On Contact Invariants in Bordered Floer Homology

Konstantinos Varvarezos[1]
(joint with Hyunki Min[1])

[1]University of California, Los Angeles

Tech Topology Conference
December 2024

Contact structures and Heegaard Floer homology

Let Y be a 3-manifold, and let ξ be a contact structure on Y.

- Y closed $\rightsquigarrow c(\xi) \in \widehat{HF}(-Y)$ [OS05]

Contact structures and Heegaard Floer homology

Let Y be a 3-manifold, and let ξ be a contact structure on Y.

- Y closed $\rightsquigarrow c(\xi) \in \widehat{HF}(-Y)$ [OS05]
- ∂Y sutured $\rightsquigarrow EH(\xi) \in SFH(-Y)$ [HKM09]

Contact structures and Heegaard Floer homology

Let Y be a 3-manifold, and let ξ be a contact structure on Y.

- Y closed $\rightsquigarrow c(\xi) \in \widehat{HF}(-Y)$ [OS05]
- ∂Y sutured $\rightsquigarrow EH(\xi) \in SFH(-Y)$ [HKM09]
- ∂Y parametrized $\rightsquigarrow c_A(\xi) \in \widetilde{CFA}(-Y)$
  $\rightsquigarrow c_D(\xi) \in \widetilde{CFD}(-Y)$ [AFH$^+$23]

Contact structures and Heegaard Floer homology

Let Y be a 3-manifold, and let ξ be a contact structure on Y.

- Y closed $\rightsquigarrow c(\xi) \in \widehat{HF}(-Y)$ [OS05]
- ∂Y sutured $\rightsquigarrow EH(\xi) \in SFH(-Y)$ [HKM09]
- ∂Y parametrized $\rightsquigarrow c_A(\xi) \in \widetilde{CFA}(-Y)$
  $\rightsquigarrow c_D(\xi) \in \widetilde{CFD}(-Y)$ [AFH+23]

ξ overtwisted $\Rightarrow$ invariants vanish

Bordered Sutured Contact Invariants

Theorem 1 (Min–V.)

Let $(Y, \Gamma, \mathcal{F})$ be a bordered sutured manifold and ξ a contact structure on the sutured manifold $(Y, \Gamma \cup \Gamma_I)$ where Γ_I is an elementary dividing set for $\mathcal{F}$. There exist contact invariants: $c_A(\xi) \in \widehat{BSA}(-Y)$ and $c_D(\xi) \in \widehat{BSD}(\mathcal{TW}^+ \cup -Y)$

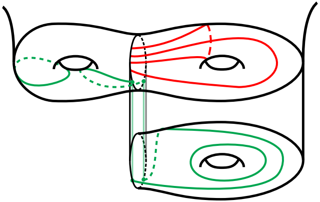
(More generally, bimodule invariants: $c_{AA}(\xi) \in \widehat{BSAA}(-Y)$, etc.)

Bordered Sutured Contact Invariants

Theorem 1 (Min–V.)

Let $(Y, \Gamma, \mathcal{F})$ be a bordered sutured manifold and ξ a contact structure on the sutured manifold $(Y, \Gamma \cup \Gamma_I)$ where Γ_I is an elementary dividing set for $\mathcal{F}$. There exist contact invariants: $c_A(\xi) \in \widehat{BSA}(-Y)$ and $c_D(\xi) \in \widehat{BSD}(\mathcal{TW}^+ \cup -Y)$

(More generally, bimodule invariants: $c_{AA}(\xi) \in \widehat{BSAA}(-Y)$, etc.)



Properties

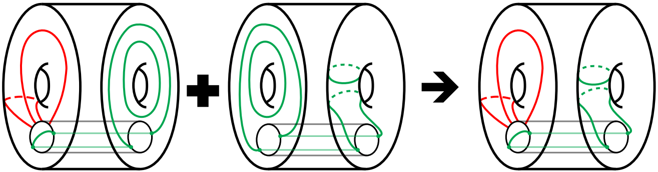
- Pairing: $[c_A(\xi_1) \boxtimes c_D(\xi_2)] = EH(\xi_1 \cup \xi_2)$

Properties

- Pairing: $[c_A(\xi_1) \boxtimes c_D(\xi_2)] = EH(\xi_1 \cup \xi_2)$
- Gluing a contact $\Sigma_\bullet \times [0,1]$ corresponds to m_2 in $\widehat{BSA}$
  $c_A(\xi \cup \xi_{a(\rho)}) = m_2(c_A(\xi), a(\boldsymbol{\rho}))$

Properties

- Pairing: $[c_A(\xi_1) \boxtimes c_D(\xi_2)] = EH(\xi_1 \cup \xi_2)$
- Gluing a contact $\Sigma_\bullet \times [0,1]$ corresponds to m_2 in $\widehat{BSA}$
  $c_A(\xi \cup \xi_{a(\rho)}) = m_2(c_A(\xi), a(\boldsymbol{\rho}))$

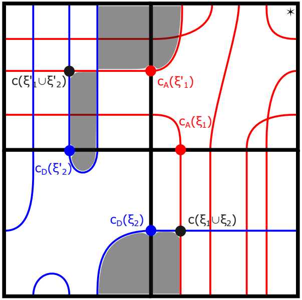


Computations

- c_A and c_D are derived from EH

Computations

- c_A and c_D are derived from *EH*
- For torus boundary, amenable to immersed curve technique of [HRW24]



Applications

- Positive contact surgery formula for knots in L-spaces

Applications

- Positive contact surgery formula for knots in L-spaces
- Classification of tight contact structures on torus knot surgeries

Applications

- Positive contact surgery formula for knots in L-spaces
- Classification of tight contact structures on torus knot surgeries
- Invariants of Legendrian satellite knots?

Thank you!

Bibliography I

Akram Alishahi, Viktória Földvári, Kristen Hendricks, Joan Licata, Ina Petkova, and Vera Vértesi.
Bordered Floer homology and contact structures.
Forum Math. Sigma, 11:Paper No. e30, 44, 2023.

Ko Honda, William H. Kazez, and Gordana Matić.
The contact invariant in sutured Floer homology.
Invent. Math., 176(3):637–676, 2009.

