2-KNOT GROUP TRISECTIONS

Sarah Blackwell

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J.W. Kirby, Klug, Longo, RuppiK
(g, K)-trisection

of (closed) 4-mfld \( X \) = decomposition \( X = X_1 \cup X_2 \cup X_3 \) st

1) \( X_i \cong \mathbb{H}^k (S' \times \beta^3) \)
2) \( X_i \cap X_j \cong H_g \)
3) \( X_1 \cap X_2 \cap X_3 \cong \Sigma_g \)
2-KNOT GROUP TRISECTIONS

\[ (g, k) \text{- trisection of (closed) 4-mfld } X \]

\[ \Sigma_g \times S^1 \times S^2 \]

\[ H_g \]

\[ \text{SVK} \]

\[ \tilde{\Pi}_1 (\Sigma_g) \rightarrow \tilde{\Pi}_1 (X) \]

\[ F_{\tilde{g}} \rightarrow F_{\tilde{k}} \]

\[ (g, k) \text{- group trisection of } \tilde{\Pi}_1 (X) \]

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2-KNOT GROUP TRISECTIONS

Figure 4. A 2-bridge trisection of an unknotted 2-sphere, depicted with the tri-plane in 3-space, along with the corresponding tri-plane diagram.

Bridge trisection for a surface in $S^3$

[Meier + Zupan]

SVK

?

$\pi_1(B_3 \setminus \delta) \rightarrow \pi_1(S^3 \setminus U_{\beta\alpha})$

$\pi_1(S^2 \setminus \mathcal{U}_\ast) \rightarrow \pi_1(B^3_2 \setminus \delta) \rightarrow \pi_1(S^3 \setminus U_{\beta\alpha})$

$\pi_1(B^3_1 \setminus \alpha) \rightarrow \pi_1(S^3 \setminus U_{\alpha\beta}) \cong \pi_1(B^4_1 \setminus D_{\alpha\beta})$

Van Kampen diagram for the complement of a surface S in $S^4$
Figure 4. A 2-bridge trisection of an unknotted 2-sphere, depicted with the tri-plane in 3-space, along with the corresponding tri-plane diagram.

Bridge trisection for a surface in $S^4$

[Meier + Zupan]

Van Kampen diagram for the complement of a surface $S$ in $S^4$

2-Knot Group Trisections
Small Quotients of Braid Groups

Noah Caplinger
Joint with Kevin Kordek

Georgia Institute of Technology

December 2020
Question

*What is the smallest finite quotient of the braid group?*
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Example 1. $B_n \to S_n$
What is the smallest finite quotient of the braid group?

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Example 2. $B_n^{\text{ab}} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. 
Main Question

**Question**

*What is the smallest finite quotient of the braid group?*

Example 1. \(B_n \rightarrow S_n\)

Example 2. \(B_n^{ab} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}.*

**Conjecture (Margalit)**

*For \(n \geq 5\), \(S_n\) is the smallest non-cyclic quotient of \(B_n\).*
Main Theorem

Theorem

For \( n = 5, 6 \), \( S_n \) is the smallest non-cyclic quotient of \( B_n \).
Totally Symmetric Sets

Definition (Kordek, Margalit)

Let $G$ be a group. A subset $S = \{g_1, \ldots, g_k\} \subset G$ is said to be a totally symmetric set if

1. The elements of $S$ pairwise commute.
2. Every permutation of $S$ can be realized by conjugation in $G$. 
Two Facts about Totally Symmetric Sets

Fact

If \( f : G \to H \) is a homomorphism, and \( S \subset G \) is totally symmetric, then \( f(S) \) is totally symmetric of cardinality \( |S| \) or 1.

Totally symmetric sets can "collapse" under homomorphisms.
Two Facts about Totally Symmetric Sets

Fact
If $f : G \to H$ is a homomorphism, and $S \subset G$ is totally symmetric, then $f(S)$ is totally symmetric of cardinality $|S|$ or 1.

