior work:

(Chen): <u>labeled</u> points in $\mathbb{R}^2, \mathbb{S}^2, S_g$

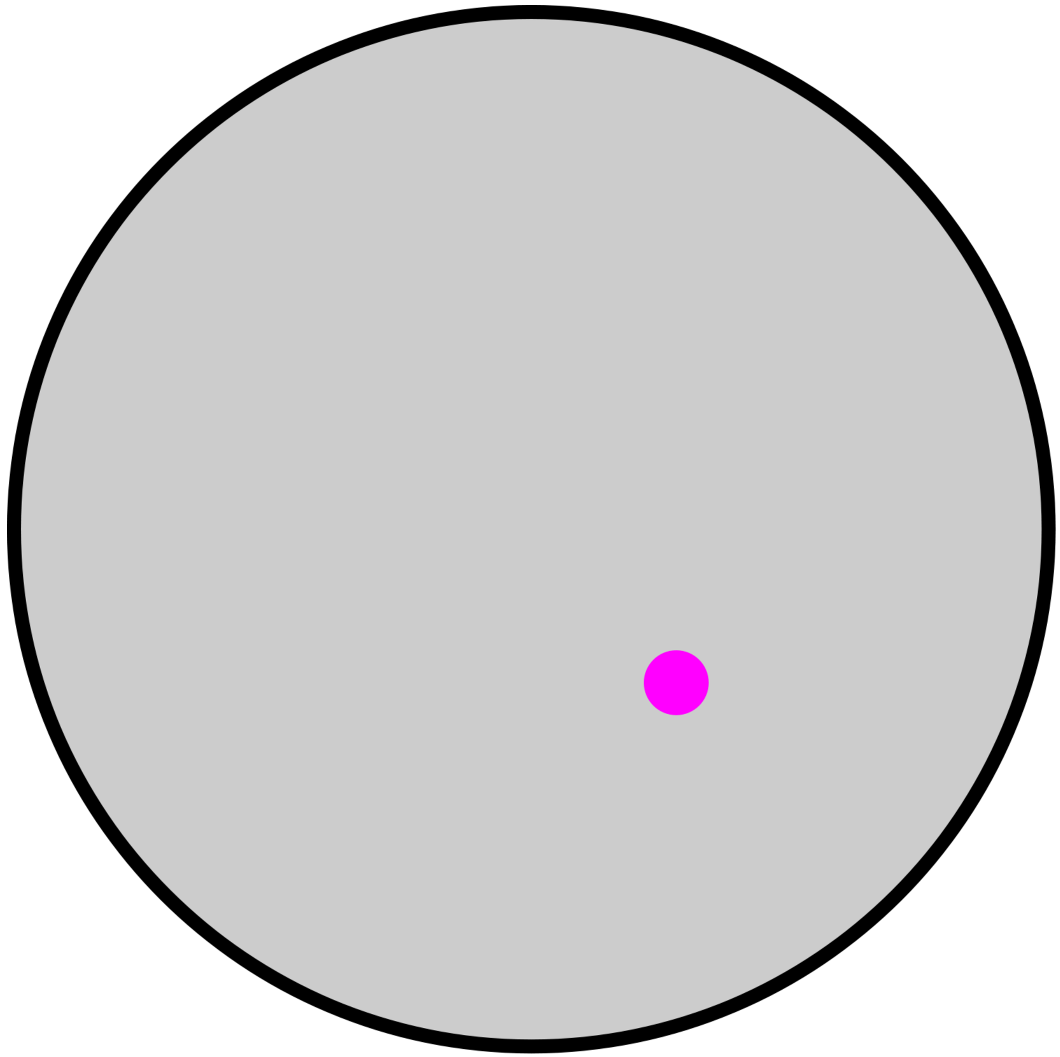
(Chen–Salter): add <u>many</u> points in $\mathbb{S}^2$

$n = 1$



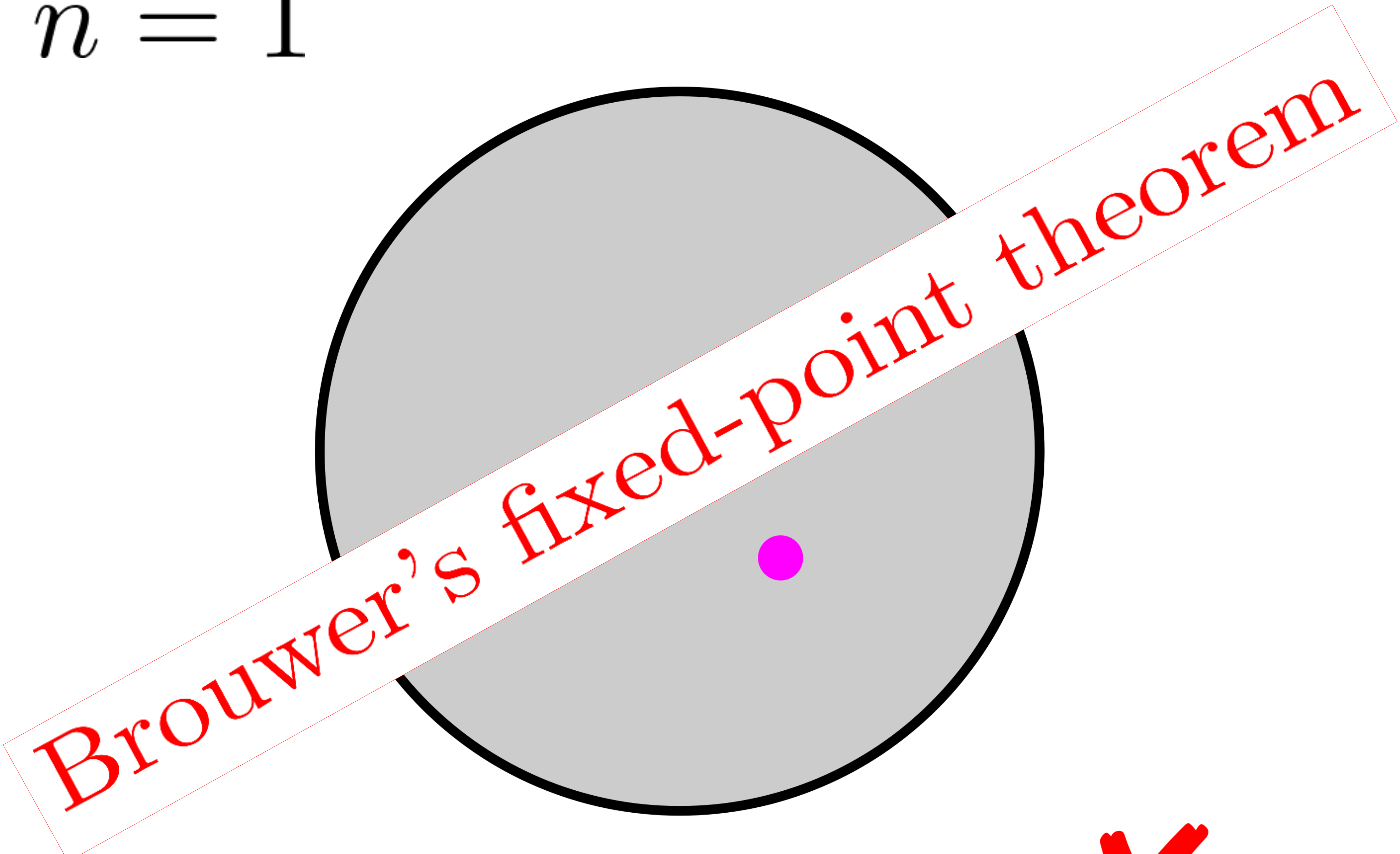
$$\mathrm{Conf}_1(\mathbb{B}^2) \to \mathbb{B}^2?$$