Bibliography II

Jonathan Hanselman, Jacob Rasmussen, and Liam Watson.
Bordered Floer homology for manifolds with torus boundary via immersed curves.
J. Amer. Math. Soc., 37(2):391–498, 2024.

Peter Ozsváth and Zoltán Szabó.
Heegaard Floer homology and contact structures.
Duke Math. J., 129(1):39–61, 2005.

Transverse invariant as Khovanov skein spectrum at its extreme Alexander grading

Nilangshu Bhattacharyya, Adithyan Pandikkadan

Louisiana State University

December 7, 2024

College of Science
Department of Mathematics

Tech Topology Conference

Overview

Here L is a closed braid representative of a transverse knot

$$G = \mathbb{Z}, \mathbb{Z}_2$$

$$1 = \psi_{Sk}(L) \in \; G \cong \; \widetilde{H}_{Sk}^{0,sl(L),\; -b(L)}(L;G) \longrightarrow \widetilde{H}_{Kh}^{0,sl(L)}(L;G) \;\ni \psi_{Kh}(L)$$

Overview

$$\boxed{G = \mathbb{Z}, \mathbb{Z}_2}$$

$$1 = \psi_{Sk}(L) \in \; G \cong \; \widetilde{H}^{0, sl(L), -b(L)}_{Sk}(L;G) \longrightarrow \widetilde{H}^{0, sl(L)}_{Kh}(L;G) \; \ni \psi_{Kh}(L)$$

$$G = \mathbb{Z}_2$$

$$1 \in \quad \mathbb{Z}_2 \cong \widehat{HFK}\left(-\Sigma(L), \widetilde{B}, -g\left(\widetilde{B}\right); \; \mathbb{Z}_2\right) \longrightarrow \widehat{HF}(-\Sigma(L); \; \mathbb{Z}_2) \quad \ni c(\xi)$$

Overview

$$\mathscr{X}^{sl(L)}_{Kh}(L)$$

$$\downarrow \widetilde{H}^0(\_;G)$$

$$\boxed{G = \mathbb{Z}, \mathbb{Z}_2}$$

$$1 = \psi_{Sk}(L) \in \; G \cong \widetilde{H}^{0,sl(L),-b(L)}_{Sk}(L;G) \longrightarrow \widetilde{H}^{0,sl(L)}_{Kh}(L;G) \ni \psi_{Kh}(L)$$

$$G = \mathbb{Z}_2 \qquad\qquad G = \mathbb{Z}_2$$

$$1 \in \; \mathbb{Z}_2 \cong \widehat{HFK}\left(-\Sigma(L), \widetilde{B}, -g\left(\widetilde{B}\right);\ \mathbb{Z}_2\right) \longrightarrow \widehat{HF}(-\Sigma(L);\ \mathbb{Z}_2) \ni c(\xi)$$

Overview

$$\mathbb{S} \longleftarrow \mathscr{X}^{sl(L)}_{Kh}(L) \ : \ \Psi_{Kh}(L)$$

$$\downarrow \widetilde{H}^0(\_;G)$$

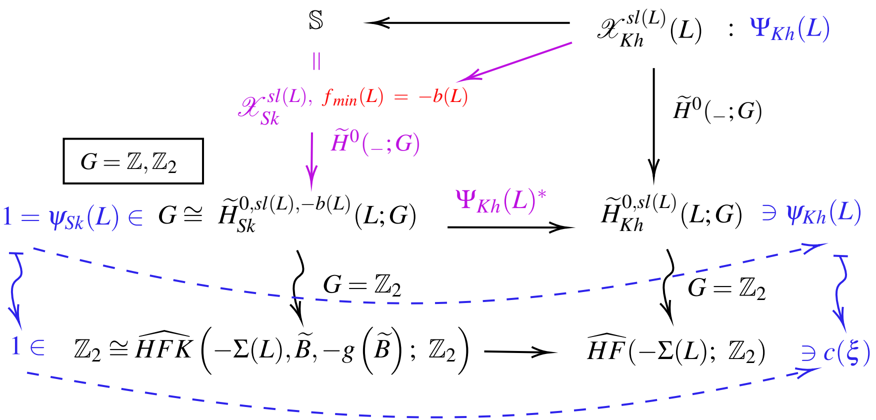
$$\boxed{G = \mathbb{Z}, \mathbb{Z}_2}$$

$$1 = \psi_{Sk}(L) \in \ G \cong \ \widetilde{H}^{0,sl(L),-b(L)}_{Sk}(L;G) \longrightarrow \widetilde{H}^{0,sl(L)}_{Kh}(L;G) \ \ni \psi_{Kh}(L)$$

$$G = \mathbb{Z}_2 \qquad\qquad G = \mathbb{Z}_2$$

$$1 \in \ \mathbb{Z}_2 \cong \widehat{HFK}\left(-\Sigma(L), \widetilde{B}, -g\left(\widetilde{B}\right);\ \mathbb{Z}_2\right) \longrightarrow \widehat{HF}(-\Sigma(L);\ \mathbb{Z}_2) \ \ni c(\xi)$$

