Spines for spineless 4-manifolds

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Tech Topology conference

December 8, 2019

Background

(Since Lisa says I have to.)

Let W^n be a compact manifold with non-empty boundary.

A **spine** is a subset S that is a deformation retract of W.

Usually, require more. Eg $\dim(S) < \dim(W)$.

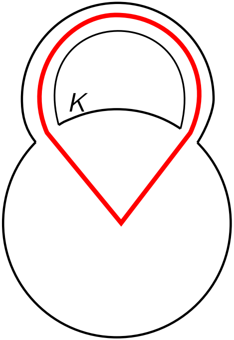
- W is PL, then S should be a subcomplex.
- Standard argument: any W has a codimension one spine.
- Lots of results (insert famous names here) about codim ≥ 3.
- Cappell-Shaneson (1974): if $n > 4$ and $\pi_1 = 1$ (or n is odd) and if $W \simeq \Sigma^{n-2}$ a PL manifold, then W has a codim 2 spine.
- Matsumoto (1975) found a spineless 4-manifold $\simeq T^2$.
- Cappell-Shaneson (1976): spineless manifolds for $n \geq 6$ even.

The canonical example

We'll focus on simply connected 4-manifolds.

Canonical example: Knot trace $X_n(K) = B^4 \cup_K h^2$ with framing n.

S is the core of the handle $\cup$ cone(K); it's PL but not locally flat.



Spines and homology cobordisms

Definition: *A topological spine of W^4 is a locally tame (= locally PL) submanifold $S \subset W$ that is a deformation retract of W.*

Assume S locally flat away from a single point:

- local model given by $(B^4, \mathrm{cone}(\mathrm{K}))$
- We say the spine has singularity (modeled on) K.

Suppose $W \simeq S^2$, with intersection form $Q_W = \langle n \rangle$.

$\partial W \sim L(n,1)$ and a neighborhood of S is homeomorphic to $X_n(K)$.

Key observation: $W - \mathrm{int}(X_n(K))$ is a homology cobordism between ∂W and $S^3_n(K)$.

Notation for homology cobordism: $\partial W \sim_H S^3_n(K)$.

A rough converse

Suppose $W \simeq S^2$, $Q_W = \langle n \rangle$, and there are

- a knot K
- a homology cobordism V between $S^3_n(K)$ and ∂W with
- $\pi_1(S^3_n(K))$ normally generating $\pi_1(V)$.

Then W has a spine S with singularity modeled on K.

What kind of spine?

- If V is just a topological H-cobordism then S is a topological spine.
- If V is PL (aka smooth) then S is a PL spine.

Spineless 4-manifolds

Theorem (Levine-Lidman, 2018). *For any $n = m^2$, $m \geq 2$, there are infinitely many smooth $W_p \simeq S^2$ with $Q_{W_p} = \langle n \rangle$ that have no PL spine.*

Uses principles above: [LL] show (via d-invariants) that ∂W_p is not H-cobordant to $S^3_n(K)$ for any knot K.

Theorem (Kim-Ruberman, 2019). *The Levine-Lidman manifolds W_p with $n = 4$ have locally tame topological spines.*

Theorem (Hayden-Piccirillo, 2019.5). *There exist smooth (PL) structures on knot traces that admit no PL spines.*

Based on different principle; ∂W is by definition surgery on a knot.

Spineless 4-manifolds

Some questions:

- What about the other LL manifolds with $n > 4$?
- Does every $W \simeq S^2$ admit a locally tame spine?

It seems there's always a (wild) 2-sphere in W; maybe it's a (wild) spine.

Finding tame spines

From now on, will only discuss the topological case.

Recall the goal: start with Levine-Lidman manifolds W_p, find knot K_p with $\partial W_p \sim_H S^3_4(K_p)$.

Main point is to find the right knot so we can solve the homology cobordism problem via topological surgery.

Levine-Lidman: Write $m = -2p - 3$; then $\partial W_p = Q_m \# - Y_p$.

- Y_p is a $\mathbb{Z}HS^3$ which we can ignore since it bounds contractible.
- Q_m is $+4$ surgery on a knot in Y_p.

Useful observation: Q_m is a spherical space form; $G_m = \pi_1(Q_m)$.

$$1 \longrightarrow \mathbb{Z}_m \longrightarrow G_m \rightleftarrows \mathbb{Z}_4 \longrightarrow 1$$

Finding tame spines

We want $K_m \subset S^3$ with $Q_m \sim_H S^3_4(K_m)$ (and π_1 condition).

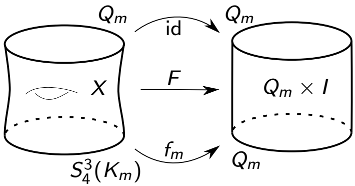
Will do in a couple of steps; K_m gets modified a few times.

1. Find K_m with $\pi_1(S^3 - K_m) \twoheadrightarrow G_m$, and $\mathbb{Z}[G_m]$ homology equivalence

$$f_m : S^3_4(K_m) \to Q_m.$$

   There aren't any in the knot tables up through 12 crossings!

