Ordered groups and n-dimensional dynamics

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$Diff^i(X, Y)$ = all which are smooth of class $i = 1, 2, \ldots, \infty$.

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If f is not the identity, then there is $x \in I$ such that either $x < f(x)$ or $x > f(x)$. In the first case, we conclude that $f(x) < f^2(x)$, $f^2(x) < f^3(x)$, etc. So by transitivity, $x < f^n(x)$ for all powers n. Similarly, in the second case, we see that $f^n(x) \neq x$ for all n.

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The case $n = 1$ was just proven. The case $n = 2$ was proved by Kerékjártó exactly a century ago.

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Now ∂I^n represents a nontrivial $(n-1)$-cycle in $Fix(f)$. If $Fix(f)$ is not all of I^n, this cycle cannot bound in $Fix(f)$, a contradiction. Therefore $Fix(f) = I^n$; in other words f is the identity. □

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In fact, there is a much stronger theorem for PL or smooth homeomorphisms. So we next consider souped-up versions of being torsion-free.

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But there are many torsion-free groups which are not left-orderable.

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Theorem (Hölder)

Bi-orderable implies locally indicable.

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If $(G,<)$, is a left-ordered group, define the positive cone to be

$$P := \{g \in G \mid g > 1\}.$$

Then (1) P is closed under multiplication and (2) for every $g \in G$ exactly one of $g \in P, g^{-1} \in P$ or $g = 1$ holds.

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Conversely, if a group G has a subset P satisfying (1) and (2), then the formula $g < h \iff g^{-1}h \in P$ defines a left-ordering of G.
A group G is bi-orderable iff it has a subset P satisfying (1) and (2) and also (3) $gPg^{-1} \subset P$ for all $g \in G$.

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Bi-orderability is also preserved under taking subgroups, but not necessarily under extensions.

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$$1 \to \mathbb{Z} \to G \to \mathbb{Z} \to 1.$$

While $\mathbb{Z}$ is bi-orderable, the same cannot be said of G. If it were, then if y were greater than the identity, the relation would imply y^{-1} is also greater than the identity, a contradiction. Similarly $y < 1$ would also lead to a contradiction.

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On the other hand, this group G is locally indicable and (hence) left-orderable and torsion-free.

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Fundamental groups of surfaces are all bi-orderable, with the exception of the Klein bottle (only left-orerable) and the projective plane, whose fundamental group is $\mathbb{Z}/2\mathbb{Z}$.
Fundamental groups of 3-manifolds which have positive first Betti number are left-orderable. As a special case, knot groups are left-orderable.

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This is conjectured to be true for torsion-free groups.

Theorem

If G is bi-orderable, and g^m and h^n commute, then so do g and h. Moreover, roots are unique: that is, if $g^n = h^n$ for some $n \neq 0$, then $g = h$.

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To see this, consider a countable dense subset $r_1, r_2, \dots$ of the interval $I = [0, 1]$ and compare two different functions $f, g \in Homeo(I, \partial I)$ by declaring that

$$f \prec g \iff f(r_i) < g(r_i)$$

at the first i for which the values $f(r_i)$ and $g(r_i)$ differ.

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Corollary

$Homeo(I, \partial I)$ is NOT bi-orderable or locally indicable.

This follows because there are many countable LO groups which are not bi-orderable or locally indicable.

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Theorem (Chehata)

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One can compare two PL functions $f, g \in PL(I, \partial I)$ by examining the "first" point x_0 (reading from the left of the interval) at which their graphs begin to differ, and declare $f \prec g$ if and only if the graph of g is above that of f just beyond that point. This is a bi-order, as this property is preserved under conjugation.

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Theorem (W. Thurston)

$Diff^1(I, \partial I)$ is locally indicable.

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Theorem (Calegari - R)

Let $n \geq 1$. *Then* $PL(I^n, \partial I^n)$ *is locally indicable, and therefore left-orderable.*

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More generally,

Theorem (Calegari - R)

If M is a connected PL n-manifold and $B \subset M$ a proper PL submanifold of codimension 0 or 1, then $PL(M, B)$ is locally indicable.

Proof that $PL(I^n, \partial I^n)$ is locally indicable

Lemma

Let G be the subgroup of $GL(n, \mathbb{R})$ consisting of all linear maps which pointwise fix an $n-1$ dimensional subspace S of $\mathbb{R}^n$ and preserve orientation. Then G is locally indicable.