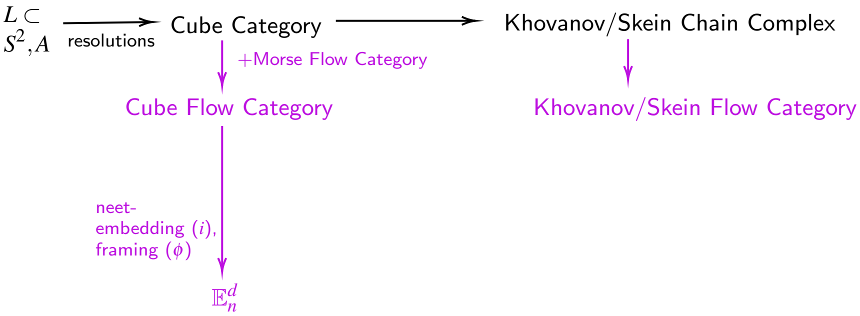
Overview


$\mathbb{S}$
$\mathscr{X}^{sl(L)}_{Kh}(L)$: $\Psi_{Kh}(L)$
$\|$
$\mathscr{X}^{sl(L),}_{Sk}$ $f_{min}(L) = -b(L)$
$\widetilde{H}^0(\_;G)$
$\widetilde{H}^0(\_;G)$
$G = \mathbb{Z}, \mathbb{Z}_2$
$1 = \psi_{Sk}(L) \in G \cong \widetilde{H}^{0,sl(L),-b(L)}_{Sk}(L;G)$
$\Psi_{Kh}(L)^*$
$\widetilde{H}^{0,sl(L)}_{Kh}(L;G) \ni \psi_{Kh}(L)$
$G = \mathbb{Z}_2$
$G = \mathbb{Z}_2$
$1 \in \mathbb{Z}_2 \cong \widehat{HFK}\left(-\Sigma(L), \widetilde{B}, -g\left(\widetilde{B}\right);\ \mathbb{Z}_2\right)$
$\widehat{HF}(-\Sigma(L);\ \mathbb{Z}_2) \ni c(\xi)$

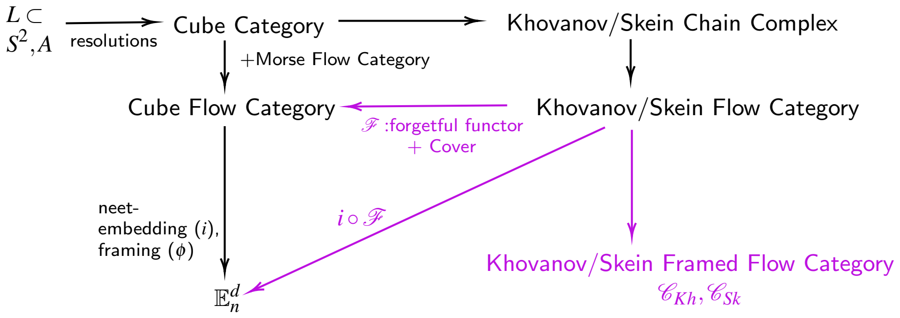
Lipshitz-Sarkar Khovanov Spectrum

$L \subset S^2, A$ $\xrightarrow{\text{resolutions}}$ Cube Category $\longrightarrow$ Khovanov/Skein Chain Complex

