Taut Foliations, Positive 3-Braids & the L-space conjecture

by Krishna

The L-space conjecture:

Suppose Y^3 is closed, connected, oriented, and irred $\mathbb{Q}HS^3$

TFAE

1) Y admits a taut foliation "geometry"
2) Y is a non-L-space "Heegaard Floer hom."
3) $\pi_1(Y)$ is left-orderable "algebra"

§ Taut Foliations:

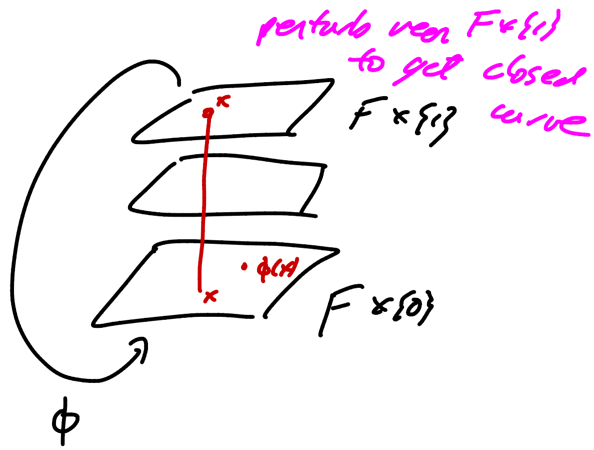
Def^n: A taut foliation of a 3-manifold Y is a particular type of decomposition into codim 1 submanifolds, called "leaves", s.t. $\exists$ a s.c.c. γ meeting each leaf $\pitchfork$ly

Ex Fibered 3-mfd

F cpt oriented connected

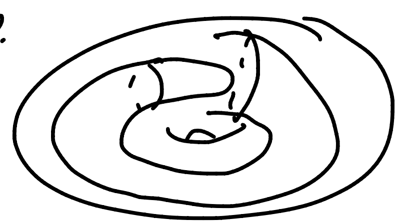
$\phi: F \circlearrowleft$ a diffeo

$M_\phi = F \times I / (x,1) \sim (\phi(x),0)$



non Ex: Reeb comp.

$D^2 \times S^1$



Claim: If $\mathcal{F}$ is a fol^n of M^3 & $\mathcal{F}$ contains the Reeb comp. as a sub fol^n then $\mathcal{F}$ isn't taut

§ L-space

Suppose Y $\mathbb{Q}HS^3$ irred

then Y is an L-space if

$$\operatorname{rk}(\widehat{HF}(Y;\mathbb{Z}/2\mathbb{Z})) = |H_1(Y;\mathbb{Z})|$$

Rmk: always "$\geq$"

L-spaces are simple from the Heegard Floer persp.

Ex: Lens spaces $\subset$ Elliptic geometry (like P.H.S.)

§ Left-orderability

Def^{n}: A group G is L.O. if $\exists$ a total order "$>$" which is preserved by left multiplication

Fact: L.O. groups are torsion-free

§ Evidence:

Th^{m} (Ozsváth - Szabó)

Y admits a taut fol^{n} $\Rightarrow$ Y is a non-L-space

Th^{m}: The L.S.C. holds for graph manifolds

Pf: Input from many people!

§ Producing Non-L-spaces via Dehn surgery

Def^{n}: Suppose $K \subset S^3$ $(K \neq U)$

K is an L-space knot if $\exists$ $r \in \mathbb{Q}_+$ s.t.

$S^3_r(K)$ is an L-space

ex: Torus knots (Moser)

Berge Knots

$K \subset S^3$
$(K \neq U)$

! (not quite right)

K is an L-space knot

$\downarrow$

Thm (Kronheimer-Mrowka-Ozsváth-Szabó, J+S Rasmussen)

$$S^3_r(K) = \begin{cases} \text{non-}L\text{-space} & r < 2g(K)-1 \\ L\text{-space} & r \geq 2g(K)-1 \end{cases}$$

K isn't an L-space knot

$\downarrow$

$\forall r \in \mathbb{Q}$, $S^3_r(K)$ is not-L-space

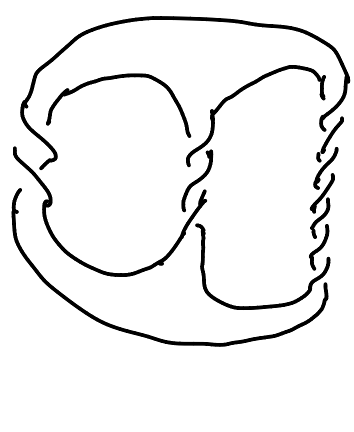
Point: Regardless of K's status $\forall r \in (-\infty, 2g(K)-1)$
$S^3_r(K)$ is a non-L-space

The L.S.C. predicts these mfds admit TFs

Thm (K):

Suppose $K \subset S^3$, $K \simeq \hat{\beta}$, $\beta \in B_3^+$ ← positive 3-braids
Then $\forall r \in (-\infty, 2g(K)-1)$, $S^3_r(K)$ has a taut foliation

Ex: $P(-2,3,7)$



"Fintushel-Stern" $g(K)=5$
admits Lens space surgery
$\Rightarrow$ L-space knot

so $\forall r \in (-\infty, 7)$, $S^3_r(K)$ is a non-L-space
closure pos 3-braid

$\text{Th}^{\underline{m}}$ (Lidman-Moore, Baker-Moore)

The only L-space ~~pretzel~~ Montesinos knots are the $P(-2,3,q)$ $q \geq 1$, q odd

These are all pos 3-braid closures!