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Lemma

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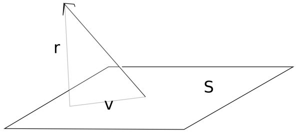
Proof: With appropriate basis, G is the group of matrices $\begin{pmatrix} Id & v \\ 0 & r \end{pmatrix}$



where Id is the $(n-1) \times (n-1)$ identity matrix, $v \in \mathbb{R}^{n-1}$ is an arbitrary (column) vector and $r \in \mathbb{R}, r > 0$.

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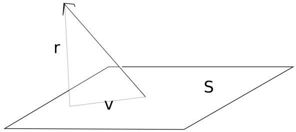

where Id is the $(n-1) \times (n-1)$ identity matrix, $v \in \mathbb{R}^{n-1}$ is an arbitrary (column) vector and $r \in \mathbb{R}, r > 0$. Determinant maps G to $\mathbb{R}_+$ and we have an exact sequence $1 \to \mathbb{R}^{n-1} \to G \to \mathbb{R}_+ \to 1$ $\square$

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To see that $PL(I^n, \partial I^n)$ is locally indicable, consider a finitely-generated nontrivial subgroup $H = \langle h_1, \dots h_k \rangle$. The set $Fix(H)$ of points fixed by all of H is the intersection of the $Fix(h_i)$, and therefore a polyhedron containing ∂I^n.

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$PL(I^n, \partial I^n)$ not biorderable in general

Proposition

$PL(I^2, \partial I^2)$ is NOT bi-orderable.

To show this, we construct two functions $f, g \in PL(I^2, \partial I^2)$ with the property that $fgf^{-1} = g^{-1}$. Such an equation cannot hold, for $g \neq 1$, in a bi-orderable group, as discussed earlier.

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We will take f to be a PL map fixed on the outer square ∂I^2 and rotating an interior square of half the size by 180 degrees. For example, we can use $f = h^6$, where h is as follows.

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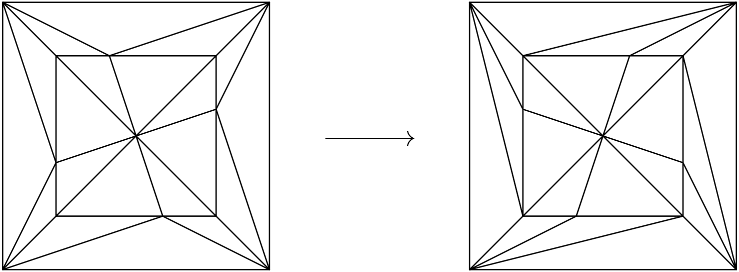

A PL map $h : I^2 \to I^2$, with the property that $f = h^6$ rotates the inner square by 180 degrees.

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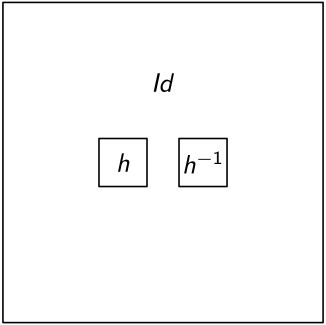
Define the function $g : I^2 \to I^2$ to be the identity, except on two small squares as illustrated. On one square take g to be a suitably scaled version of h. On the other take g to be h^{-1}. Then $fgf^{-1} = g^{-1}$ $\square$.



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Proposition

There is an isomorphic embedding of groups $Homeo(I^n, \partial I^n) \to Homeo(I^{n+1}, \partial I^{n+1})$ *and* $PL(I^n, \partial I^n) \to PL(I^{n+1}, \partial I^{n+1})$.

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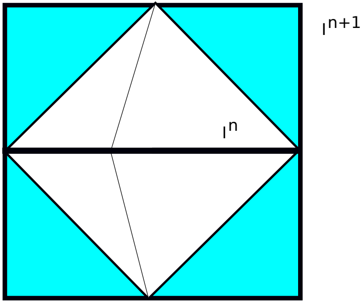

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In higher dimensions, $PL(I^n, \partial I^n)$ is locally-indicable, but not bi-orderable.
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It had long been an open question whether $Homeo(I^2, \partial I^2)$ is left-orderable.

Conclusion

Recently, James Hyde showed that these higher-dimensional results do not hold for the topological category.

Theorem (Hyde)

The group $Homeo(I^2, \partial I^2)$ is not left-orderable. Therefore the same is true for $Homeo(I^n, \partial I^n)$ for all $n \geq 2$.

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His proof is to construct a family of homeomorphisms of I^2, fixed on the boundary, which obey a certain relation in the group $Homeo(I^2, \partial I^2)$. Then he argues that such a relation cannot hold in a left-orderable group.

Thank you !