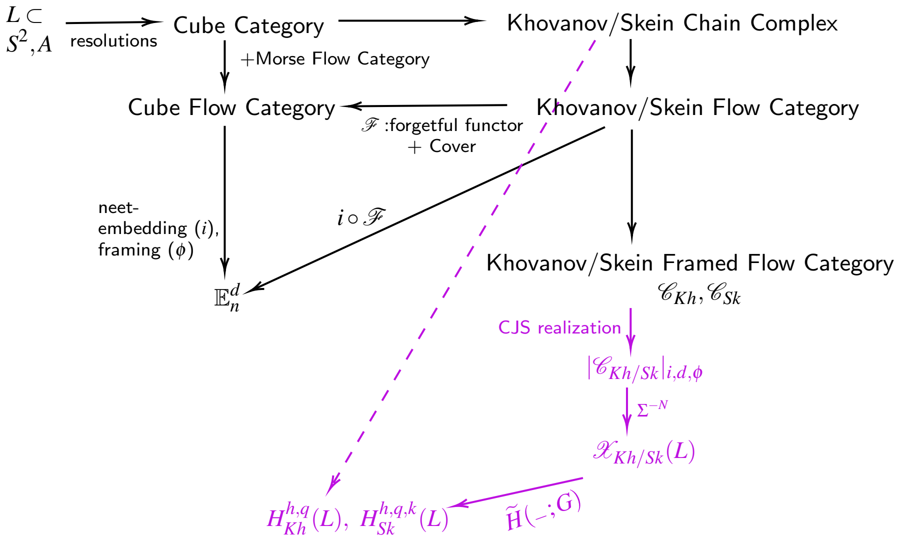
Lipshitz-Sarkar Khovanov Spectrum



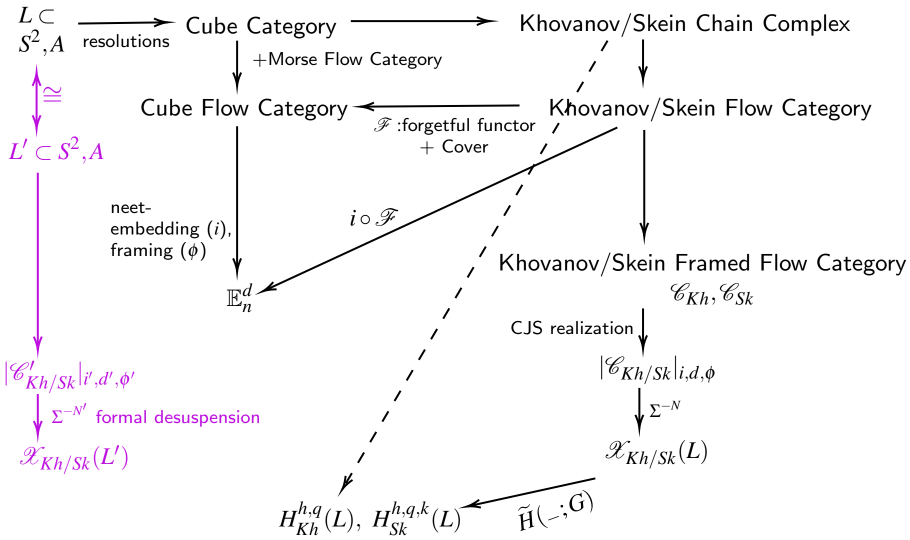
Lipshitz-Sarkar Khovanov Spectrum



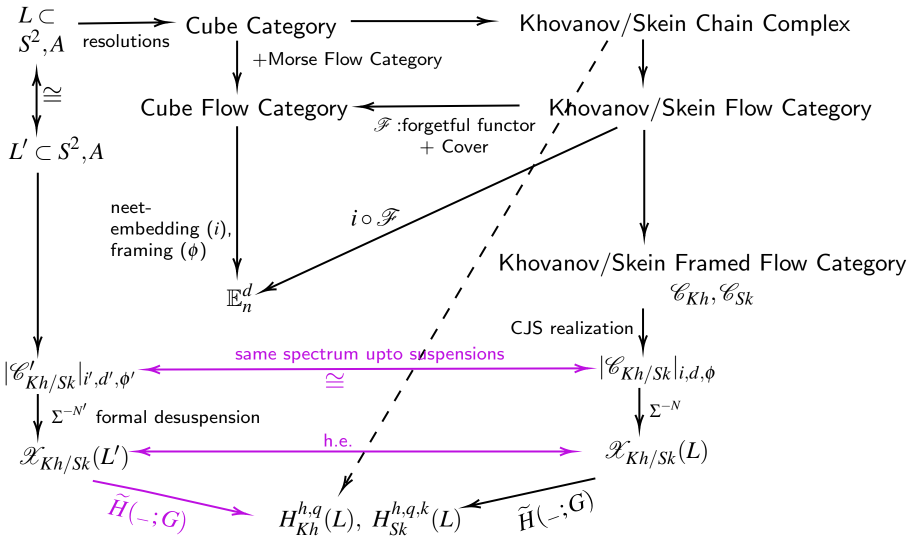
Lipshitz-Sarkar Khovanov Spectrum



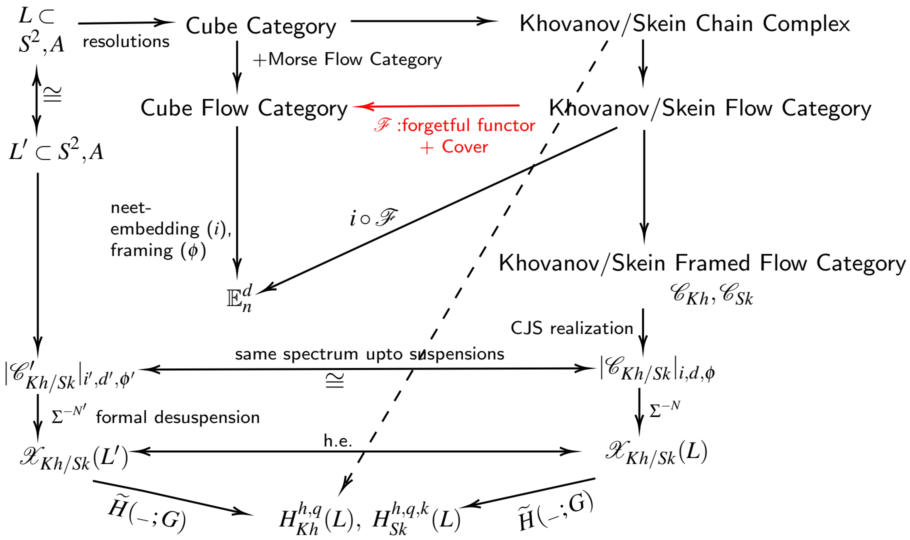
Lipshitz-Sarkar Khovanov Spectrum



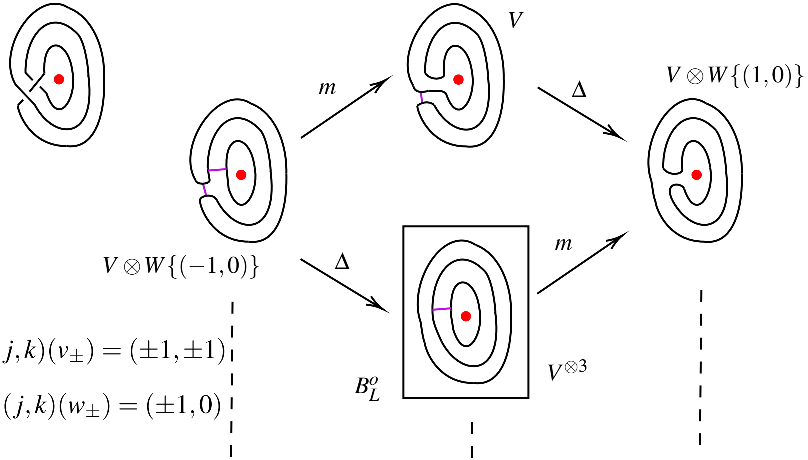
Lipshitz-Sarkar Khovanov Spectrum



Lipshitz-Sarkar Khovanov Spectrum



Khovanov skein homology



$(j,k)(v_\pm) = (\pm 1, \pm 1)$

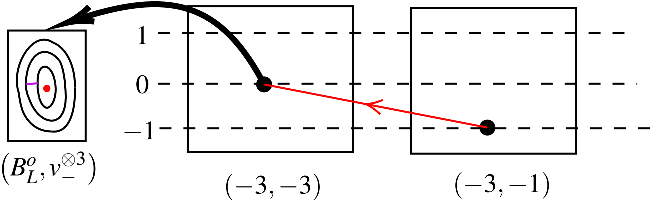
$(j,k)(w_\pm) = (\pm 1, 0)$

$$0 \longrightarrow C^{-1,j,k}_{Sk} \longrightarrow C^{0,j,k}_{Sk} \longrightarrow C^{1,j,k}_{Sk} \longrightarrow$$

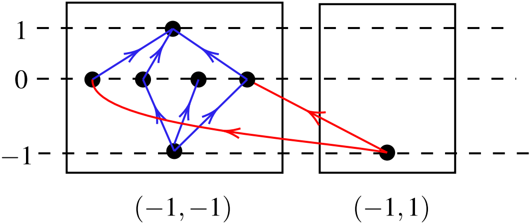
$\mathscr{X}^j_{Kh} \to \mathscr{X}^{j,k_{min}(j)}_{Sk}$

$$\partial_{Kh} = \partial_{Kh,0} + \partial_{Kh,-2} \qquad \partial_{Sk} = \partial_{Kh,0}$$