$n = 1$



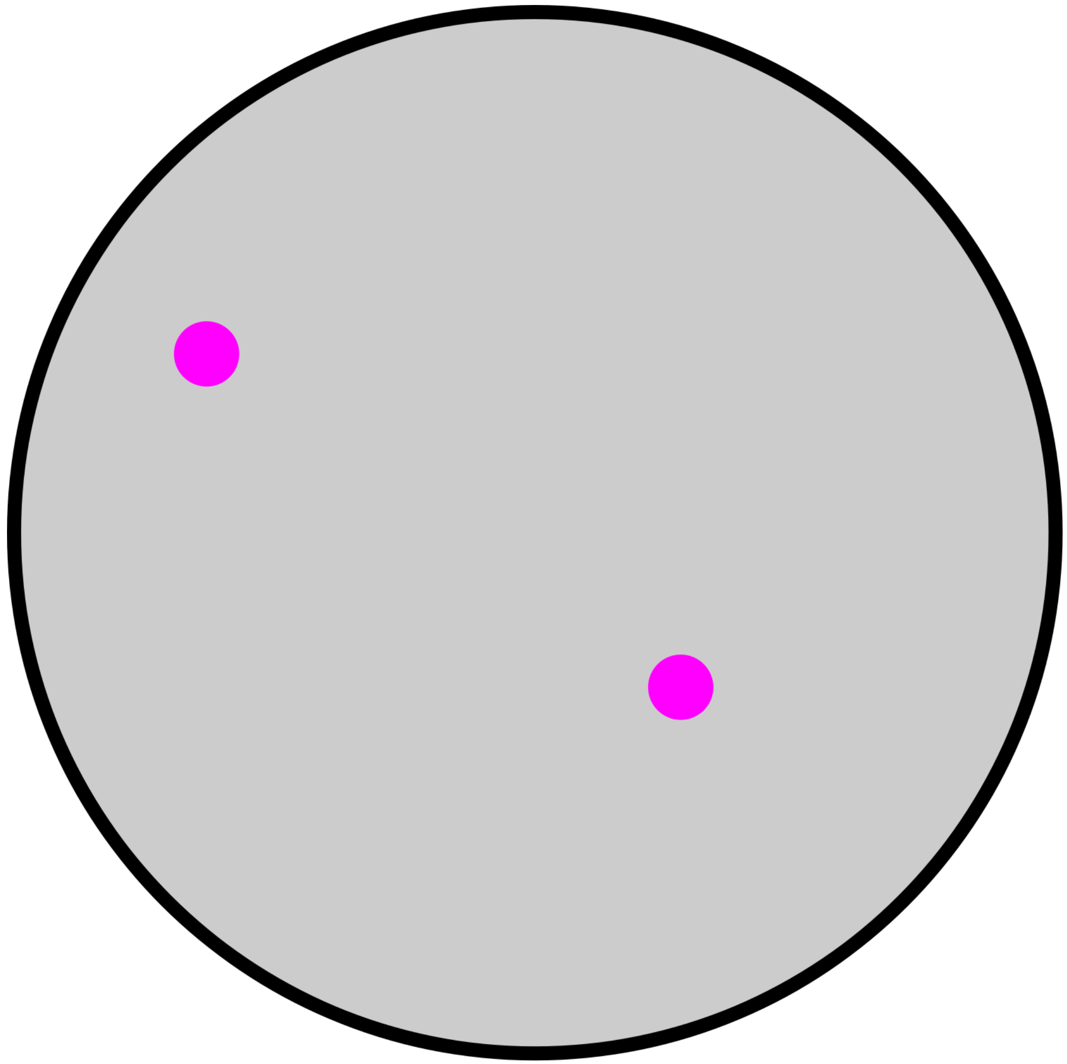
$\mathbb{B}^2 \to \mathbb{B}^2?$

$n = 1$



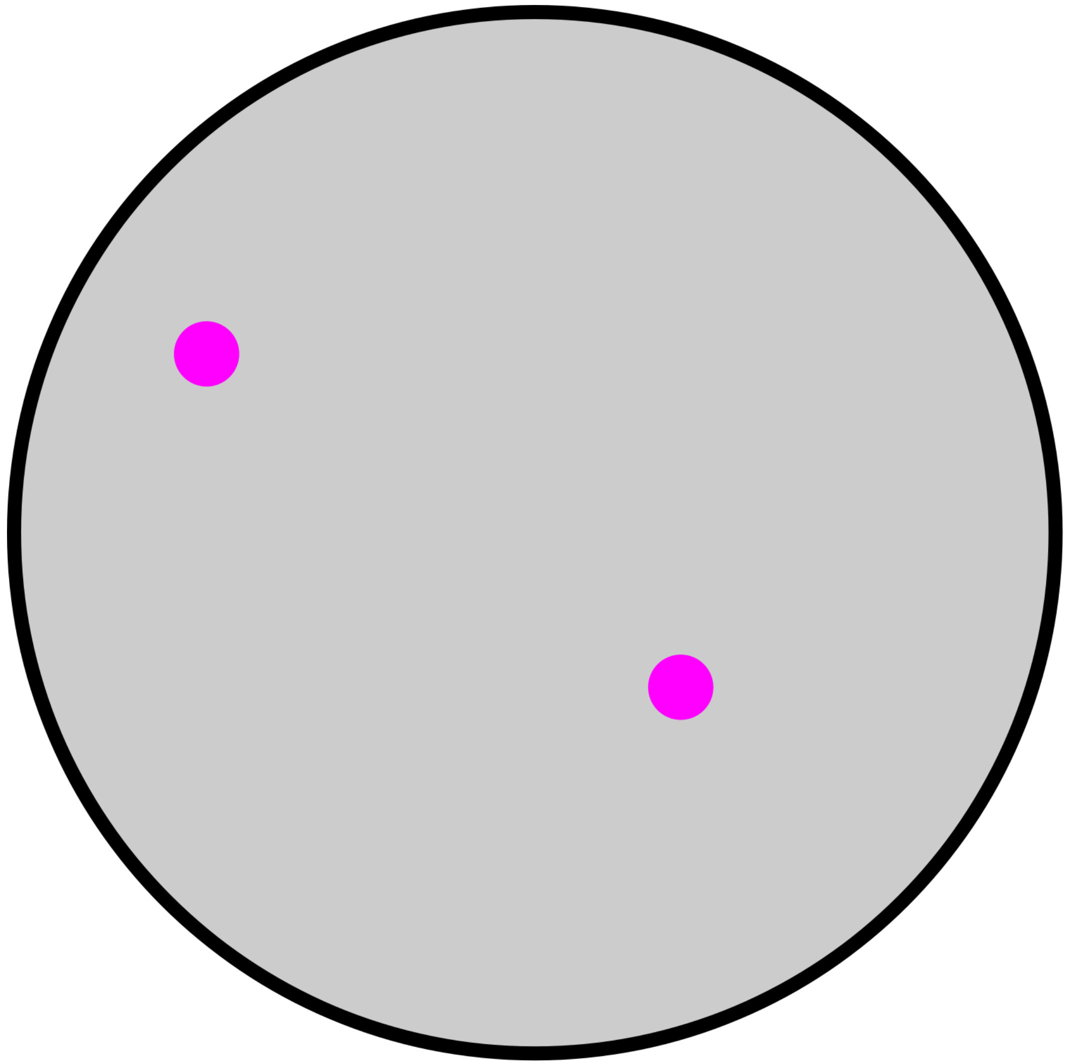
$\mathbb{B}^2 \to \mathbb{B}^2$ ✗

$n = 2$



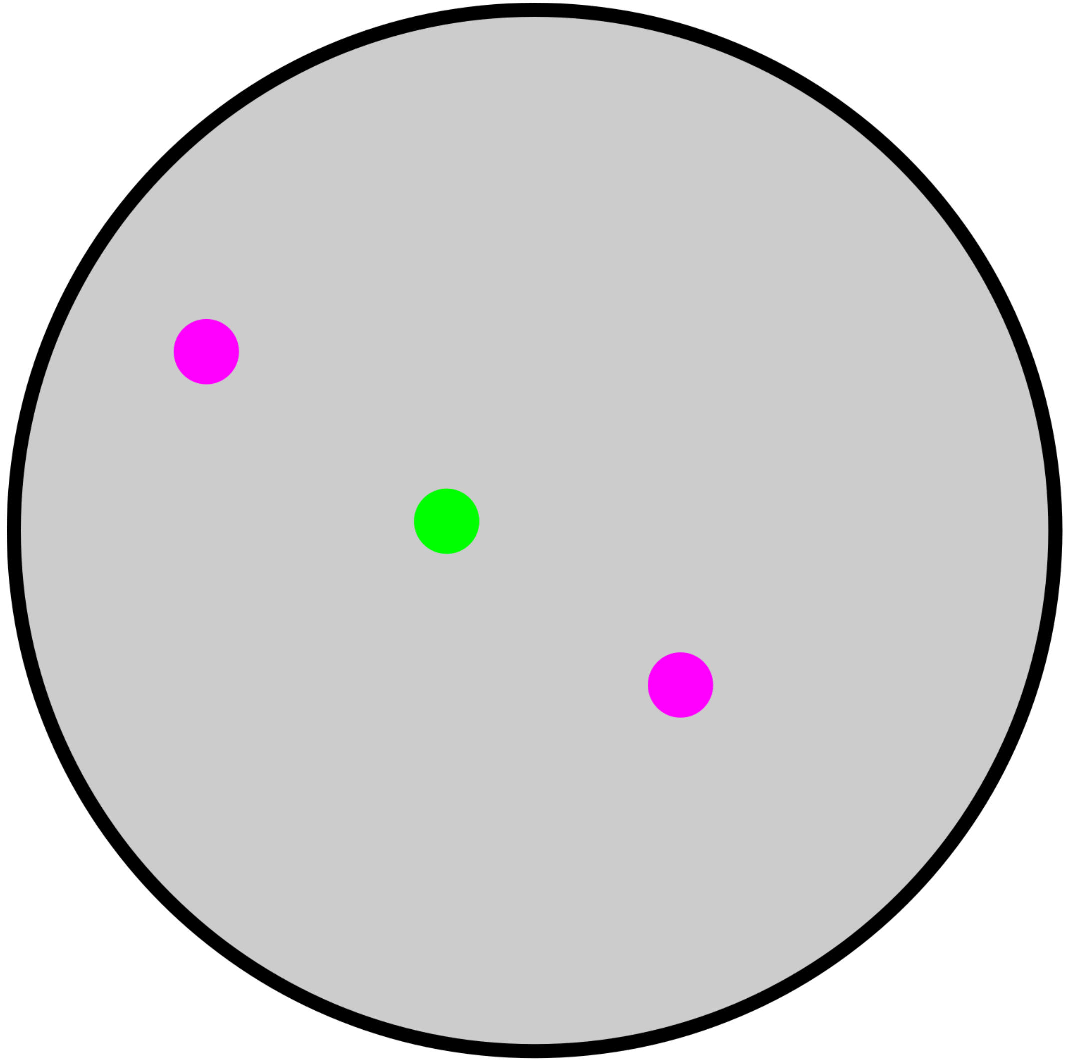
$$\mathrm{Conf}_2(\mathbb{B}^2) \to \mathbb{B}^2?$$

$n = 2$



$\mathrm{Conf}_2(\mathbb{B}^2) \to \mathbb{B}^2$ ✓

$n = 2$



$\mathrm{Conf}_2(\mathbb{B}^2) \to \mathbb{B}^2$ ✓

points	add?
1	✗
2	✓
3	
4	
5	
⋮	

Difficulties:

- $\mathrm{Conf}_n(\mathbb{B}^m) \simeq \mathrm{Conf}_n(\mathbb{R}^m)$
- forget is not a fibration for $\mathbb{B}^m$

Theorem (Chen–Gadish–L)

points	add?
1	✗
2	✓
3	✗
4	✗
5	✗
⋮	⋮

Theorem (Chen–Gadish–L)

For $m \geq 2$, the forgetful map

$$\mathrm{Conf}_{n,1}(\mathbb{B}^m) \to \mathrm{Conf}_n(\mathbb{B}^m)$$

has a continuous section if and only if $n = 2$.

Idea of proof:

- Classify Σ_n-equivariant sections for $\mathrm{PConf}_n(\mathbb{R}^m)$ via cohomology.
- Any section for $\mathrm{Conf}_n(\mathbb{B}^m)$ yields a "point on boundary" section.
- Pull back a cohomology class in two ways, get two different answers.

Things to try:

- use other spaces
- add more points
- classify sections

Things to try:

- use other spaces
- add more points
- classify sections

Thanks

Cryptographic Applications of Braids

Sinem Çelik Onaran

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December 7-9, 2018

Cryptosystems

- Private-key cryptosystem: Alice and Bob have secret key f_K, f_K^{-1}. Alice wants to send the message m.

Alice	sends $f_K(m)$
Bob	computes $f_K \circ f_K^{-1}(m) = m$

- Public-key cryptosystem: Bob makes f_B public, keeps f_B^{-1} secret.

Public key	f_B
Alice	sends $f_B(m)$
Bob	computes $f_B \circ f_B^{-1} = m$

Widely used protocol

The Diffie-Hellmann **key** exchange protocol

- **Problem:** Given g^x mod p for a prime p and x, $g \in \mathbb{Z}_p$, compute x.