Totally symmetric sets can "collapse" under homomorphisms.

Fact
A well-chosen totally symmetric set $X_n \subset B_n$ collapses under a quotient of $B_n$ if and only if the quotient is cyclic.
Proof Strategy

If $H$ has no totally symmetric sets of cardinality $|X_n|$, it cannot be a non-cyclic quotient of $B_n$. 
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If $H$ has no totally symmetric sets of cardinality $|X_n|$, it cannot be a non-cyclic quotient of $B_n$.

Bad idea: get a computer to check for totally symmetric sets in every group or order up to $n!$. 
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Bad idea: get a computer to check for totally symmetric sets in every group or order up to $n!$.

Better idea: Check for totally symmetric sets in simple groups of small order, then leverage this information to say something about braid groups.
For $n = 5, 6, 7, 8$, the alternating group $A_n$ is the smallest non-trivial quotient of the commutator subgroup $B'_n$. 
For $n = 5, 6, 7, 8$, the alternating group $A_n$ is the smallest non-trivial quotient of the commutator subgroup $B'_n$.

For $n = 5, 6$, $S_n$ is the smallest non-cyclic quotient of $B_n$. 
Link Detection Results for Knot Floer Homology

Fraser Binns,

joint work with Gage Martin

Boston College

Tech Topology Conference 2020
Can we distinguish links?

Question
If I meet two links in the wild, can I distinguish them?
Knot Floer Homology is an invariant of links which takes values in the category of bi-graded \( \mathbb{Z}/2 \)-vector spaces.
What is knot Floer homology?

Knot Floer Homology is an invariant of links which takes values in the category of bi-graded \( \mathbb{Z}/2 \)-vector spaces.

**Theorem (Ni, Ozsváth-Szabó)**

\[ \hat{HFK}(L) \text{ determines the genus of } L. \]
What is knot Floer homology?

Knot Floer Homology is an invariant of links which takes values in the category of bi-graded $\mathbb{Z}/2$-vector spaces.

**Theorem (Ni, Ozsváth-Szabó)**

$\widehat{HFK}(L)$ determines the genus of $L$.

**Theorem (Ghiggini, Ni)**

$\widehat{HFK}(L)$ determines whether or not $L$ is fibered.
What is link Detection?

**Definition**

We say \( \widehat{HFK} \) detects \( L \) if whenever \( \widehat{HFK}(L') \cong \widehat{HFK}(L) \), \( L' \) is isotopic to \( L \).
What is link Detection?

**Definition**

We say $\widehat{\text{HFK}}$ detects $L$ if whenever $\widehat{\text{HFK}}(L') \cong \widehat{\text{HFK}}(L)$, $L'$ is isotopic to $L$.

**Theorem (B-Martin)**

*Knot Floer homology detects $T(2,4)$.*
Detection Results for knot Floer homology

Knot Floer homology detects:

- The unknot (Ozsváth-Szabó '04)
- The Hopf link (Ozsváth-Szabó '04, Ni '07)
- The trefoil, figure eight (Ghiggini '08)
- Unlinks (Ni '14, Hedden-Watson '18)

Knot Floer homology cannot distinguish:

- Infinitely many knots in each concordance class (Hedden-Watson '18)
- Non-trivial band sums of split links, where the bands differ by a twist (Wang '20)
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Which links are good candidates for detection results?

Definition

A $2$-cable link is one which bounds an embedded annulus.