$j = -3:$

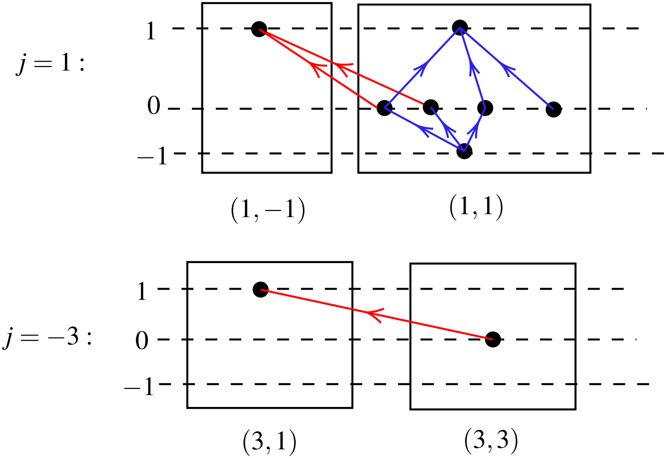


$j = -1:$



$\mathscr{X}_{Kh}^{j} \to \mathscr{X}_{Sk}^{j,k_{min}(j)}$

$$\partial_{Kh} = \partial_{Kh,0} + \partial_{Kh,-2} \qquad \partial_{Sk} = \partial_{Kh,0}$$



Result:

Theorem (with Adithyan Pandikkadan)

For an oriented closed braid diagram $B_L \subset A$ there is a map

$$\Psi^j(B_L) : \mathscr{X}^j_{Kh}(B_L) \to \mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L),$$

Result:

Theorem (with Adithyan Pandikkadan)

For an oriented closed braid diagram $B_L \subset A$ there is a map

$$\Psi^j(B_L) : \mathscr{X}^j_{Kh}(B_L) \to \mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L),$$

such that the induced map on the reduced cohomology

$$\Psi^j(B_L)^* : \widetilde{H}^i\left(\mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L);G\right) \to \widetilde{H}^i\left(\mathscr{X}^j_{Kh}(B_L);G\right)$$

Result:

Theorem (with Adithyan Pandikkadan)

For an oriented closed braid diagram $B_L \subset A$ *there is a map*

$$\Psi^j(B_L) : \mathscr{X}^j_{Kh}(B_L) \to \mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L),$$

such that the induced map on the reduced cohomology

$$\Psi^j(B_L)^* : \widetilde{H}^i\left(\mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L);G\right) \to \widetilde{H}^i\left(\mathscr{X}^j_{Kh}(B_L);G\right)$$

is the same map as $i^* : H^{i,j,f_{min}(B_L,j)}_{Sk}(L;G) \to H^{i,j}_{Kh}(L;G)$ *for the embedding* $L \subset A \times I \subset S^3$*, for* $G = \mathbb{Z}_2$ *or* $\mathbb{Z}$*.*

Result:

Theorem (with Adithyan Pandikkadan)

For an oriented closed braid diagram $B_L \subset A$ there is a map

$$\Psi^j(B_L) : \mathscr{X}^j_{Kh}(B_L) \to \mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L),$$

such that the induced map on the reduced cohomology

$$\Psi^j(B_L)^* : \widetilde{H}^i\left(\mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L);G\right) \to \widetilde{H}^i\left(\mathscr{X}^j_{Kh}(B_L);G\right)$$

is the same map as $i^* : H^{i,j,f_{min}(B_L,j)}_{Sk}(L;G) \to H^{i,j}_{Kh}(L;G)$ *for the embedding* $L \subset A \times I \subset S^3$*, for* $G = \mathbb{Z}_2$ *or* $\mathbb{Z}$.
In particular, when $j = sl(L)$*, we get* $\Psi^{sl(L)} : \mathscr{X}^{sl(L)}_{Kh}(B_L) \to \mathscr{X}^{sl(L),-b(B_L)}_{Sk}(B_L) = \mathbb{S}$

Result:

Theorem (with Adithyan Pandikkadan)

For an oriented closed braid diagram $B_L \subset A$ *there is a map*

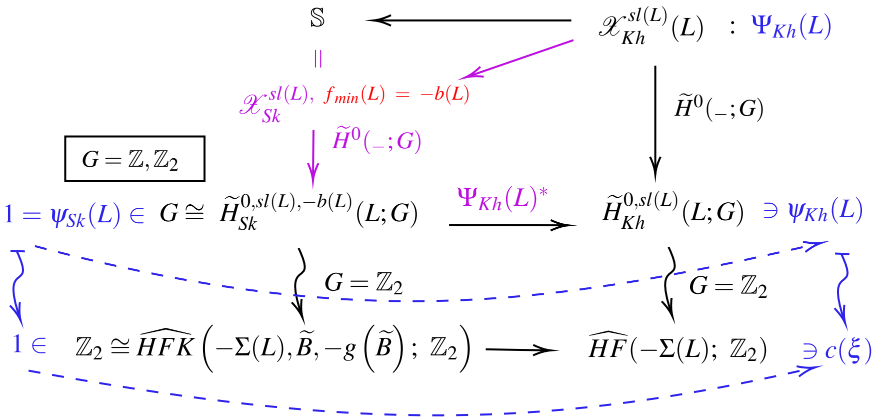
$$\Psi^j(B_L) : \mathscr{X}^j_{Kh}(B_L) \to \mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L),$$

such that the induced map on the reduced cohomology

$$\Psi^j(B_L)^* : \widetilde{H}^i\left(\mathscr{X}^{j,f_{min}(B_L,j)}_{Sk}(B_L);G\right) \to \widetilde{H}^i\left(\mathscr{X}^j_{Kh}(B_L);G\right)$$

is the same map as $i^* : H^{i,j,f_{min}(B_L,j)}_{Sk}(L;G) \to H^{i,j}_{Kh}(L;G)$ *for the embedding* $L \subset A \times I \subset S^3$*, for* $G = \mathbb{Z}_2$ *or* $\mathbb{Z}$*.*
In particular, when $j = sl(L)$*, we get* $\Psi^{sl(L)} : \mathscr{X}^{sl(L)}_{Kh}(B_L) \to \mathscr{X}^{sl(L),-b(B_L)}_{Sk}(B_L) = \mathbb{S}$ *such that the induced map* $\Psi^{sl(L)}(B_L)^* : G \cong H^{0,sl(L),-b(B_L)}_{Sk}(L;G) \to H^{0,sl(L)}_{Kh}(L;G)$ *satisfies* $\Psi^{sl(L)}(B_L)^*(\psi_{Sk}(B_L)) = \psi_{Kh}(B_L)$*, for* $G = \mathbb{Z}_2$ *or* $\mathbb{Z}$*.*

Transverse invariant



Associated to a braid diagram K is a map

$$\Psi(K) : \mathscr{X}_{Kh}^{sl(K)}(K) \to \mathbb{S},$$

$$\Psi(K)^* : \mathbb{Z} = \widetilde{H}^0(\mathbb{S}) \to \widetilde{H}^0\left(\mathscr{X}_{Kh}^{sl(K)}(K)\right) \cong H_{Kh}^{0,sl(K)}(K)$$

sends a generator of $\mathbb{Z}$ to the graded Plamenevskaya invariant $[\psi_{Kh}(K)]$.

