Skinning Maps

and a

Lost Theorem of Thurston

Yair Minsky , Tech Topology 2021

joint work with R. Canary & K. Bromberg

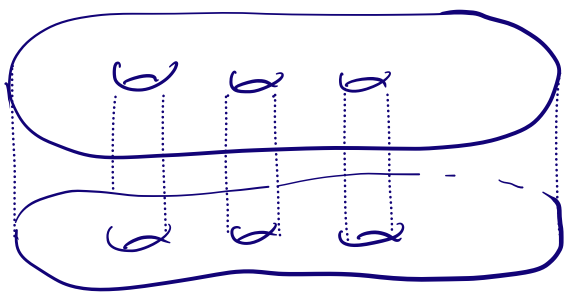
Plan:

- 3-Manifolds and hyperbolic structures
- The skinning map and the gluing problem
- Thurston's lost theorem
- Uniform models and the proof

3-manifolds in this talk:

compact, oriented, irreducible, atoroidal,
with incompressible boundary.

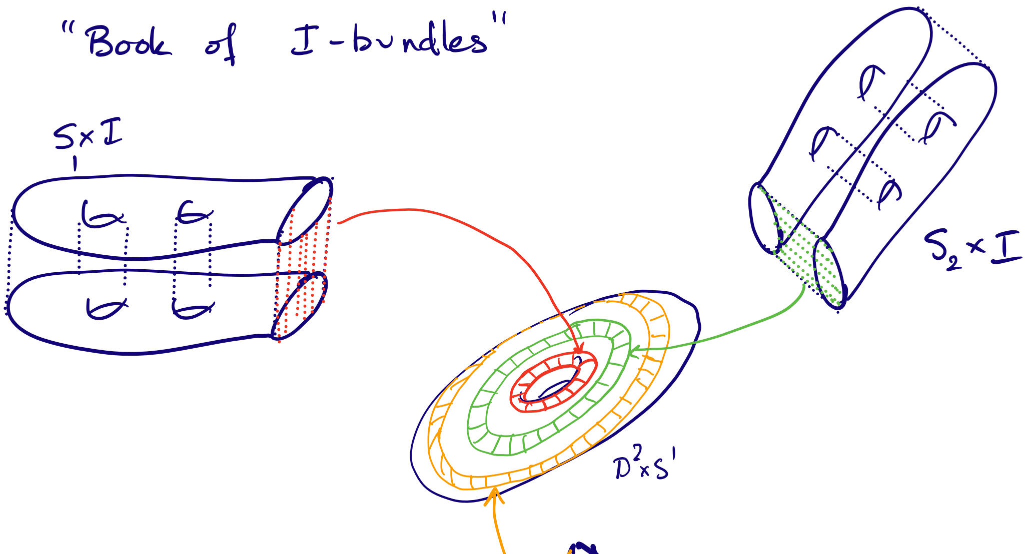
Examples to keep in mind:

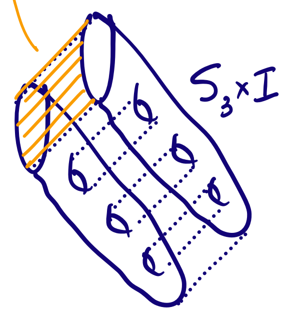


"product I-bundle"

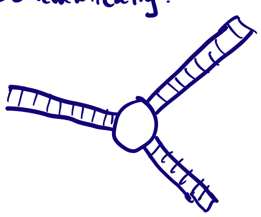
$S \times [0,1]$

"Book of I-bundles"

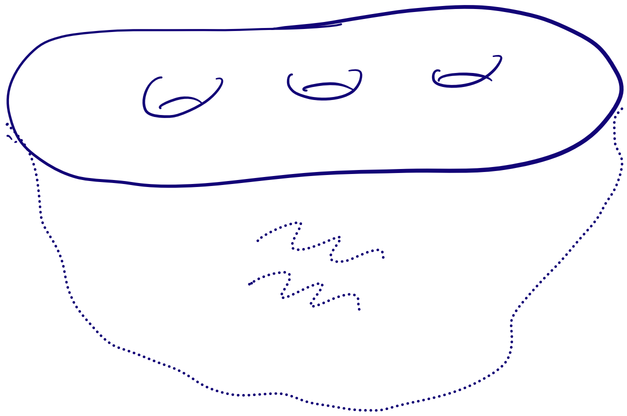




Schematically:



Acylindrical manifolds



no properly embedded essential $(S^1 \times I, \partial(S^1 \times I)) \to (M, \partial M)$