Public key	$g \in \mathbb{Z}_p$
Private key Alice	$x \in \mathbb{Z}_p$
Private key Bob	$y \in \mathbb{Z}_p$
Alice	sends g^x
Bob	sends g^y
Alice & Bob	computes the key $K = (g^y)^x = (g^x)^y$

Conjugator search problem for braids

Protocols have problems.

- The same method is used for data transfer.
- Future is quantum computers.

For solving such problems, people look for new public cryptosystems.

- New braid-base public cryptosystems are introduced.

Conjugator search problem for braids

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- The same method is used for data transfer.
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- New braid-base public cryptosystems are introduced.

Two braids σ_1, σ_2 are conjugate if $\sigma_2 = b\sigma_1 b^{-1}$ for some braid b.

- **Conjugator search problem:** Given two conjugate braids σ_1, σ_2, find b such that $\sigma_2 = b\sigma_1 b^{-1}$.

The Anshel-Anshel-Fisher-Goldfeld Scheme

Public key	braids $\{\sigma_1, \sigma_2, \ldots, \sigma_m\} \subset B_n$
Private key Alice	$a \in <\sigma_1, \sigma_2, \ldots, \sigma_m>$
Private key Bob	$b \in <\sigma_1, \sigma_2, \ldots, \sigma_m>$
Alice	sends $(a\sigma_1 a^{-1}, \ldots, a\sigma_m a^{-1})$
Bob	sends $(b\sigma_1 b^{-1}, \ldots, b\sigma_m b^{-1})$
Alice & Bob	computes the key $K = aba^{-1}b^{-1}$

The Diffie-Hellmann like Scheme

- Let $LB_n = < \sigma_1, \ldots, \sigma_{m-1} >$, and $UB_n = < \sigma_{m+1}, \ldots, \sigma_{n-1} >$ with $m = \lfloor n/2 \rfloor$.

Public key	$x \in B_n$
Private key Alice	$a \in LB_n$
Private key Bob	$b \in UB_n$
Alice	sends axa^{-1}
Bob	sends bxb^{-1}
Alice & Bob	computes the key $abxb^{-1}a^{-1} = baxa^{-1}b^{-1}$

What my student wants

- The protocols use normal form, greedy normal form of braids (from the work of Garside and developed by Dehorny)

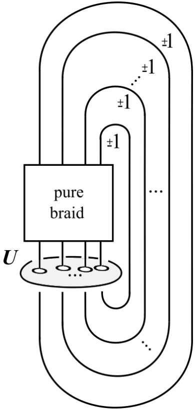
He wants to improve the protocols, he wants to find faster algorithms to compute the normal forms of braids.

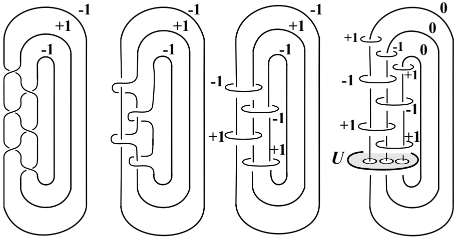
$$\text{2 steps/ efficient} \qquad \begin{array}{rcc} & 2 & 3 \\ + & 2 & 3 \\ \hline & 4 & 6 \end{array}$$

$$\text{23 steps/ inefficient} \quad 23, 24, 25, \ldots, 44, 45, 46$$

What I want

I want to use Kirby Calculus in protocols.





Contact structures on hyperbolic 3-manifolds

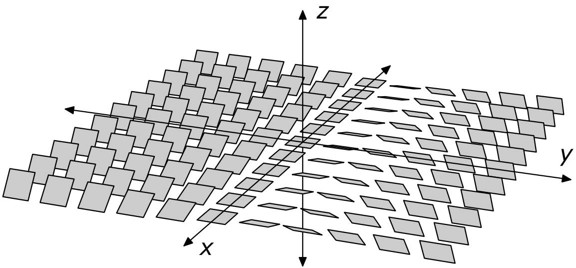
Hyunki Min

Georgia Tech

Joint work with James Conway

Contact structures

- A contact structure on a 3-manifold M is a plane field $\xi = \ker \alpha$ where $\alpha \in \Omega^1(M)$, $\alpha \wedge d\alpha > 0$.



Tight & overtwisted contact structures

- An overtwisted disk is an embedded disk tangent to the contact planes along the boundary.
- A contact structure is called overtwisted if it contains an overtwisted disk.
- A contact structure is called tight if it does not contain an overtwisted disk.

Theorem (Eliashberg 1989)

There is a one to one correspondence between overtwisted contact structures (up to isotopy) and plane fields (up to homotopy).

Tight contact structures

Goal

Classify tight contact structures up to isotopy.

Prime manifolds

- Seifert fibration: Many results
- Toroidal: Many results
- Hyperbolic: No result

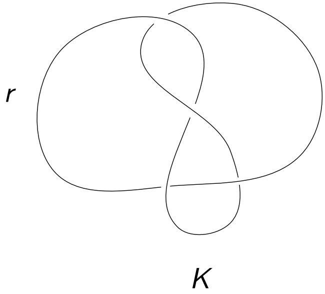
Hyperbolic manifolds

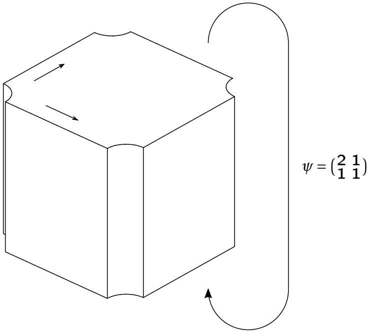
Why is it hard?

- Need a 'good' decomposition.
- Analyze contact structures in each piece.
- Most decompositions for hyperbolic manifolds are not simple enough.

Figure-8 knot

- Surgeries on the figure-8 knot yield hyperbolic manifolds.
- $S^3_r(K) \setminus N(K^*)$ is a punctured torus bundle over S^1 with a pseudo-Anosov monodromy.





Results I

Theorem (M – Conway 2018)

There are exactly two Stein fillable and universally tight contact structures on $S^3_{1/n}(K)$ for $n < -1$.

Results II

Theorem (M – Conway 2018)

Let r be a rational number. Then $S_r(K)$ supports

$$\begin{cases} 2\Phi(r), & r \in [1,4) \cup [5,\infty) \\ \Phi(r) + \Psi(r), & r \in (-\infty,-4) \cup [-3,0) \end{cases}$$

tight contact structures, where

$$-\frac{1}{r} = [r_0, \ldots, r_n] = r_0 - \cfrac{1}{r_1 - \cfrac{1}{\ddots - \frac{1}{r_n}}},$$

$$\Phi(r) = \left| r_0(r_1+1)\cdots(r_n+1) \right|.$$

$$\Psi(r) = \begin{cases} 0, & r \geq -3 \\ \Phi(-\frac{1}{r+3}), & r < -3. \end{cases}$$

Thank you!

