# The trace embedding lemma & spinelessness

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Fox-Milnor '50: PL embeddings of $\Sigma_g \hookrightarrow X^4_{sm}$

think of as smooth with cone singularities

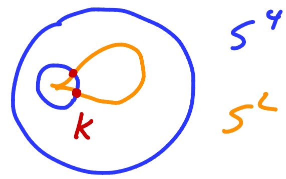
Obs: If $S^2 \xhookrightarrow[PL]{} S^4$ unique sing, then

$\partial \nu(\text{sing}) = K \subseteq S^3$

s.t. $K$ bounds

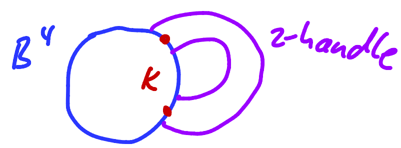
sm $D^2 \hookrightarrow B^4_1$ (top loc flat)

such a knot is slice (top)



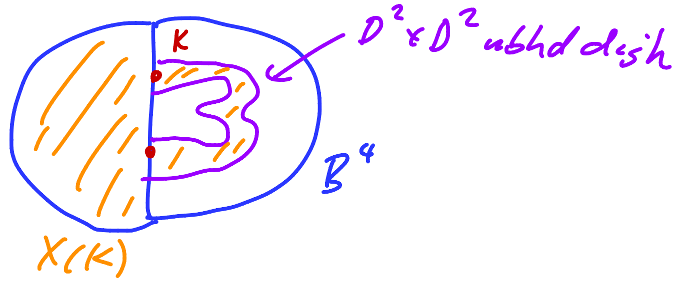

$B^4 \cong \nu(\text{sing})$

Def$^{n}$: a knot trace $X(K)$ is $B^4 \cup$ 2-handle along $K$ with framing $0$




Trace embedding lemma: $K$ is $\overset{sm}{\underset{top}{\text{slice}}} \Leftrightarrow X(K) \xhookrightarrow[sm/top]{} S^4$ (or $\mathbb{R}^4$)

$(\Rightarrow)$




$(\Leftarrow)$ exercise

Knot concordance $\xleftrightarrow{TEL}$ smooth 4-manifold topology

$(\rightsquigarrow)$ Th$^{m}$: $\exists\ W^4_{sm} \cong_{TOP} \mathbb{R}^4,\ W^4 \not\cong_{sm} \mathbb{R}^4$

Fact: (Freedman-Donaldson): $\exists\ K$ which TOP slice not SM slice

Sketch: first $K$ TOP not SM slice

$$TEL \Rightarrow X(K) \underset{TOP}{\hookrightarrow} \mathbb{R}^4,$$

define $\underset{TOP}{Z} := \mathbb{R}^4 \setminus X(K)$ Freedman-Quinn give sm str $Z_{sm}$

set $W_{sm} := Z_{sm} \cup X(K)$

$W_{sm} \cong_{TOP} \mathbb{R}^4$ suppose $W^4 \cong_{sm} \mathbb{R}^4$

$\Rightarrow X(K) \hookrightarrow \mathbb{R}^4 \Rightarrow K$ sm slice ✗

($\Leftarrow$) Problem: given $K$ is it smoothly slice?

Usually: use sliceness invts

Thm (P '18): The Conway knot (on which all known sliceness invts vanish) is not slice

Proof: Build $J$ st. $X(J) \cong_{sm} X(\text{Conway})$

TEL $\Rightarrow$ $J$ slice $\Leftrightarrow$ Conway slice

show $J$ not slice using std invts //

Defn: $\Sigma_g \underset{PL}{\hookrightarrow} X^4$ is a spine if $\phi$ induces homotopy equivalence.

$X^4 \simeq_{h.e.} \Sigma_g$ is spineless if no such embeddings exist

Problem: Does there exist spineless $X^4$?

Thm (Matsumoto '77): $\exists$ spineless 4-mflds h.e. $T^2$

later w/ same techniques get higher dims

Thm (Levine-Lidman '18, Kirby Problem 4.25)

$\exists$ spineless $X^4$ h.e. $S^2$.

<u>Fact</u>: hard to obstruct PL subsurfaces

above thm's corollaries of

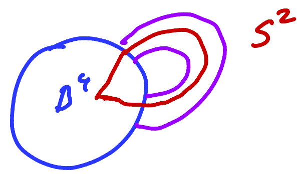
Thm (Matsumoto, Lidman-Levine)

$\forall g \in \mathbb{N}, \exists M^3_g$ st.

① $M = \partial X_g$, $X_g \simeq_{he} \Sigma_g$

② Any such $X_g$ is spineless

example: for any $K$, $X(K)$ has a spine




if $K$ slice then $X(K)$ has a smooth spine

<u>Thm (Hayden-P '19)</u>: $\forall n \in \mathbb{Z}$, $g \in \mathbb{N}$, $\exists W^4$ which is homeo to $W'$ which has a smooth $\Sigma_g$ spine, $W$ is spineless, $Q_W = [n]$

<u>Cor</u>: When $n = 0 = g$, get $W \cong_{TOP} W'$ st.

| | |
|---|---|
| • $W \not\cong_{sm} X(J)$ for any $J$ | • $W'$ is $X(K)$ for $K$ slice |
| • spineless | • has sm spine |
| • is Stein | • not a strong sympl. filling |
| • is not g.s.c. | • g.s.c |

Def$^n$: $X^4$ is <u>geometrically simply connected</u> (g.s.c.) if it admits a handle decomposition w/ no 1-handles

Obs: g.s.c. $\Rightarrow$ simply connected

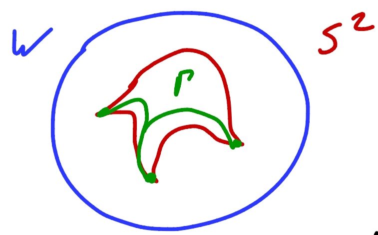
Problem 4.18: If $W$, closed, compact, oriented does s.c. $\Rightarrow$ g.s.c.

Sketch proof of Thm when $n=g=0$:

Build $W$ s.t.

1. $W \xhookrightarrow[\text{sm}]{} S^4$
2. $W$ Stein
3. $W$ homeo $X(K)$ some slice $K$

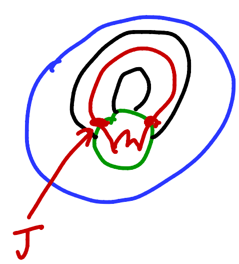
given this, suppose $W$ has $S^2$ spine




$\Gamma$ = graph containing all sings

nbhd $\Gamma \cong B^4$

$B^4$ isotopic to std $B^4$

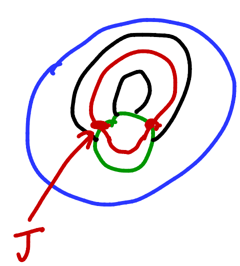



$\Rightarrow X(J) \hookrightarrow W$

we know $W \hookrightarrow S^4$

$\overset{\text{TEL}}{\Rightarrow}$ $J$ slice

$\Rightarrow$ can remove sings & replace with smooth $D^2$




so $\exists$ smooth $S^2 \hookrightarrow W$ a h.e.

contradiction via ② //