Remark

The torus links $T(2, 2n)$ are the $2$-cables such that both components are unknotted.
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\[ T(2, 2n) \]
\[ T(2, 4), \ T(2, 6) \]
Detection results

Theorem (B-Martin)

*Knot Floer Homology detects:*

\[ T(2, 2n), \quad T(2, 4), \quad T(2, 6), \quad T(3, 3), \quad L7n1 \]
Mapping class groups vs. handlebody groups

Marissa Miller
University of Illinois at Urbana-Champaign
Mapping class group

$S_g$ closed, orientable, genus $g$ surface:

\[ MCG(S_g) = \text{Homeo}^+(S_g) / \text{isotopy} \]
**Curve graph** $C(S_g)$

**Vertices:** Isotopy classes of essential simple closed curves

**Edge:** If two isotopy classes can be made disjoint
Curve graph $C(S_g)$

**Vertices:** Isotopy classes of essential simple closed curves

**Edge:** If two isotopy classes can be made disjoint
Handlebody group

Handlebody, $V_g$: 3-ball with $g$ 1-handles attached (a 3-manifold)

$$H_g = MCG(V_g) = \text{Homeo}^+(V_g)/\text{isotopy}$$
Disk graph $D(V_g)$

**Vertices:** Isotopy classes of essential simple closed curves on $\partial V_g$ bounding disks in $V_g$

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**Disk graph** $D(V_g)$

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Disk graph $D(V_g)$

Vertices: Isotopy classes of essential simple closed curves on $\partial V_g$ bounding disks in $V_g$

Edge: If two isotopy classes can be made disjoint.

$D(V_g) \subseteq C(\partial V_g)$
A closer look...

1. $H_g \subset MCG(\partial V_g)$, but is badly distorted
2. $D(V_g) \subset C(\partial V_g)$, but is badly distorted

The geometries of $H_g$ and $D(V_g)$ do not reflect the ambient geometries of $MCG(\partial V_g)$ and $C(\partial V_g)$
Hierarchically hyperbolic spaces?

\textbf{HHS} \approx \text{Almost hyperbolic; obstructed by product regions}
Hierarchically hyperbolic spaces?

HHS ≈ Almost hyperbolic; obstructed by product regions

Mapping class groups: inspiration for HHSs (Behrstock-Hagen-Sisto)
Hierarchically hyperbolic spaces?

**HHS** $\approx$ Almost hyperbolic; obstructed by product regions

**Mapping class groups:** inspiration for HHSs (Behrstock-Hagen-Sisto)

**Handlebody groups:**
- Yes for genus two! (Miller)
- No for higher genus (Hamenstädt-Hensel, Behrstock-Hagen-Sisto)
Characterization of stable subgroups

**Stable subgroup** ∼ subgroups of finitely generated groups that exhibit hyperbolic-like behavior
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**Stable subgroup** \(\approx\) subgroups of finitely generated groups that exhibit hyperbolic-like behavior

**Mapping class groups**: stable \(\Leftrightarrow\) quasi-isometrically embed in curve graph
(Durham-Taylor, Hamenstädt, Kent-Leininger)
Characterization of stable subgroups

**Stable subgroup** \(\approx\) subgroups of finitely generated groups that exhibit hyperbolic-like behavior

**Mapping class groups:** stable \(\iff\) quasi-isometrically embed in curve graph (Durham-Taylor, Hamenstädt, Kent-Leininger)

**Handlebody groups:**

- Genus two: stable \(\iff\) quasi-isometrically embed in disk graph (Miller)
- Higher genus: exist quasi-isometrically embedded subgroups that aren’t stable (Miller)
Thank you!
Symmetric unions and reducible fillings

Feride Ceren Kose

Tech Topology Conference 2020
Symmetric unions

Definition

A symmetric union \((D \cup -D)(n_1, \ldots, n_k)\) \((n_i \in \mathbb{Z})\) is a knot diagram defined as follows:

\[
\begin{align*}
0: & \quad () \\
1: & \quad \begin{array}{c}
\times \\
-1: \quad \begin{array}{c}
\times
\end{array}
\end{array} \\
2: & \quad \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array} \\
-2: & \quad \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array}
\end{align*}
\]
Examples

Figure: 11n139

Figure: 11n132
Symmetric unions

Theorem (Kinoshita-Terasaka ’57, Lamm ’00)

Symmetric unions are ribbon.
Symmetric unions

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*Symmetric unions are ribbon.*

**Question**

*Is every ribbon knot a symmetric union?*
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Yes for

- all prime ribbon knots with up to 10 crossings (Lamm ’00)
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- 122 of 137 prime ribbon knots with 11 and 12 crossings (Seeliger ’14)
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Symmetric unions are ribbon.