Link homology, bridge trisections and knotted surface invariants

Adam Saltz (University of Georgia)

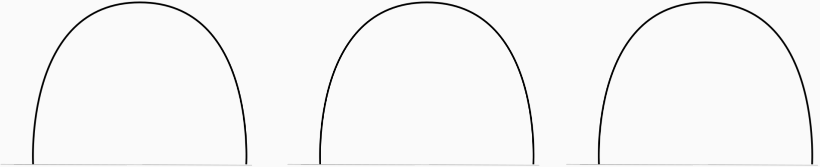
December 8, 2018

Georgia Tech
Tech Topology Conference

Meier and Zupan: any closed surface in S^4 can be divided into three nice sets of disks.

Theorem (Meier, Zupan)

$$\frac{\textit{surfaces in } S^4}{\textit{isotopy}} \cong \frac{\textit{triples of trivial tangles with } t_i\bar{t}_j \textit{ unlinked}}{\textit{some moves}}$$



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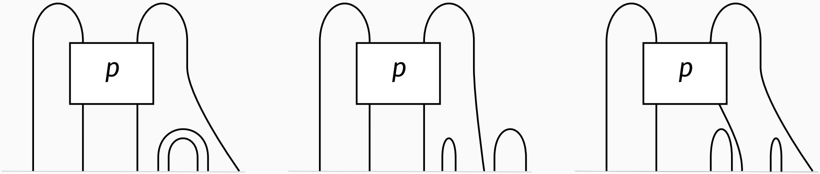


Link homology and surfaces

Link $L \longrightarrow$ group $\mathrm{Kh}(L)$

Cobordism $\Sigma\colon L \to L' \longrightarrow$ map $F_\Sigma\colon \mathrm{Kh}(L) \to \mathrm{Kh}(L')$

Not interesting for closed surfaces! (Rasmussen; Tanaka)

Question: Can we use link homology and bridge trisections to obtain interesting invariants of knotted surfaces?

Theorem (S.)

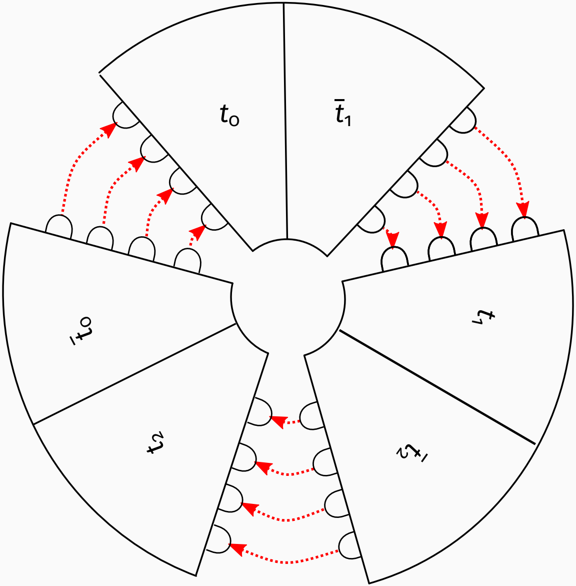
Let $\mathbf{t}$ *be a bridge trisection for the surface* Σ*.*

There is an invariant $q(\mathbf{t}) \in \mathbb{Z}/2\mathbb{Z}$ *of* Σ *defined using link homology.*

This invariant distinguishes the unknotted sphere from spun $(2, p)$*-torus knots.*

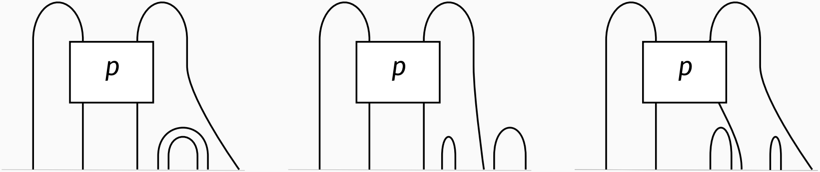
Uses a link homology theory due to Sarkar, Seed, Szabó (building on Bar-Natan and also Szabó)

The big picture



Relating the spun $(2, p)$-torus knots?

Write S_p for the spun $(2, p)$-torus knot.



Lemma (S.)

$q(S_p) = q(S_{p-1})$

What's the topological connection?

Towards a higher-dimensional construction of stable/unstable Lagrangian laminations

Sangjin Lee

University of California, Los Angeles

Tech Topology Conference, Dec 8, 2018

Surface case

- A surface automorphism $\psi : S \xrightarrow{\sim} S$ is of **pseudo-Anosov type** if there is a transversal pair of singular foliations which are preserved by ψ.

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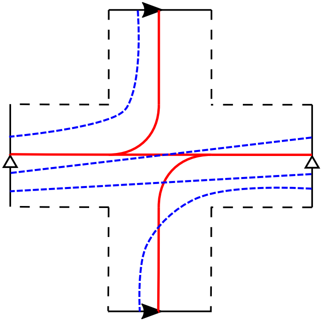
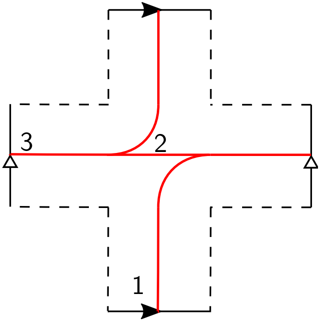
Surface case

- A surface automorphism $\psi : S \xrightarrow{\sim} S$ is of **pseudo-Anosov type** if there is a transversal pair of singular foliations which are preserved by ψ.
- Thurston iterated ψ on an isotopy class of a closed curve $c \subset S$.
- The sequence $\psi^n(c)$ can be encoded with a small amount of data τ_ψ, which is called a **train track**.

An example of a train track





Surface case

- One can construct a geodesic lamination $\mathcal{L}$ from τ_ψ such that $\lim_{n\to\infty} \psi^n(c) = \mathcal{L}$.

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- One can extend the stable geodesic lamination to a stable singular foliation.

Generalizations of train tracks

Theorem 1 (L, in preparation.)

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Generalizations of train tracks

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Let M be a symplectic manifold and let $\psi : M \overset{\sim}{\to} M$ be a symplect automorphism of generalized Penner type.
Then there exists a Lagrangian branched submanifold $\mathcal{B}_\psi$ such that if L is a Lagrangian submanifold which is carried by $\mathcal{B}_\psi$, $\psi^m(L)$ is carried by $\mathcal{B}_\psi$ for all $m \in \mathbb{N}$.

Braids

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- There are braid sequences corresponding to $\psi^n(L)$ and their limits.

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- We defined data for L on $\mathcal{B}_\psi$, which are called **braids**, containing information about the singularities.
- A braid is assigned on a boundary of each sector, which is a connected component of $\mathcal{B}_\psi \setminus \{\text{branching loci}\}$.
- There are braid sequences corresponding to $\psi^n(L)$ and their limits.
- One can construct a stable Lagrangian lamination from the limits of braid sequences.

Theorems

Theorem 2 (L, in preparation.)

Let M be a symplectic manifold and let $\psi : M \stackrel{\simeq}{\to} M$ be a symplectic automorphism of generalized Penner type.
Then there is a Lagrangian lamination $\mathcal{L}$ such that
if L is a Lagrangian submanifold of M which is carried by $\mathcal{B}_\psi$, then there is a Lagrangian submanifold L_m which is Hamiltonian isotopic to $\psi^m(L)$ and

$$\lim_{m \to \infty} L_m = \mathcal{L}.$$

Theorems

Theorem 3 (L, in preparation.)