2. Find normal cobordism (X, F) between f_m and id_{Q_m}.



Normal means F is covered by map of stable normal bundles.

Finding tame spines

Basic invariant: intersection form on

$$\ker\left(F_* : H_2(X;\mathbb{Z}[G_m]) \to H_2(Q_m \times I;\mathbb{Z}[G_m])\right)$$

defines element $\theta(F) \in L_4(\mathbb{Z}[G_m])$ (a large group).

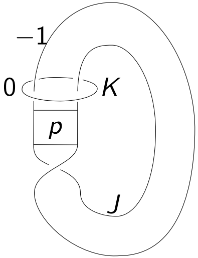
3. Modify K_m to get K'_m for which $\theta(F)$ vanishes in L_4; surger to get H-cobordism V with $\pi_1(V) = G_m$.
4. $V \cup X_4(K'_m) \approx W_p$.

Items 1 and 3 are harder–done by similar technique.

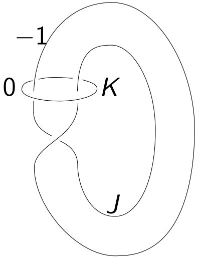
Item 2 has a $\mathbb{Z}_2$ obstruction; turns out it's 0. Last bit from S. Boyer.

First approximation to K_m

Q_m is $+4$ surgery on indicated knot in $Y_p = -1$ surgery on $T(2, 2p+1)$.
(Remember $m = -4p - 3$.)



Q_m



$S^3_4(K)$

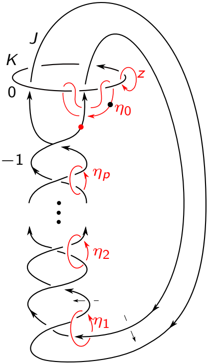
Want K_m so that $S^3_4(K_m)$ maps onto Q_m with degree one.

First approximation to K_m

Lemma (Boileau-Wang 1996) *Suppose the knot $\eta \subset M$ is null-homotopic. For any r, the Dehn surgered manifold $M_r(\eta)$ maps to M with degree 1.*

We can turn $T(2, 2p+1)$ into an unknot by crossing changes, ie surgeries along curves η_i. Make these null-homotopic by adding copies of η_0 and z.

Surger the resulting curves η_i'.



Second approximation to K_m

Now we have knots K_m with degree 1 maps $f_m : S^3_4(K_m) \to Q_m$, with f_m a $\mathbb{Z}$ homology equivalence.

But for surgery to apply, $\theta(F)$ must be non-singular over $\mathbb{Z}[G_m]$.

This means that f_m should be a $\mathbb{Z}[G_m]$ homology equivalence.

How to understand $H_1(S^3_4(K_m); \mathbb{Z}[G_m])$?

Answer: The G_m cover is obtained from S^3 by surgery on lifts of the η_i.

Its homology is determined by their linking numbers (and framings).

Do finger moves (downstairs) to change the (upstairs) linking matrix into an invertible one.

Get new knot, new $\mathbb{Z}[G_m]$ homology equivalence $f_m : S^3_4(K_m) \to Q_m$.

We finally get to do some surgery theory

Lemma: *The map f_m is normally cobordant to* id_{Q_m}. *The intersection form of the cobordism, say W, is the surgery obstruction $\theta \in L_4(\mathbb{Z}[G_m])$.*

By adding a copies of the E_8 manifold, we can assume $\mathrm{sign}(\theta) = 0$. Hence by a change of basis, the intersection form *over* $\mathbb{Z}$ is a sum of hyperbolics.

To kill the surgery obstruction, add 2-handles to W along $S^3_4(K_m)$ that 'realize' the intersection form $-\theta$.

A nice surprise: Can do this so the new boundary is of the form $S^3_4(K'_m)$.

The resulting cobordism has trivial surgery obstruction and hence can be surgered to a homology cobordism.

Glue in $X_4(K'_m)$ to get (top) manifold with a spine.

Killing the surgery obstruction

Proposition: *Let $Y = S^3_m(J)$ and $\alpha : \pi_1(Y) \to G$. Suppose that A is a (nonsingular, Hermitian) matrix with $\mathbb{Z}[G]$ coefficients such that*

$$\epsilon(A) = \bigoplus_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} := \ H^n.$$

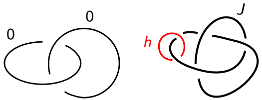
Then there is a cobordism W from Y to $Y' = S^3_m(J')$ with $\mathbb{Z}[G]$ intersection form A.

Augmentation ϵ means image of A under $\mathbb{Z}[G] \to \mathbb{Z}$.

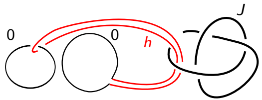
$$\text{Typical } A = \begin{pmatrix} g + g^{-1} - (k + k^{-1}) & h \\ h^{-1} & 0 \end{pmatrix}.$$

Killing the surgery obstruction

Proof of the proposition: First add 0-framed Hopf links (still have $Y = S^3_m(J)$).



To create h entries in A, do finger moves guided by h.



If you ignore J, it's still a Hopf link so you still get S^3.

But J gets changed to J'. □

Thanks from our mascot!

Spike