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Is every ribbon knot a symmetric union?

Yes for

- all prime ribbon knots with up to 10 crossings (Lamm ’00)
- 122 of 137 prime ribbon knots with 11 and 12 crossings (Seeliger ’14)
- all 2-bridge ribbon knots (Lamm ’05)
Some classical results

Theorem (Fox-Milnor ’58)

\[ K \text{ is slice } \Rightarrow \Delta_K(t) = f(t)f(t^{-1}) \]
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Corollary

\[ K \text{ is slice } \Rightarrow \det(K) \text{ is a perfect square} \]
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**Corollary**

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**Theorem (KT '57, Lamm '00)**

\[ K = (D \cup -D)(n_1, \ldots, n_k) \text{ with } n_i \in 2\mathbb{Z} \Rightarrow \Delta_K(t) = (\Delta_D(t))^2 \]
Some classical results

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**Theorem (KT ’57, Lamm ’00)**

\[ K = (D \cup -D)(n_1, \ldots, n_k) \Rightarrow \text{det}(K) = (\text{det}(D))^2 \]
Minimal twisting number $tw(K)$

**Definition**

The *minimal twisting number* of a symmetric union $K$, denoted by $tw(K)$, is the smallest number of twisting regions in all symmetric union presentations of $K$. 

Remarks:

- $tw(K \# - K) = 0$
- $K$ is prime $\Rightarrow tw(K) > 0$
- $tw(11_{n139}) = 1$ and $1 \leq tw(11_{n132}) \leq 2$

**Theorem (Tanaka '15)**

$tw(11_{n132}) = 2$
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**Theorem (Tanaka '15)**

$tw(11n132) = 2$
Main result

**Theorem (Tanaka ’19, K.’20)**

Let $K$ be a composite ribbon knot that admits a symmetric union diagram. If $tw(K) = 1$, then $K = K_1 \# K_2 \# - K_2$ where $K_1$ is a symmetric union with $tw(K_1) = 1$ and $K_2$ is a nontrivial knot.
Main result

Theorem (Tanaka '19, K.'20)

Let $K$ be a composite ribbon knot that admits a symmetric union diagram. If $\text{tw}(K) = 1$, then $K = K_1 \# K_2 \# - K_2$ where $K_1$ is a symmetric union with $\text{tw}(K_1) = 1$ and $K_2$ is a nontrivial knot.

Corollary

$\text{tw}(3_1 \# 8_{10}) > 1$. 
3-manifold topology

**Definition**

An oriented compact 3-manifold $M$ is said to be *irreducible* if any embedded 2-sphere bounds a 3-ball. Otherwise, it is *reducible.*
3-manifold topology

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**Theorem** ($(\Rightarrow)$ Waldhausen '69, $(\Leftarrow)$ Kim-Tollefson '80)

$\Sigma_2(K)$ is irreducible $\iff K$ is prime
3-manifold topology

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An oriented compact 3-manifold $M$ is said to be *irreducible* if any embedded 2-sphere bounds a 3-ball. Otherwise, it is *reducible*.

**Theorem ((⇒) Waldhausen ’69, (⇐) Kim-Tollefson ’80)**

$\Sigma_2(K)$ is irreducible $\iff K$ is prime

**Theorem (Gordon-Luecke ’96)**

Let $M$ be an orientable and irreducible 3-manifold with a torus boundary. If $M(\pi)$ and $M(\gamma)$ are reducible for distinct slopes $\pi$ and $\gamma$, then $\Delta(\pi, \gamma) = 1$. 
Sketch of the proof