Let $\psi : M \xrightarrow{\sim} M$ be a symplectic automorphism and let $\mathcal{B}_\psi$ be a Lagrangian branched submanifold such that $\psi(\mathcal{B}_\psi)$ is carried by $\mathcal{B}_\psi$. Moreover if $\mathcal{B}_\psi$ admits a decomposition into singular and regular disks,
then there is a Lagrangian lamination $\mathcal{L}$ such that
if L is a Lagrangian submanifold of M which is carried by $\mathcal{B}_\psi$, then there is a Lagrangian submanifold L_m which is Hamiltonian isotopic to $\psi^m(L)$ and

$$\lim_{m\to\infty} L_m = \mathcal{L}.$$

Theorems

Theorem 4 (L, in progress.)

Let M be a plumbing space of cotangent bundles of spheres $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_l$ such that

$$\alpha_i \cap \alpha_j = \emptyset, \beta_i \cap \beta_j = \emptyset \text{ for all } i \neq j.$$

Moreover if $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_j$ generate the (compact) Fukaya category of M,
then a symplectic automorphism ψ of generalized Penner type induces an pseudo-Anosov autoequivalence on the (compact) Fukaya category of M.

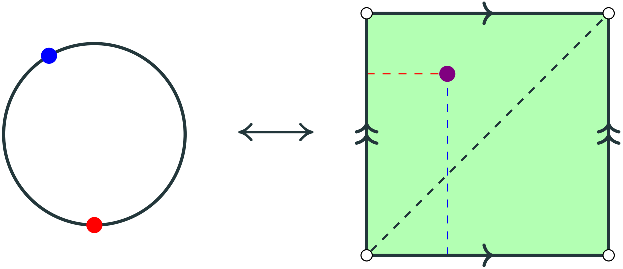
Orthoscheme Configuration Spaces

Michael Dougherty (Grinnell College)

December 8, 2018

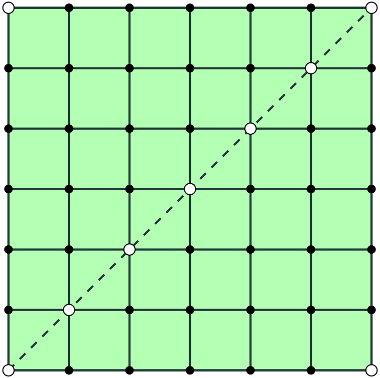
Tech Topology Conference

The configuration space $\textsc{Conf}_n(X)$ of n points in X is the space of n-tuples in X^n with distinct entries:



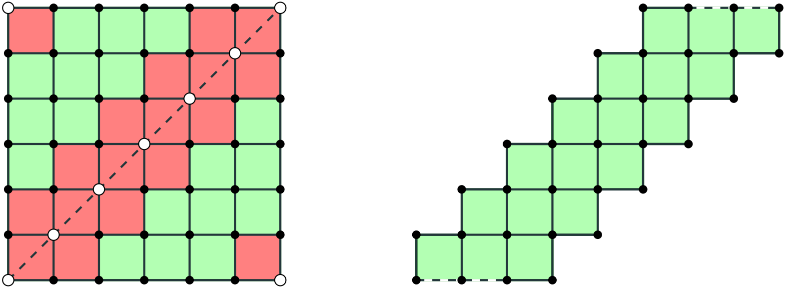
$\textsc{Conf}_2(\mathbb{S}^1)$ is the interior of an annulus.

What if X is a graph? (e.g. a hexagon)



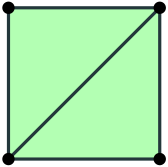
Removing the "diagonal" destroys the cell structure!

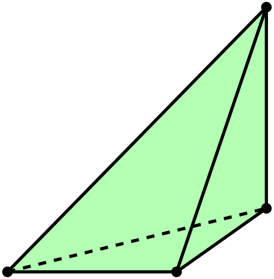
Solution: remove all cells touching the diagonal



Theorem: (Abrams '00) The graphical configuration space is a cube complex with CAT(0) universal cover.

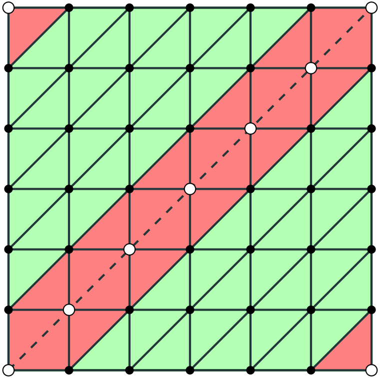
What if we want a simplicial complex instead?



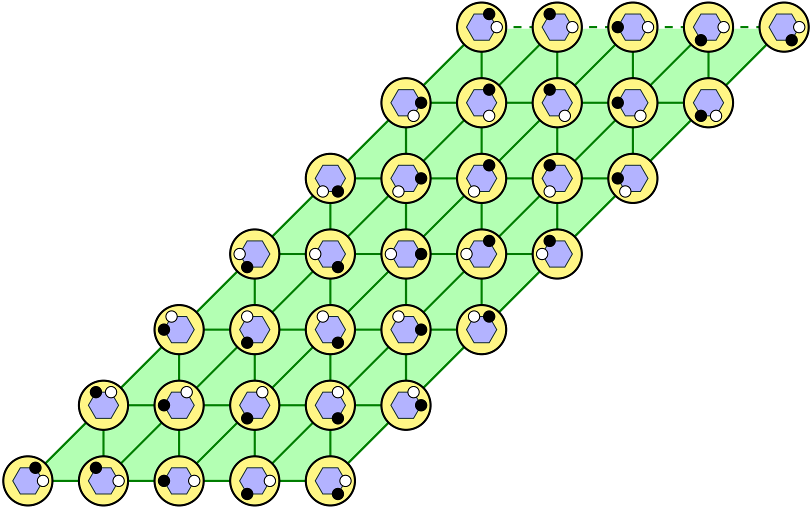


Idea: split each n-cube into $n!$ orthoschemes

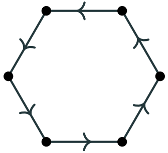
Again, remove all cells touching the diagonal:



The orthoscheme configuration space of two points in an oriented hexagon is a closed annulus.



Theorem: (D-McCammond-Witzel) The orthoscheme configuration space of k points in an oriented n-cycle is $\Delta^{k-1} \times \mathbb{S}^1$ and its universal cover is CAT(0).



(oriented 6-cycle)

What about...*any* other directed graph?

Movies of singular fibrations on 4-manifolds (in 5 minutes)

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Definitions

Let K be a knot in S^3.
Recall that K is *fibered* if $S^3 \setminus \nu(K)$ is fibered over S^1.
K is *ribbon* if K bounds a ribbon disk in B^4, where $S^3 = \partial B^4$.

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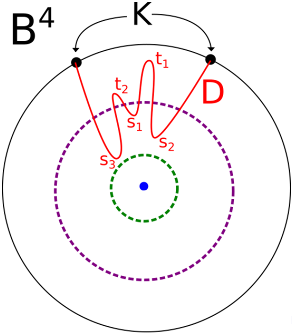
K is *ribbon* if K bounds a ribbon disk in B^4, where $S^3 = \partial B^4$.
(A disk D is ribbon if it has no maxima with respect to the radial Morse function h.)