These all admit hyperbolic structures:

$$\operatorname{int} M \cong \mathbb{H}^3/\Gamma$$

and in fact convex-cocompact ones:

$$\partial_\infty \mathbb{H}^3 = \Lambda \sqcup \Omega$$

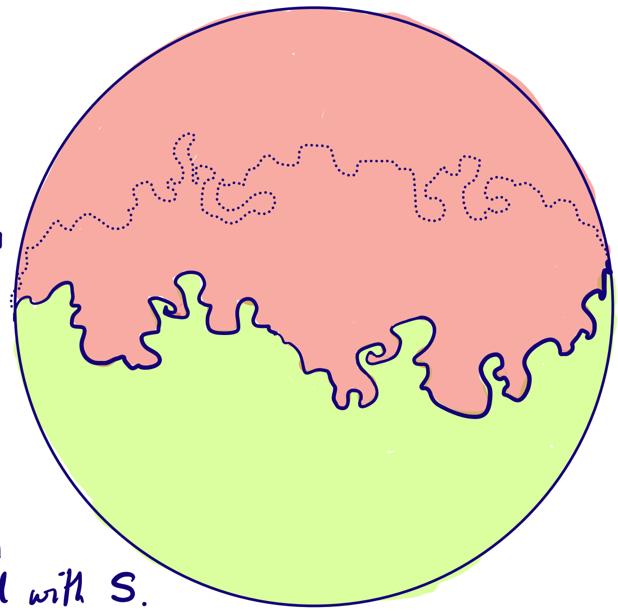
Λ = limit set (closed)

Ω = domain of discontinuity

$$M \cong (\mathbb{H}^3 \cup \Omega)/\Gamma$$

Pictures of limit sets

I-bundles:

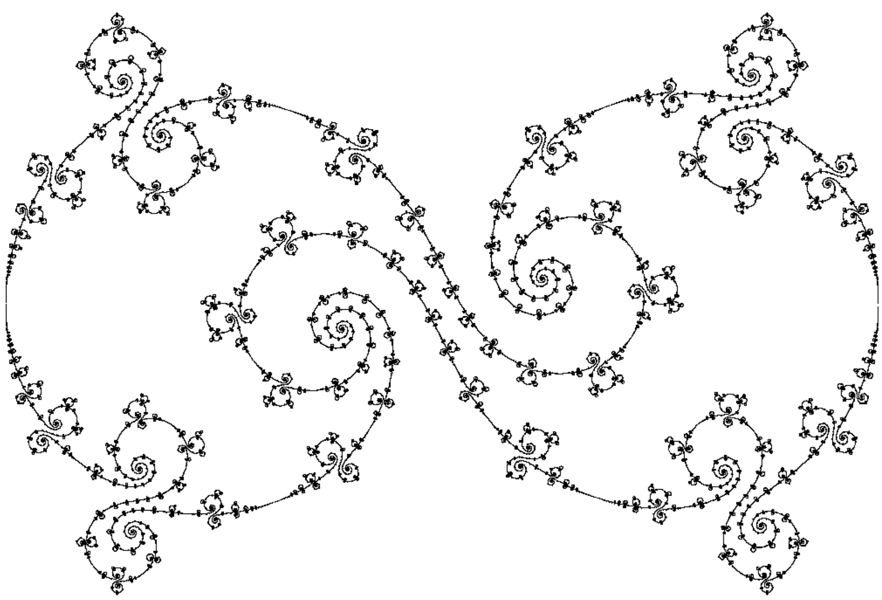


Γ is <u>quasifuchsian</u>

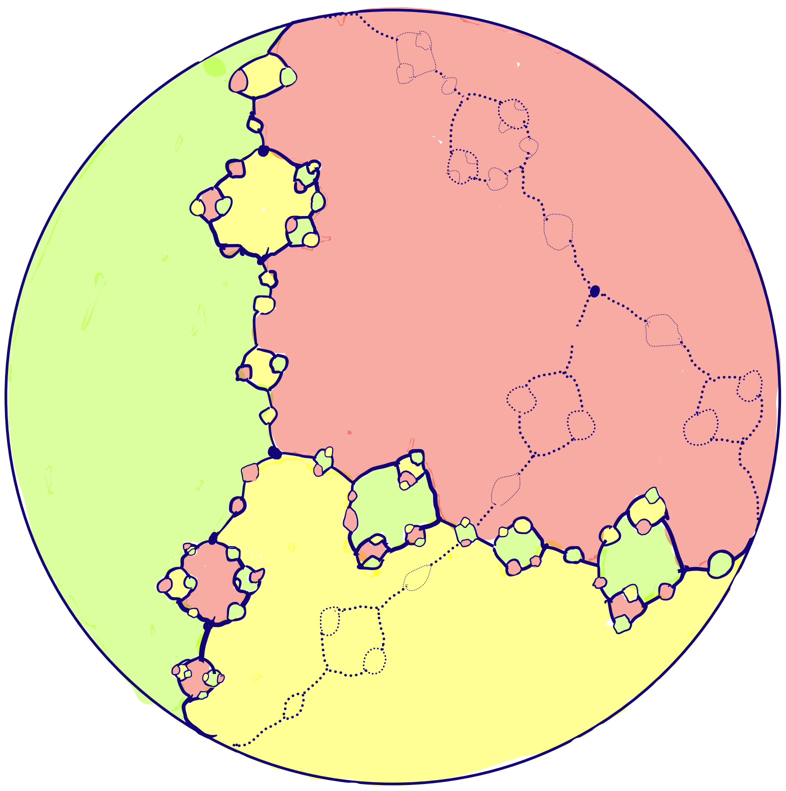
$\Omega = \Omega_1 \sqcup \Omega_2$

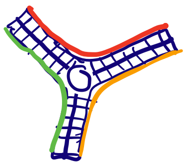
Ω_1/Γ, Ω_2/Γ

are two Riemann surface identified with S.