Let \( K = (D \cup -D)(n) \)
Sketch of the proof

Let $K = (D \cup -D)(n)$

- $tw(K) = 1 \implies \Sigma_2(K) = M(\frac{1}{n})$
Sketch of the proof

Let $K = (D \cup -D)(n)$

- $tw(K) = 1 \Rightarrow \Sigma_2(K) = M(\frac{1}{n})$
- $K$ is composite $\Rightarrow M(\frac{1}{n})$ is reducible
Sketch of the proof

Let $K = (D \cup -D)(n)$

- $tw(K) = 1 \Rightarrow \Sigma_2(K) = M(\frac{1}{n})$
- $K$ is composite $\Rightarrow$ $M(\frac{1}{n})$ is reducible
- $K \neq K_1\#K_2\# - K_2 \Rightarrow M$ is irreducible
Sketch of the proof

Let $K = (D \cup -D)(n)$

- $tw(K) = 1 \Rightarrow \Sigma_2(K) = M\left(\frac{1}{n}\right)$
- $K$ is composite $\Rightarrow M\left(\frac{1}{n}\right)$ is reducible
- $K \neq K_1 \# K_2 \# -K_2 \Rightarrow M$ is irreducible
- By the symmetry:
  $-K = (D \cup -D)(-n) \Rightarrow \Sigma_2(-K) = M\left(-\frac{1}{n}\right)$
Sketch of the proof

Let $K = (D \cup -D)(n)$

- $tw(K) = 1 \Rightarrow \Sigma_2(K) = M\left(\frac{1}{n}\right)$
- $K$ is composite $\Rightarrow M\left(\frac{1}{n}\right)$ is reducible
- $K \neq K_1\#K_2\# - K_2 \Rightarrow M$ is irreducible
- By the symmetry:
  $-K = (D \cup -D)(-n) \Rightarrow \Sigma_2(-K) = M\left(-\frac{1}{n}\right)$
- Two distinct reducible slopes $\frac{1}{n}$ and $-\frac{1}{n}$, but
  $\Delta\left(\frac{1}{n}, -\frac{1}{n}\right) = 2|n| \neq 1$
The end.
On embeddings of 3 manifolds in symplectic 4 manifolds

Anubhav Mukherjee

Georgia Institute of Technology

Dec. 2020
Outline

1. Conjecture

2. Why is such a Conjecture interesting?

3. Main Results

4. Main Results
Conjecture (Etnyre, Min, M.)

*Every closed, oriented smooth 3-manifold smoothly embeds in a symplectic 4-manifold.*
The embedding of 3-manifolds in higher dimensional space has always been a fascinating problem.
Whitney’s embedding theorem says that every closed oriented 3-manifold smoothly embeds in $\mathbb{R}^6$. 
Why is such a Conjecture interesting?

Whitney’s embedding theorem says that every closed oriented 3-manifold smoothly embeds in $\mathbb{R}^6$.

Hirsch improved this result by proving that every 3-manifold can be smoothly embedded in $S^5$. 
Why is such a Conjecture interesting?

- Whitney’s embedding theorem says that every closed oriented 3-manifold smoothly embeds in $\mathbb{R}^6$.
- Hirsch improved this result by proving that every 3-manifold can be smoothly embedded in $S^5$.
- Meanwhile, Lickorish and Wallace proved that every 3-manifold can be smoothly embedded in some 4-manifold, and in fact, a generalization of their arguments shows that every 3-manifold can be smoothly embedded in the connected sum of copies of $S^2 \times S^2$. 
Conjecture

Why is such a Conjecture interesting?

Main Results

Freedman proved that all integer homology 3-spheres can be embedded topologically, locally flatly in $S^4$. 
Why is such a Conjecture interesting?

- Freedman proved that all integer homology 3-spheres can be embedded topologically, locally flatly in $S^4$.

- On the other hand, the Rokhlin invariant $\mu$ and Donaldson’s diagonalization theorem show that some integer homology spheres cannot smoothly embed in $S^4$. 
Why is such a Conjecture interesting?