Slice-Ribbon conjecture

Conjecture (Fox's Slice-Ribbon conjecture, 1962)

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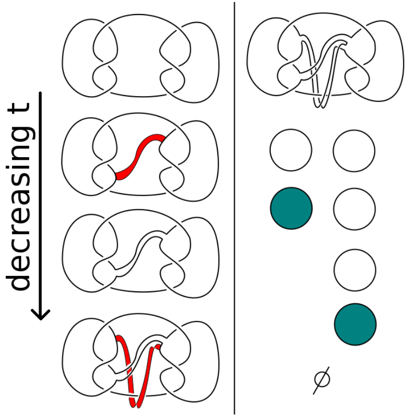
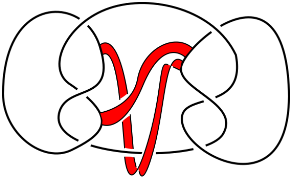
My interest: Suppose $V \cong B^4$. Is $E = D$?

How to draw a ribbon disk

A ribbon disk D for K is defined by a set of bands attached to K.





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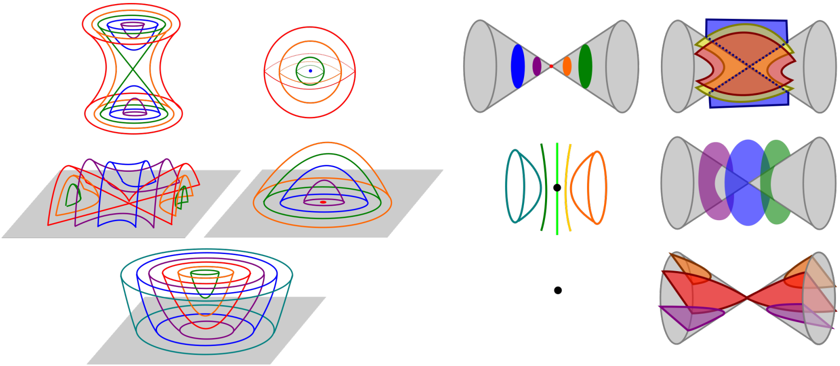
Corollary

If K is fibered and bounds a ribbon disk D with two minima, then $B^4 \setminus \nu(D)$ is fibered by handlebodies.

The proof is constructive, so we can describe the handlebody in B^4 explicitly.

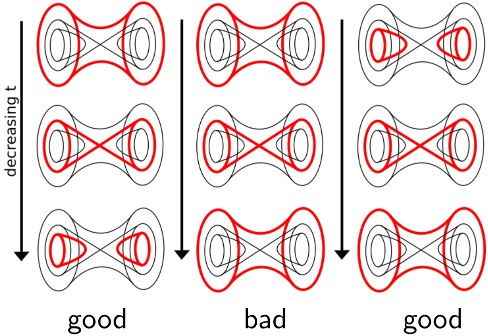
Main idea: movies of singular fibrations

We build a fibration $\mathcal{F} = \mathcal{F}_t \mid_{t \in [0,1]}$ of $B^4 \setminus \nu(D)$, where each $\mathcal{F}_t$ is a singular fibration of $h^{-1}(t)$.



Main idea: movies of singular fibrations

We want the fibers of $\mathcal{F}_t$ to be smooth 3-manifolds.



Main content: keeping track of how these singularities resolve (similar to Cerf theory).

Build a library

We build a library of basic movies of singular fibrations on simple 4-manifolds. By composing several movies, we can fibrate complicated 4-manifolds.

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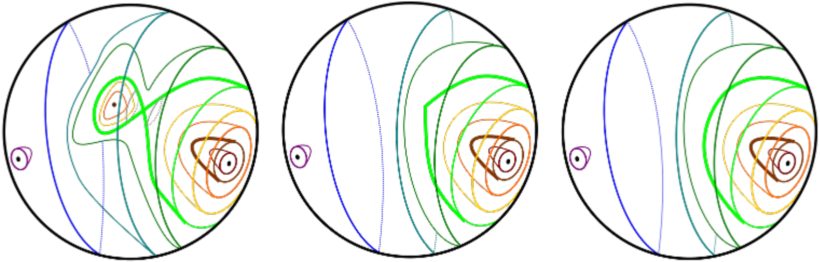
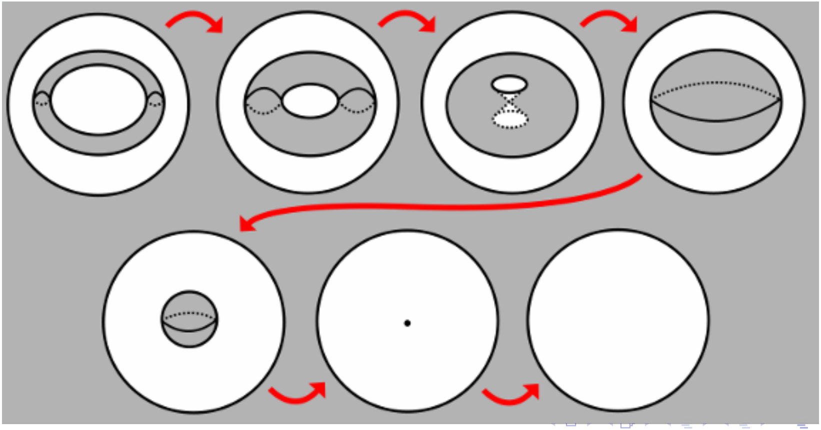
We build a library of basic movies of singular fibrations on simple 4-manifolds. By composing several movies, we can fibrate complicated 4-manifolds. One important simple movie can cancel singularities.

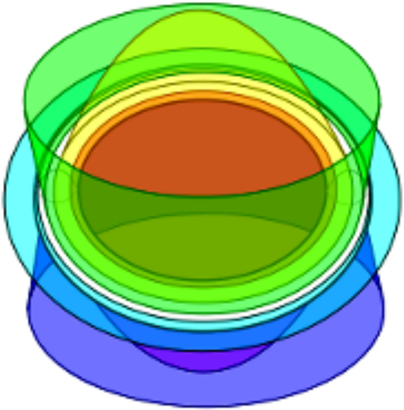


Example: minimum movie

Given a singular fibration on B^3, we build a movie of singular fibrations on $(B^3 \times I) \setminus$ (trivial disk).