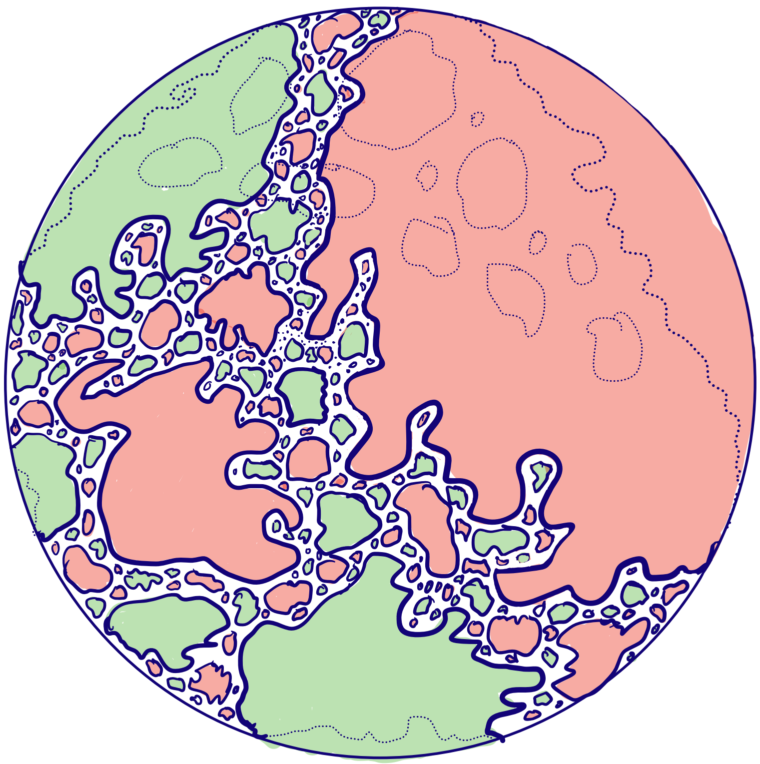


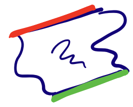
Book of I-bundles:

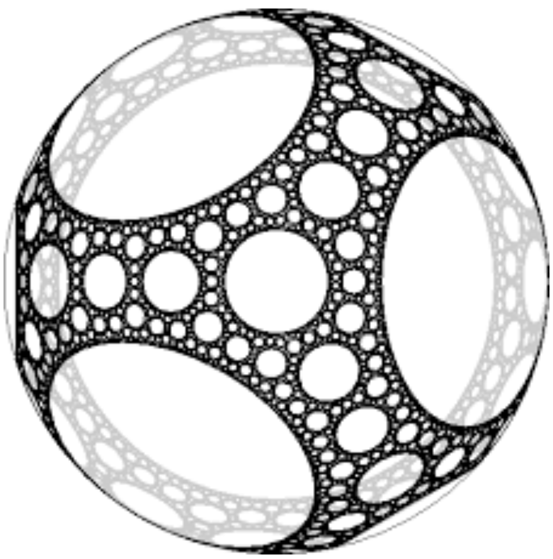




Acylindrical Manifold







Bers' Simultaneous Uniformization Theorem

Ahlfors Maskit Kra Marden Sullivan

$CC(M) = \{$ convex-cocompact hyp. structures on $M\}$

Teichmüller space

$$\cong \mathcal{T}(\underline{\partial M})$$

$$\left(M \cong \mathbb{H}^3/\Gamma\right) \longleftrightarrow \left(\partial M \cong \Omega/\Gamma\right)$$

In particular for $M = S \times I$

$$\mathcal{T}(\partial M) = \mathcal{T}(S) \times \mathcal{T}(\bar{S})$$

The problem of uniform models:

How do we, given data in $\mathcal{T}(\partial M)$,
(e.g. a minimal-length pants decomposition)
answer geometric questions about the hyperbolic structure?
(e.g. volume, diameter, injectivity radii...)

One kind of solution when $M = S \times [0,1]$:
(w. Masur, Brock & Canary)

$\nu = (\nu^+, \nu^-) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \rightsquigarrow$ minimal length pants decompositions in $S, \bar{S}$

$\rightsquigarrow$ "hierarchical paths" in curve complex of S

$\rightsquigarrow$ Explicit "model" $\mathbb{M}_\nu$

K-bilipschitz homeo. $f : \mathbb{M}_\nu \to \mathbb{H}^3/\Gamma_\nu$

"Uniform" means: K depends only on S.