Question

Does there exist a compact 4-manifold in which all 3-manifolds embed?
Why is such a Conjecture interesting?

**Question**

*Does there exist a compact 4-manifold in which all 3-manifolds embed?*

Shiomi gave a negative answer to this question.
Why is such a Conjecture interesting?

Question

Does there exist a compact 4-manifold in which all 3-manifolds embed?

Shiomi gave a negative answer to this question. Thus one can ask, what is an interesting class of 4-manifolds in which all 3-manifolds embed?
Theorem (M.)

Given a closed, connected, oriented 3-manifold $Y$ there exists a simply-connected symplectic closed 4-manifold $X$ such that $Y$ can be embedded topologically, locally flatly (i.e. it has collar neighbourhood) in $X$. 
Main Results

Conjecture
Why is such a Conjecture interesting?
Main Results

Theorem (M.)

Given a closed, connected, oriented 3-manifold $Y$ there exists a simply-connected symplectic closed 4-manifold $X$ such that $Y$ can be embedded topologically, locally flatly (i.e. it has collar neighbourhood) in $X$. This embedding can be made a smooth embedding after one stabilization, that is $Y$ can smoothly embed in $X \# (S^2 \times S^2)$. 

Anubhav Mukherjee On embeddings of 3 manifolds in symplectic 4 manifolds Dec.2020 12/29
Main Results

As an application of the proof of the last Theorem, we get followings...
Let $Y_0$ and $Y_1$ be smooth, oriented, closed 3-manifolds. A **cobordism** from $Y_0$ to $Y_1$ is a compact 4-dimensional smooth, oriented, compact manifold $W$ with $\partial W = -Y_0 \sqcup Y_1$. 
Main Results

We say $Y_0$ and $Y_1$ are $R$-homology cobordant, if $H_*(W, Y_i; R) = 0$ for $i = 0, 1$. 
Main Results

- We say $Y_0$ and $Y_1$ are $R$-homology cobordant, if $H_\ast(W, Y_i; R) = 0$ for $i = 0, 1$.

- We call this integral homology cobordism when $R = \mathbb{Z}$ and rational homology cobordism when $R = \mathbb{Q}$. This is an equivalence relation.
Main Results

We say $Y_0$ and $Y_1$ are $R$-homology cobordant, if $H_*(W, Y_i; R) = 0$ for $i = 0, 1$.

We call this integral homology cobordism when $R = \mathbb{Z}$ and rational homology cobordism when $R = \mathbb{Q}$. This is an equivalence relation.

So one can define

$$\Theta^3_R = \{ Y \text{ closed 3-manifold with } H_*(Y; R) = 0 \} / \sim$$

where $R$ is a fixed commutative ring.
Main Results

- We say $Y_0$ and $Y_1$ are $R$-homology cobordant, if $H_*(W, Y_i; R) = 0$ for $i = 0, 1$.

- We call this *integral homology cobordism* when $R = \mathbb{Z}$ and *rational homology cobordism* when $R = \mathbb{Q}$. This is an equivalence relation.

- So one can define

$$\Theta^3_R = \{ Y \text{ closed 3-manifold with } H_*(Y; R) = 0 \} / \sim$$

where $R$ is a fixed commutative ring.

- We give $\Theta^n_R$ the structure of a group where summation is given by the connected sum operation. The zero element of this group is given by the class of $S^n$, and the inverse of the class of $[Y]$ is given by the class of $Y$ with reversed orientation.
In low-dimensional topology the study of $\Theta^3_\mathbb{Z}$ and $\Theta^3_\mathbb{Q}$ are of special interest.
In low-dimensional topology the study of $\Theta_3^\mathbb{Z}$ and $\Theta_3^\mathbb{Q}$ are of special interest.

- Livingston showed that these groups are generated by irreducible 3-manifolds.
Main Results

In low-dimensional topology the study of $\Theta_3^Z$ and $\Theta_3^Q$ are of special interest.