<u>Point:</u> We obtain the $1^{\underline{st}}$ examples of

Y is a non-L-space $\Leftrightarrow$ Y admits a TF

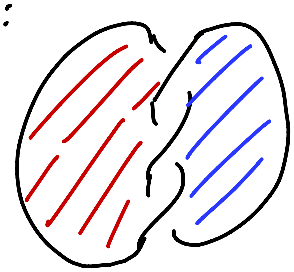
for every mfd obtained by Dehn surgery along an ∞ family of hyp knots

§ Glimpse of proof via branched surface

<u>$\text{Th}^{\underline{m}}$ (Roberts '01):</u>

$\forall\, r \in (-\infty, 1)$ $S^3_r(RHT)$ admits taut $\text{fol}^{\underline{n}}$

<u>Pf</u>:



$\leftarrow$ F a surface

$F \cong$

a diff perspective (a la Rudolph)

$\leftarrow$ pos side of F

K is a fibered knot

$X_K = S^3 - \mathring{\nu}(K)$

$= F \times I/_\phi$ $\qquad \phi = \tau_b \circ \tau_a$

what is $\phi(\alpha)$

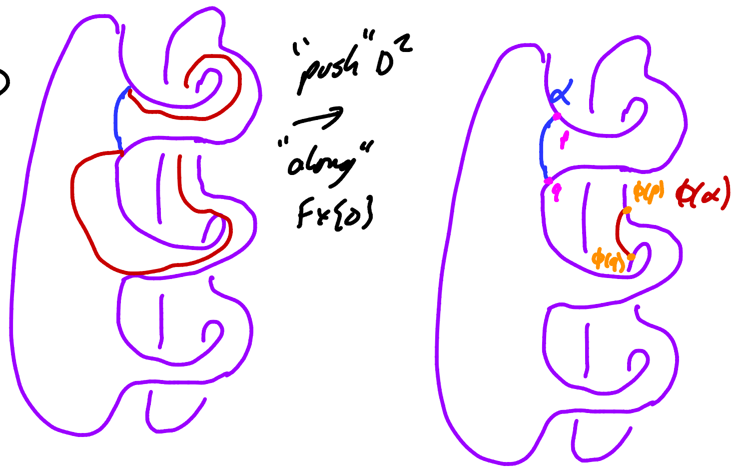
pushing α through $(X_K - (F \times \{0\}))$

$\cong F \times I$

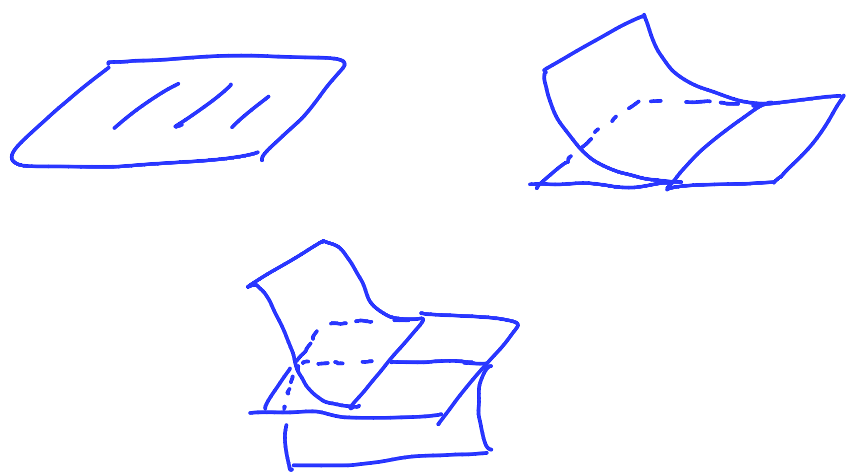
yields $\alpha \times I = D^2$

Define $S = F \times \{0\} \cup D^2$ a 2-complex in X_K

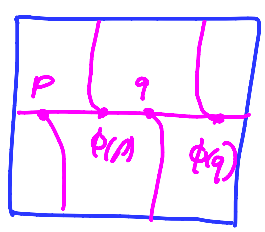
Redraw w/o $\mathring{D}^2$



Co-orienting D^2 yields a branched surface B, locally modeled by



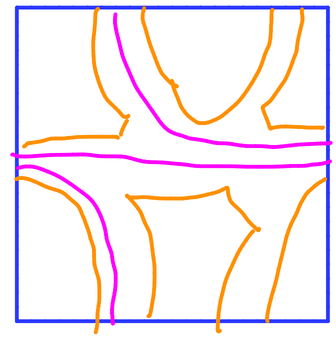
Moreover, B meets ∂X_K in a train track τ_B



τ_B carries all slopes in $(-\infty, 1)$

"recall": τ carries a slope r if a s.c.c. of slope r can be isotoped to live in $\nu(\tau)$

Ex



Point: In this case, B is laminar (L, '02 '03)
and so $\forall r \in (-\infty, 1)$, $S^3_r(K)$ has an essential lamination

More work $\Rightarrow$ promote to a taut foliation $\boxed{//}$

<u>Strategy:</u> Build a branched sfc B w/ fiber surface and disks st

1) laminar
2) τ_B carries all slopes $(-\infty, 2g(K)-1)$

<u>Point!</u>
this reduces building taut fol$^{\underline{n}}$ to checking some comb. conditions