The skinning map and Thurston's gluing problem

S component of ∂M

inclusion $\pi_1 S \longrightarrow \pi_1 M$

induces $CC(M) \longrightarrow CC(S \times [0,1])$

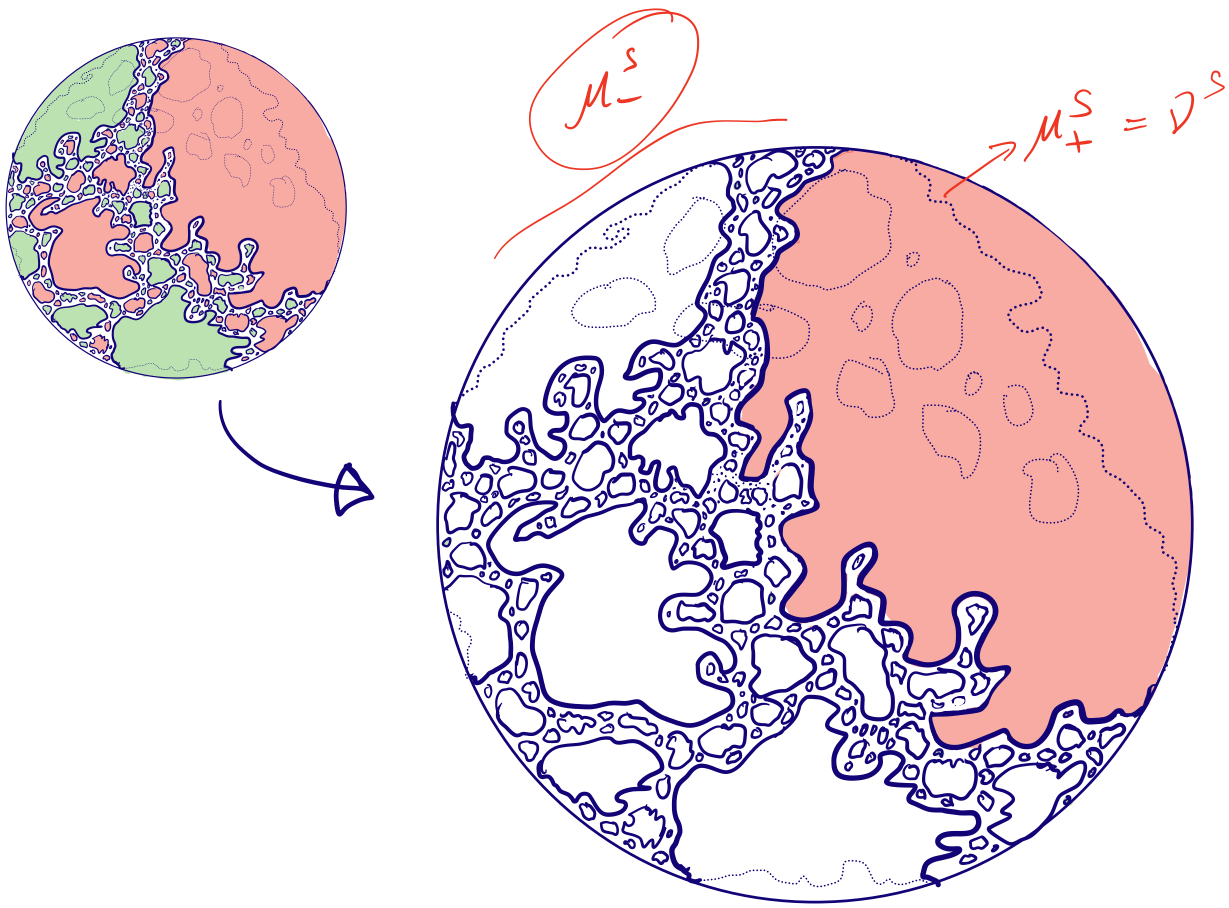
by passage to covers.

Bers parameters:

$$\nu \in T(\partial M) \longrightarrow (\mu_S^+, \mu_S^-) \in T(S) \times T(\bar{S})$$

in fact $\mu_S^+ = \nu_S$

$\boxed{\mu_S^- \text{ is a (holomorphic) function of } \nu}$



μ_-^s
$\mu_+^S = \nu^s$

Combining over all components of ∂M,

$$\sigma_M : \mathcal{T}(\partial M) \longrightarrow \mathcal{T}(\overline{\partial M})$$
$$(\nu_S) \longmapsto (\mu_S^-)$$

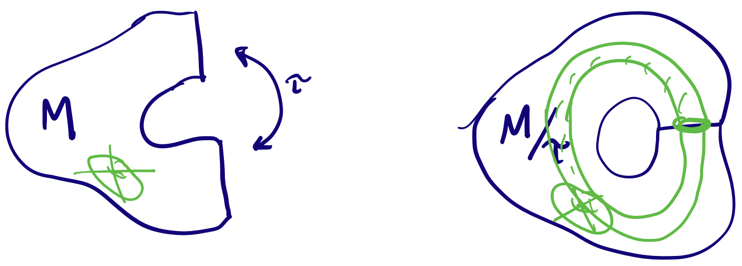
The "skinning map".