Associated to a braid diagram K is a map

$$\Psi(K) : \mathscr{X}_{Kh}^{sl(K)}(K) \to \mathbb{S},$$

$$\Psi(K)^* : \mathbb{Z} = \widetilde{H}^0(\mathbb{S}) \to \widetilde{H}^0\left(\mathscr{X}_{Kh}^{sl(K)}(K)\right) \cong H_{Kh}^{0,sl(K)}(K)$$

sends a generator of $\mathbb{Z}$ to the graded Plamenevskaya invariant $[\psi_{Kh}(K)]$.
If K' is another braid diagram representing the same transverse link type then there is a commutative diagram

$$\begin{array}{ccc} \mathscr{X}_{Kh}^{sl(K)}(K) & \xrightarrow{\Psi(K)} & \mathbb{S} \\ \Phi\big\downarrow \simeq & & \big\downarrow \\ \mathscr{X}_{Kh}^{sl(K)}(K') & \xrightarrow{\Psi(K')} & \mathbb{S}. \end{array}$$

(Here, Φ is the homotopy equivalence induced by a sequence of transverse Markov moves connecting K and K', and the map $\mathbb{S} \to \mathbb{S}$ is a self-homotopy equivalence of the sphere spectrum.)

Thank you!

Urysohn 1-Width and Covers

- Arka Banerjee
Postdoc., Auburn U.

Joint work (in progress) with

- Hannah Alpert (Auburn U.)
- Panos Papasoglu (Oxford U.)

Informally,

A metric space has a 'small' Urysohn k-width $\approx$ The metric space is 'close' to being a k-dimensional simplicial complex.

Informally, A metric space has a 'small' Urysohn k-width $\approx$ The metric space is 'close' to being a k-dimensional simplicial complex.

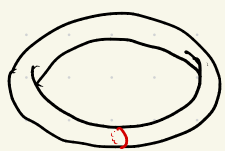
Formally,

Urysohn k-width of a metric space X, denoted by $UW_k(X)$, is $\leq \varepsilon$ if there exists a continuous map $f: X \to Y$, where Y is a k-dimensional simplicial complex and $\mathrm{diam}(f^{-1}(y)) \leq \varepsilon$ for all $y \in Y$.

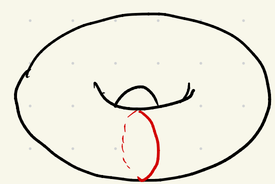
Informally, A metric space has a 'small' Urysohn k-width $\approx$ The metric space is 'close' to being a k-dimensional simplicial complex.

Formally,

Urysohn k-width of a metric space X, denoted by $UW_k(X)$, is $\leq \varepsilon$ if there exists a continuous map $f: X \to Y$, where Y is a k-dimensional simplicial complex and $\operatorname{diam}(f^{-1}(y)) \leq \varepsilon$ for all $y \in Y$.



Smaller UW_1

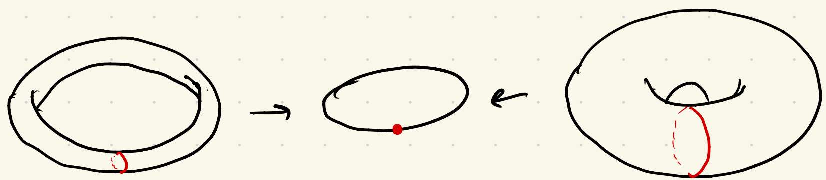


Bigger UW_1

Informally, A metric space has a 'small' Urysohn k-width $\approx$ The metric space is 'close' to being a k-dimensional simplicial complex.

Formally,

Urysohn k-width of a metric space X, denoted by $UW_k(X)$, is $\leq \varepsilon$ if there exists a continuous map $f: X \to Y$, where Y is a k-dimensional simplicial complex and $\operatorname{diam}(f^{-1}(y)) \leq \varepsilon$ for all $y \in Y$.



Smaller UW_1 Bigger UW_1

Example: $UW_0(X) = \mathrm{diam}(X)$ if X is connected.

$UW_0\Bigg($  $\Bigg) = \pi r$

Example: $UW_0(X) = \mathrm{diam}(X)$ if X is connected.

$UW_0\Big($ 

$\Big) = \pi r = \dfrac{\pi}{\sqrt{\frac{1}{r^2}}} = \dfrac{\pi}{\sqrt{\text{Scalar curvature}}}$

Example: $UW_0(X) = \mathrm{diam}(X)$ if X is connected.

$UW_0\Big($ 

$\Big) = \pi r = \dfrac{\pi}{\sqrt{\frac{1}{r^2}}} = \dfrac{\pi}{\sqrt{\text{Scalar curvature}}}$

Moral: higher positive scalar curvature
$\Rightarrow$ smaller UW_0 for 2-sphere.

Example: $UW_0(X) = \text{diam}(X)$ if X is connected.

$UW_0\Big($  $\Big) = \pi r = \dfrac{\pi}{\sqrt{\frac{1}{r^2}}} = \dfrac{\pi}{\sqrt{\text{Scalar curvature}}}$

Moral: higher positive scalar curvature
$\Rightarrow$ smaller UW_0 for 2-sphere.

Conjecture (Gromov): If the scalar curvature of a complete Riemannian manifold M^n is $\geq \sigma^2 > 0$. Then $UW_{n-2}(M^n) \leq \dfrac{c_n}{\sigma}$ for some dimensional constant c_n.