- Livingston showed that these groups are generated by irreducible 3-manifolds.
- Myers showed that these groups are generated by hyperbolic 3-manifolds.
Main Results

Theorem (M.)

The homology cobordism groups $\Theta_3^\mathbb{Z}$ and $\Theta_3^\mathbb{Q}$ are generated by Stein fillable 3-manifolds.
Main Results

Theorem (M.)

*If an L-space $Y$ smoothly embeds in a closed symplectic 4-manifold $X$ then it has to be separating. Moreover, if $X = X_1 \cup_Y X_2$ then one of the $X_i$ has to be a negative-definite 4-manifold.*
Main Results

Theorem (M.)

There exists a 3-manifold $Y$ which cannot be embedded(*) in any compact symplecteic 4-manifold with (weakly) convex boundary.
Main Results

Conjecture
Why is such a conjecture interesting?

Main Results

Theorem (M.)

There exists a 3-manifold \( Y \) which cannot be embedded(*) in any compact symplectic 4-manifold with (weakly) convex boundary.

Example of 3-manifolds without symplectic fillings were known before by the work of Lisca–Matic, Etnyre–Honda.
Main Results

Theorem (M.)

*There exists a 3-manifold $Y$ which cannot be embedded (*) in any compact symplectic 4-manifold with (weakly) convex boundary.*

- Example of 3-manifolds without symplectic fillings were known before by the work of Lisca–Matic, Etnyre–Honda.
- This above result is stronger in the sense that there exists 3-manifolds which cannot even embed(*) in (weak) filling of any 3-manifolds.
The smooth v/s topological embeddings of 3-manifolds can be used to study exotic structure on 4-manifolds.
Theorem (M.)

There exists compact 4-manifolds with boundary $X$ and $X'$ such that $b_2(X) = b_2(X') = 1$ that are homeomorphic but not diffeomorphic.
Main Results

Conjecture

Why is such a Conjecture interesting?

Main Results

Theorem (M.)

There exists compact 4-manifolds with boundary $X$ and $X'$ such that $b_2(X) = b_2(X') = 1$ that are homeomorphic but not diffeomorphic.

- Akbulut proved existence of such 4-manifolds first.
- The above is an alternative proof of that result.
Thank you!
Deep and shallow slice knots in 4-manifolds

Joint work with Michael Klug

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Def.: 

K is a knot in the boundary of a 4-manifold $X^4$. 

Slice disk $\Delta^2: D^2 \to X$ with $\partial \Delta^2 = K \subset \partial X$. 

$K$ is **deep slice** in $X$ if the disk "needs to use the extra topology of $X"$, i.e. there is no slice disk for $K$ in a collar $\partial X \times [0,1] \subset X$ of the $\partial X$. 

Collar
Non-example: There are no deep slice knots in $\#^k S^1 \times D^3$.

$\#^k S^1 \times D^3 = \text{thickening of} \quad \includegraphics[width=2cm]{knot}

\quad \xleftarrow{4\text{-dim. 1-handles}}

\includegraphics[width=4cm]{4d_with_handles}

Any slice disk generically avoids the spine $\includegraphics[width=2cm]{knot}$

$\quad \xrightarrow{\text{lives in a collar neighborhood of the boundary}}$
Example: \( X^4 = D^4 \cup (2\text{-handles}) \) has deep slice knots in boundary (which are nullhomotopic in \( \partial X \), but not contained in a 3-ball).

Two cases

\[ \pi_1(\partial X) = \{1\} \text{ and thus } \partial X = S^3 \]

\[ \pi_1(\partial X) \text{ non-trivial} \]

We use a theorem of Rohlin on the genus of embedded surfaces representing 2-dim. homology classes in \( \hat{X} = X \cup (4\text{-handle}) \).

Use Wall's self-intersection number with values in \( \mathbb{Z}[\pi_1(\partial X)] \)

\[ \langle g = g^{-1}, 1 \rangle \]

of the track of a homotopy in \( \partial X \times [0,1] \)