What is it good for?

The gluing problem:

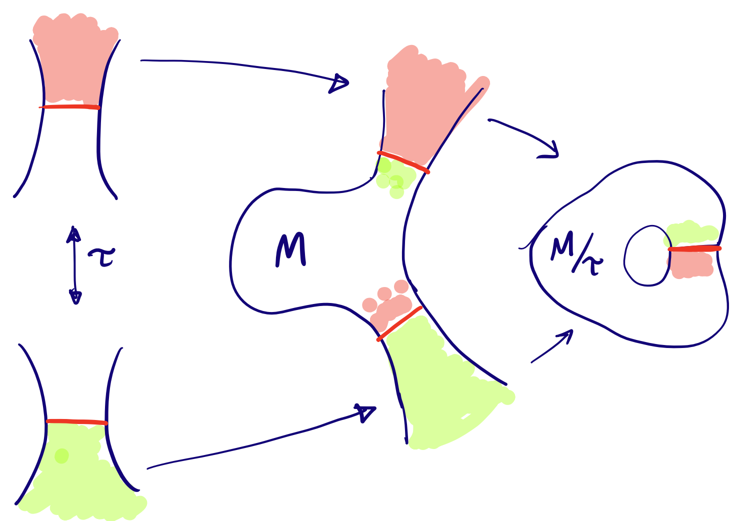
(inductive step in Thurston's proof of hyperbolization of Haken manifolds)

$\tau : \partial M \longrightarrow \partial M$ orientation-reversing, fixed point free involution



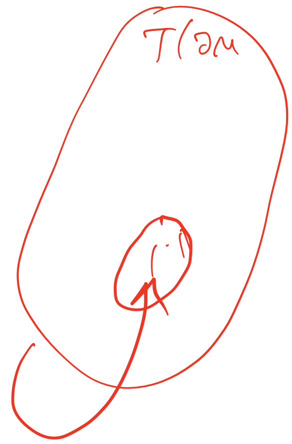
Theorem (Thurston): If M admits convex-cocompact hyperbolic structures and M/τ has no essential tori, then M admits a hyperbolic structure.

Gluing and fixed points



A solution is given by a fixed point of the map $\tau_* \circ \sigma_M$.

Thurston proved a fixed point exists.
He also made a stronger statement:



Theorem Given M there exists $k > 0$ such that, if M/τ is atoroidal,

$$(\tau_* \circ \sigma_M)^k (\mathcal{T}(\partial M))$$

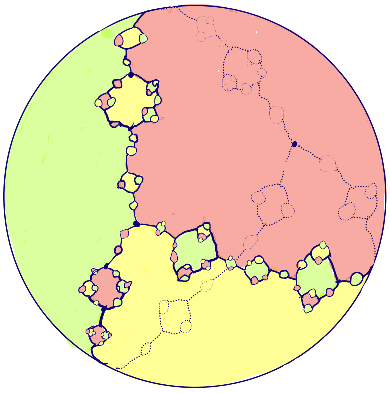
is bounded in $\mathcal{T}(\partial M)$.

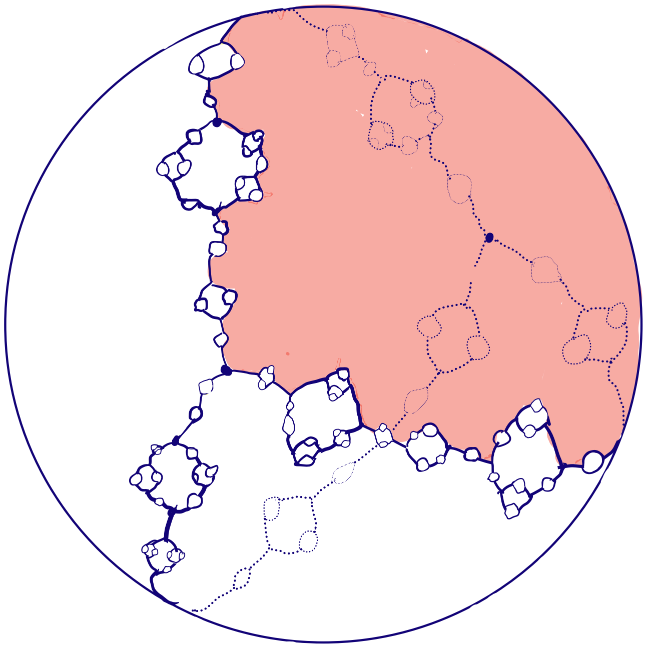
When M is acylindrical:

M/τ is always atoroidal,
and $\sigma_M(\mathcal{T}(\partial M))$ is bounded.
(Thurston's Bounded Image Theorem,
full proof A. Kent 2010)

In the cylindrical case

$\operatorname{diam} \sigma_M(\tau(\partial M)) = \infty$

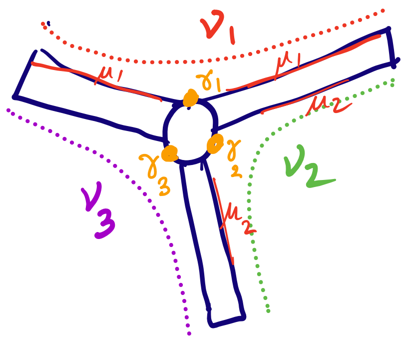




About the proof :

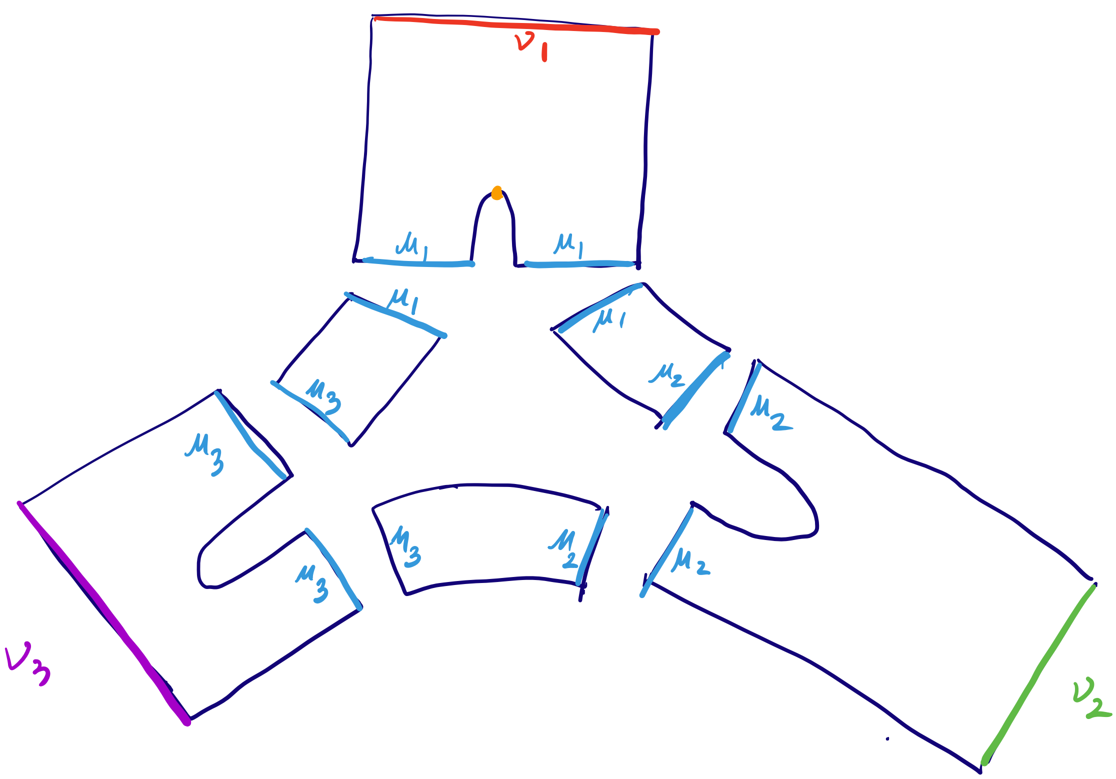
I. Main geometric part: model building

$\gamma_i \sim$ core of solid torus.



ν projects combinatorially to give a marking μ on $S \setminus \gamma$. Subsurface Projection

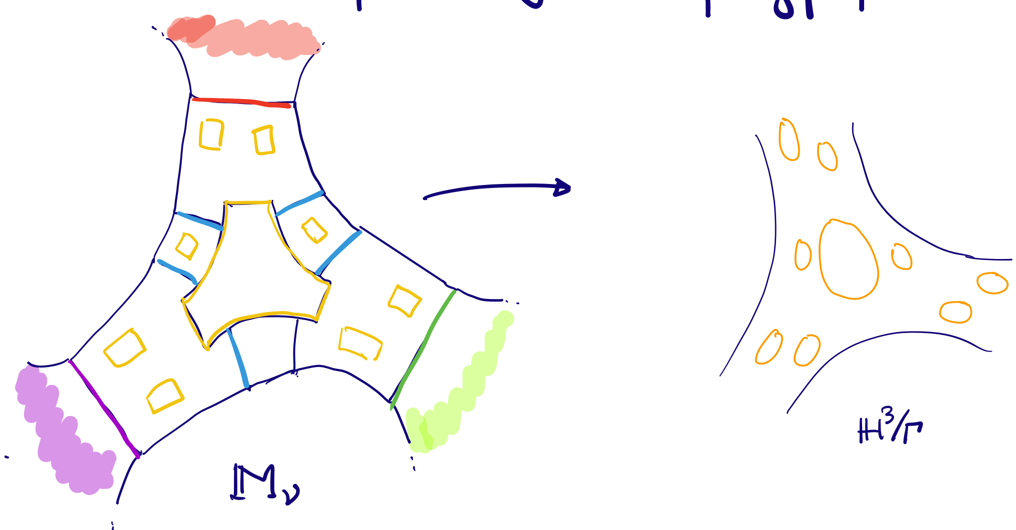
use BCM models for I- bundles :