Question: Let M^n be a Riemannian n-manifold and $UW_1(\tilde{M}^n) \leq 1$. Does that imply $UW_1(M^n) \leq c_n$ for some dimensional const c_n?

Question: Let M^n be a Riemannian n-manifold and $UW_1(\tilde{M}^n) \leq 1$. Does that imply $UW_1(M^n) \leq c_n$ for some dimensional const c_n?

Enemy Scenario: For any $R>0$, there is a Riemannian surface M_R and a cover $\hat{M}_R$ of M_R such that $UW_1(\hat{M}_R) \leq 1$ and $UW_1(M_R) \geq R$. (Alpert-Balitskiy-Guth).

Question: Let M^n be a Riemannian n-manifold and $UW_1(\tilde{M}^n) \leq 1$. Does that imply $UW_1(M^n) \leq c_n$ for some dimensional const c_n?

Question: Let M^n be a Riemannian n-manifold and $UW_1(\tilde{M}^n) \leq 1$. Does that imply $UW_1(M^n) \leq c_n$ for some dimensional const c_n?

Theorem (Alpert – B – Papasoglu):

- The answer is yes for $n=2$ and $c_2 = 1$
- If the answer is no for some n, it is no for $n=4$.

Question: Let M^n be a Riemannian n-manifold and $UW_1(\tilde{M}^n) \leq 1$. Does that imply $UW_1(M^n) \leq c_n$ for some dimensional const c_n?

Theorem (Alpert – B – Papasoglu):

- The answer is yes for $n = 2$ and $c_2 = 1$
- If the answer is no for some n, it is no for $n = 4$.

- What about $n = 3$??

Thank You!

Sectorial Decompositions of Symmetric Products of Surfaces and Homological Mirror Symmetry

Xinle (Clair) Dai
Harvard University

2024 Tech Topology Conference, Georgia Institute of Technology

December, 2024

Liouville Sectors: definition and examples

Definition (Ganatra-Pardon-Shende)

Let X be a Liouville manifold with boundaries, we say X is a *Liouville sector* iff there exists a function $I : \partial X \to \mathbb{R}$ such that it satisfies the following conditions:

- I is *linear at infinity*, meaning $ZI = I$ outside a compact set, where Z denotes the Liouville vector field.
- $dI|_{\text{char.fol.}} > 0$, where the characteristic foliation C of ∂X is oriented so that $\omega(N, C) > 0$ for any inward pointing vector N.

Liouville Sectors: definition and examples

Definition (Ganatra-Pardon-Shende)

Let X be a Liouville manifold with boundaries, we say X is a *Liouville sector* iff there exists a function $I : \partial X \to \mathbb{R}$ such that it satisfies the following conditions:

- I is *linear at infinity*, meaning $ZI = I$ outside a compact set, where Z denotes the Liouville vector field.
- $dI|_{\text{char.fol.}} > 0$, where the characteristic foliation C of ∂X is oriented so that $\omega(N, C) > 0$ for any inward pointing vector N.

Example

- Any T^*Q for any compact manifold-with-boundary Q.
- A punctured bordered Riemann surface S is a Liouville sector if and only if every component of ∂S is homeomorphic to $\mathbb{R}$ (i.e., none is homeomorphic to S^1).

Main Theorem

Definition (Quadratic Stein structure)

Let Σ be a Riemann surface and φ a proper plurisubharmonic function on Σ (i.e., $\varphi(z) \to \infty$ as $z \to +\infty$, and $dd^c\varphi > 0$).

Let $\{s_i\}_{i \in I}$ denote the set of saddles of φ, and let $\mathcal{N}(\gamma_i)$ denote the tubular neighborhood of the stable manifold γ_i of the saddle s_i.

We say (Σ, φ) is a Riemann surface with a *quadratic Stein structure* if $\varphi|_{\mathcal{N}(\gamma_i)}$ is quadratic.

Main Theorem

Definition (Quadratic Stein structure)

Let Σ be a Riemann surface and φ a proper plurisubharmonic function on Σ (i.e., $\varphi(z) \to \infty$ as $z \to +\infty$, and $dd^c\varphi > 0$).

Let $\{s_i\}_{i \in I}$ denote the set of saddles of φ, and let $\mathcal{N}(\gamma_i)$ denote the tubular neighborhood of the stable manifold γ_i of the saddle s_i.

We say (Σ, φ) is a Riemann surface with a *quadratic Stein structure* if $\varphi|_{\mathcal{N}(\gamma_i)}$ is quadratic.

Proposition

For any topological surface Σ with disjoint proper embedded arcs $\gamma_i i \in I$, we can build a quadratic Stein structure with one saddle s_i along each γ_i and one minimum m_j on each component of $\Sigma \setminus \bigcup i \in I \gamma_i$.

Main Theorem

Theorem (D.,in progress)

In this setting, the quadratic Stein structure determines a sectorial decomposition for $\mathrm{Sym}^2(\Sigma) = \bigcup_{H_{s_i,m_j}} U_{m_i,m_k}$*, where* U_{m_i,m_k} *(with* $i \leq k$*) are Liouville sectors with corners, and* H_{s_i,m_j} *are smooth hypersurfaces separating these sectors.*

Main Theorem

Theorem (D.,in progress)

In this setting, the quadratic Stein structure determines a sectorial decomposition for $\mathrm{Sym}^2(\Sigma) = \bigcup_{H_{s_i,m_j}} U_{m_i,m_k}$*, where* U_{m_i,m_k} *(with* $i \leq k$*) are Liouville sectors with corners, and* H_{s_i,m_j} *are smooth hypersurfaces separating these sectors.*

Corollary

For a 2-dimensional Liouville sector X*,* $\mathrm{Sym}^2(X)$ *is deformation equivalent to a Liouville sector, which have corners if* ∂X *has more than one component.*

Main Theorem

Theorem (D.,in progress)

In this setting, the quadratic Stein structure determines a sectorial decomposition for $\mathrm{Sym}^2(\Sigma) = \bigcup_{H_{s_i,m_j}} U_{m_i,m_k}$*, where* U_{m_i,m_k} *(with* $i \leq k$*) are Liouville sectors with corners, and* H_{s_i,m_j} *are smooth hypersurfaces separating these sectors.*