Example: minimum movie



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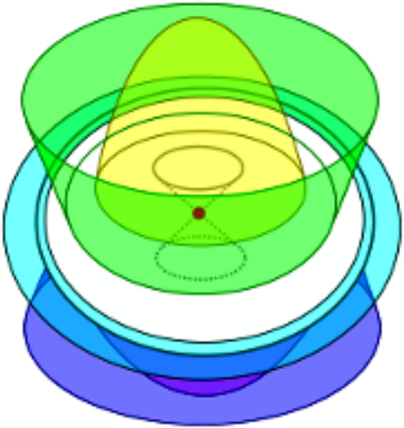

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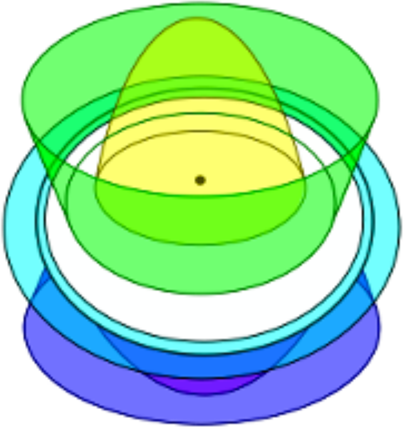

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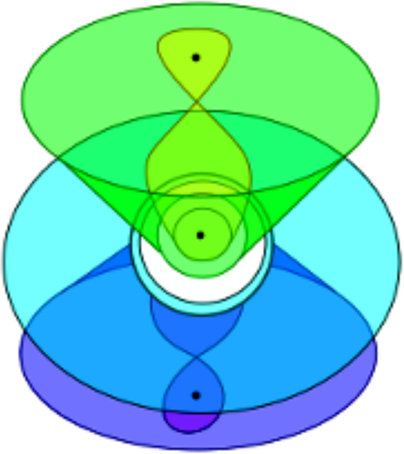

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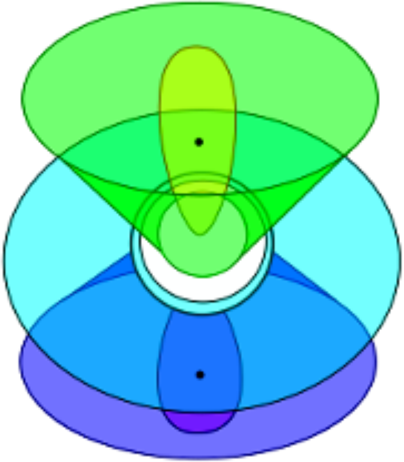

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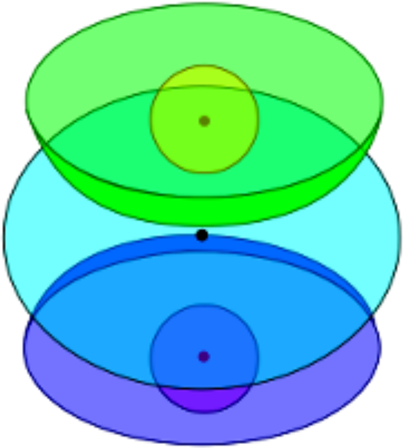

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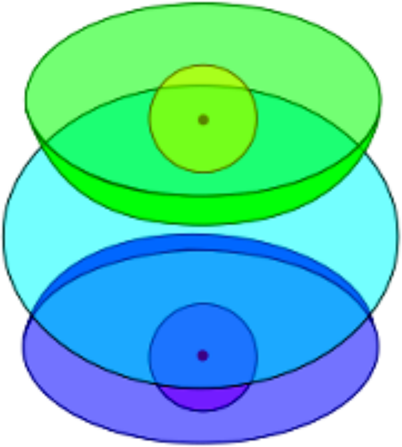

Thanks!

A. J. Casson and C. McA. Gordon, *A loop theorem for duality spaces and fibred ribbon knots*, Invent. Math. **74** (1983), 119–137.

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M. Miller, *Extending fibrations on knot complements to ribbon disk complements*, arXiv:1811.09639 [math.GT], Nov. 2018.

Blair's Conjecture and Contact Dynamics

Surena Hozoori

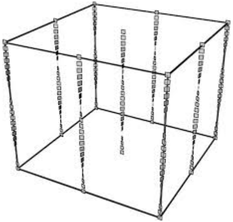
Georgia Institute of Technology

Contact Manifolds

- Let M be a closed oriented (2n+1)-manifold during this talk.
- A 1-form α on M is called a (positive) contact form if $\alpha \wedge (d\alpha)^n > 0$.
- We call $\xi := \ker \alpha$ a (co-orientable) contact structure on M.
- Any contact form α with $\ker \alpha = \xi$ defines a unique vector field X_α (Reeb vector field) such that:

  i) $d\alpha(X, .) = 0$

  ii) $\alpha(X) = 1$
- e.g: (1) $\xi_n = \ker(2\pi nzdx + 2\pi nzdy)$ on $\mathbb{T}^3$
- e.g. (2) $\xi_{strd} = T\mathbb{S}^3 \cap JT\mathbb{S}^3$ on unit $\mathbb{S}^3$ in $\mathbb{C}^2$ and Hopf fibration is the corresponding Reeb vector field.



Compatible Geometry of Contact Manifolds

- Now given a contact (2n+1)-manifold (M, ξ) we can naturally define a Riemannian metric on M by defining

$$g(u, v) = \frac{1}{\theta'} d\alpha(u, Jv) + \alpha(u)\alpha(v)$$

- where α is a contact form (i.e. $\ker \alpha = \xi$), θ' is a positive number ("instantaneous rotation") and J is complex structure on ξ (compatible with $d\alpha$ as symplectic structure), naturally extended to TM.
- We call such a Riemannian metric a compatible Riemannian metric for ξ.
- X_α is orthonormal to ξ and moreover is geodesic field.
- e.g: The round metric and flat metrics are compatible with $(\mathbb{S}^3, \xi_{std})$ and $(\mathbb{T}^3, \xi_n)$ respectively.

Blair's Conjecture

Blair's Conjecture

There is no non-flat compatible metric of non-positive curvature.

Blair's Conjecture: Previous Work

Theorem (Zeghib 95, Rukimbira 98)

A compact contact manifold cannot admit any compatible metric of strictly negative curvature.

Theorem (Blair 76)

A contact manifold of dimension ≥ 5 *cannot admit any flat compatible metric.*

Theorem (Rukimbira 98)

Characterized flat contact manifolds in dimension 3.

Theorem (Etnyre-Komendraczyk-Massot 12, 16)

Blair's conjecture holds for overtwisted contact manifolds.

Theorem (H. 18)

Better than Blair's conjecture for overtwisted contact manifolds holds!

Conley Zehnder Index and Vertical Sectional Curvatures

Theorem (H. 18)

Let g be a compatible metric for (M, ξ), a closed contact 3-manifold, such that

$$k(e, X_\alpha) \leq \left(\frac{\theta'}{2} - \sqrt{\frac{\theta'^2}{4} - Ricc(X_\alpha)}\right)^2 \text{ for every unit vector } e \in \xi$$

(in particular if the sectional curvature of any plane containing X_α is non positive), then:

1) $2c(\xi) = 0$.
2) $\mu_{CZ}(\gamma) = 0$ *for every contractible periodic orbit* γ *of* X_α.
3) If we have strict inequality, all the periodic orbits are non-degenerate and hyperbolic.
4) ξ *is not overtwisted (i.e. is tight).*

Remarks

- This yields Blair's conjecture for overtwisted contact 3-manifolds.
- Also proves tightness for flat contact manifolds.
- Improves Zeghib-Rukimbira theorem for overtwisted case.
- Same proof seems to work in higher dimensions!

Thank you!