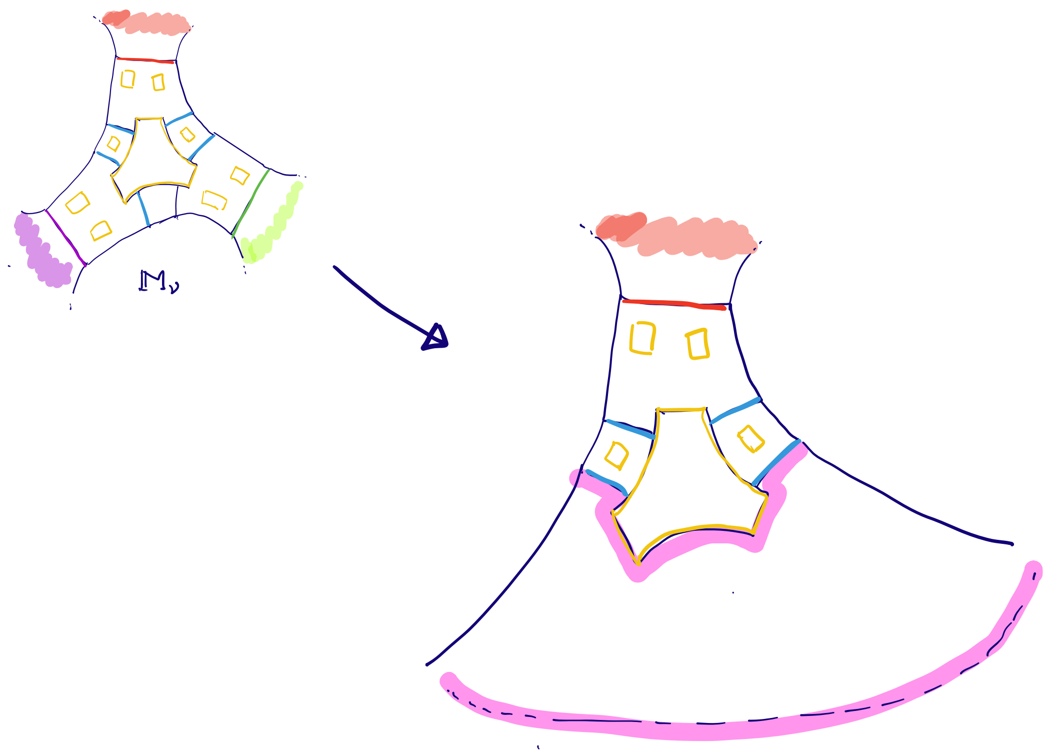
Theorem: (Bromberg-Canary-M)

This model admits a K-Lipschitz proper hom. eq. to $\mathbb{H}^3/\Gamma$ which respects the thick-thin decomposition.

K depends only on topology of M



II. Estimating the skinning image from the model



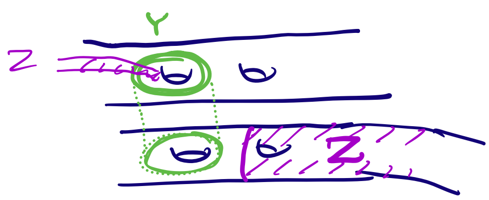
III. Topological Endgame: propagating control

Def. $Z, Y \subset \partial M$. essential subsurfaces

<u>Y escapes from Z</u> if:

$\exists \quad h : (Y \times [0,1],\ Y \times \{0,1\}) \longrightarrow (M, \partial M)$

- $h(\cdot, 0) = id_Y$
- $h(Y \times 1) \cap Z = \phi$
- h not homotopic into ∂M as map of pairs.



Let $w(Z)$ = *maximal essential subsurface that escapes from* Z

$$\psi(Z) = \partial M \setminus w(Z)$$

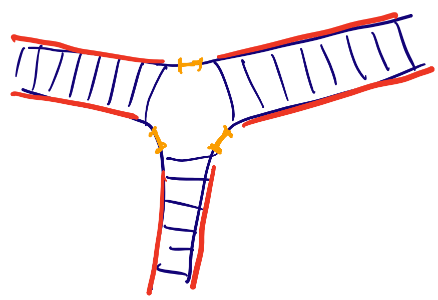
= subsurface "controlled by Z"

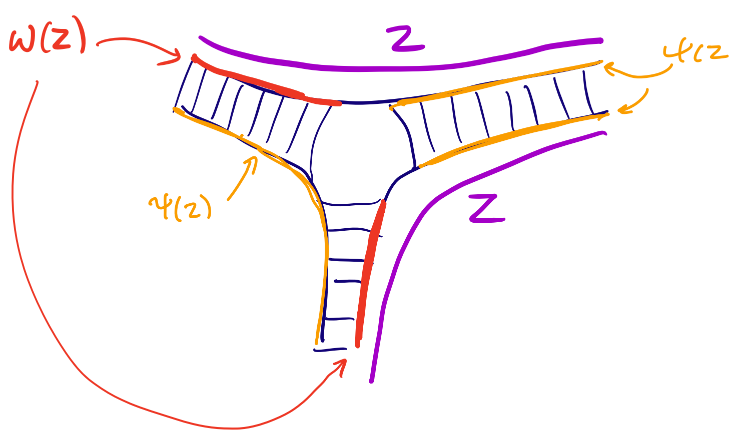
examples:

$Z = \emptyset$

$w(Z)$ = I-bundle boundaries

$\psi(Z)$ = solid torus annuli





ψ monotonic: $W \subseteq Z \Rightarrow \psi(W) \subseteq \psi(Z)$

$Z \subset \partial M$ let $r_Z : \mathcal{T}(\partial M) \to \mathcal{T}(Z)$

be restriction map. $\mathcal{T}(\emptyset) = \{*\}$

Geometry in (I) & (II) suffices to give:

<u>Theorem</u> (Bromberg-Canary-M)

Given $Z \subset \partial M$, $K \subset \mathcal{T}(Z)$ compact

$\exists$ $L \subset \mathcal{T}(\Psi(Z))$ compact

$$\sigma_M(r_Z^{-1}(K)) \subset r_{\Psi(Z)}^{-1}(L)$$

"control on $Z \Rightarrow$ control on $\Psi(Z)$ after skinning"

Now given τ, define $Z_0 \subseteq Z_1 \subseteq Z_2 \cdots$

$$Z_0 = \emptyset$$

$$Z_{i+1} = \tau(\psi(Z_i))$$

The theorem gives compact $L_i \subset T(Z_i)$ such that

$$(\tau \circ \sigma_M)^i (T(S)) \subset r_{Z_i}^{-1}(L_i)$$

Eventually $Z_{n+1} = Z_n$. (note n depends on topology of ∂M only)

if $Z_n = \partial M$

$$r_{Z_n}^{-1}(L_n) = L_n \quad \text{and we are done}$$

If $Z_n \neq \partial M$: (and must be nonempty)

Then ∂Z_n bounds a union of <u>annuli</u> in M which assemble to essential tori in M/τ.

Illustration for book of I-bundles:

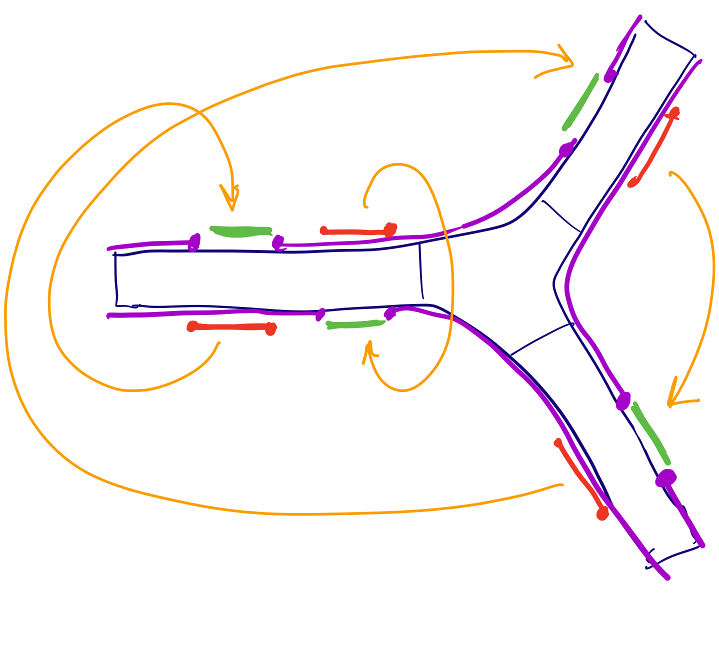
$Z_0 = \phi$

$Z_1 = \Psi(\phi) =$ annuli $\gamma_1\ \gamma_2\ \gamma_3$

$\vdots$

Z_n contains $\gamma_1 \cup \gamma_2 \cup \gamma_3$ so $Z_n^c \subset I$-bundles

$Z_{n+1} = Z_n$



$\omega(Z_n)$

Z_n

$\tau(\omega(Z_n)) = Z_{n+1}^c$

$= Z_n^c$

THANKS!