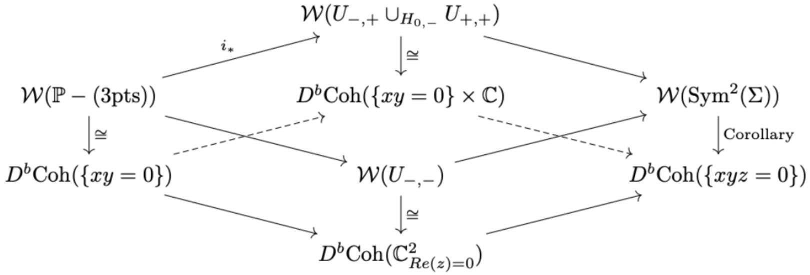
Corollary

For a 2-dimensional Liouville sector X*,* $\mathrm{Sym}^2(X)$ *is deformation equivalent to a Liouville sector, which have corners if* ∂X *has more than one component.*

This follows from the fact that every Weinstein sector of complex dimension 1 is deformationally equivalent to an open subset of a Stein Riemann surface.

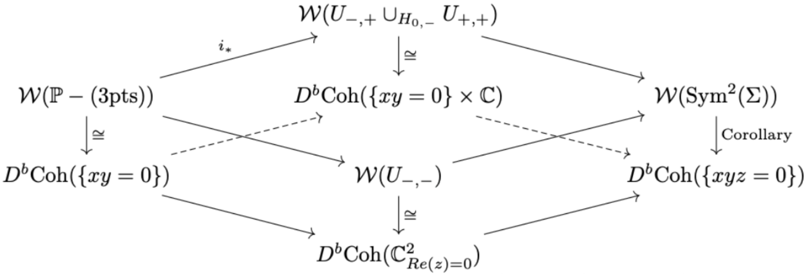
Application to Homological Mirror Symmetry

For $\Sigma := \mathbb{P}^1 - 4\text{points}$, $\text{Sym}^2(\Sigma)$ is a two-dimensional pair of pants.



Application to Homological Mirror Symmetry

For $\Sigma := \mathbb{P}^1 - 4\text{points}$, $\mathrm{Sym}^2(\Sigma)$ is a two-dimensional pair of pants.

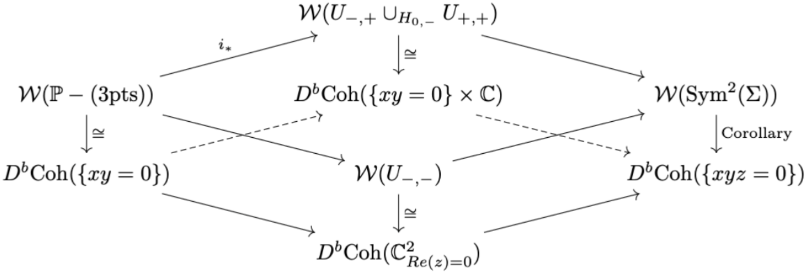


- The top square is commutative as a push-out diagram by Ganatra-Pardon-Shende.
- All the vertical arrows are isomorphisms except the last one by Homological Mirror Symmetry.
- The square on the side is commutative by Gammage-Shende.
- We can get the Mirror Symmetry of

$\mathrm{Sym}^2(\Sigma)$ and $\{xyz = 0\}$ as a corollary of the main theorem.

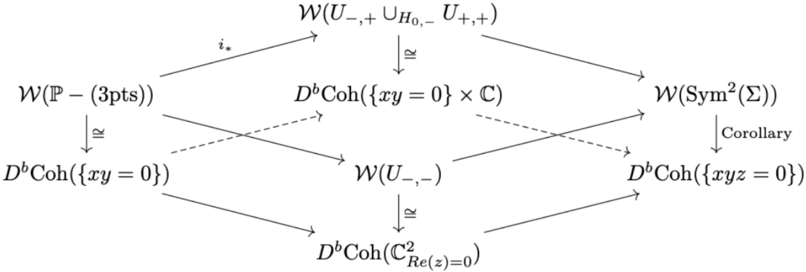
Application to Homological Mirror Symmetry

$\Sigma = \mathbb{P}^1 - 4\text{points}$



Application to Homological Mirror Symmetry

$\Sigma = \mathbb{P}^1 - 4\text{points}$



Remark

The HMS of $\mathrm{Sym}^2(\Sigma)$ and $\{xyz = 0\}$ was proved by Lekili-Polishchuk. However, the sectorial decomposition we get for $\mathrm{Sym}^2(\Sigma)$ matches a geometric decomposition of its mirror, providing a geometric interpretation of the HMS.

Thank you!

A Slice-Bennequin Inequality for $\mathbb{RP}^3$ s-invariant

Ivan So

Michigan State University

MICHIGAN STATE
UNIVERSITY

Slice-Bennequin Inequality and Friends

Rudolph: For a transverse knot in $(S^3, \xi_{\mathrm{std}})$,

$$\boxed{sl(K) \leq 2g_4(K) - 1}$$

Slice-Bennequin Inequality and Friends

Rudolph: For a transverse knot in (S^3, ξ_{std}),

$$\boxed{sl(K) \le 2g_4(K) - 1}$$

- Plamenevskaya (2004): In (S^3, ξ_{std}), $sl(K) \le 2\tau(K) - 1$.

Slice-Bennequin Inequality and Friends

Rudolph: For a transverse knot in (S^3, ξ_{std}),

$$\boxed{sl(K) \leq 2g_4(K) - 1}$$

- Plamenevskaya (2004): In (S^3, ξ_{std}), $sl(K) \leq 2\tau(K) - 1$.
- Plamenevskaya (2006), Shumakovitch (2007): In (S^3, ξ_{std}), $sl(K) \leq s(K) - 1$.

Slice-Bennequin Inequality and Friends

Rudolph: For a transverse knot in (S^3, ξ_{std}),

$$\boxed{sl(K) \leq 2g_4(K) - 1}$$

- Plamenevskaya (2004): In (S^3, ξ_{std}), $sl(K) \leq 2\tau(K) - 1$.
- Plamenevskaya (2006), Shumakovitch (2007): In (S^3, ξ_{std}), $sl(K) \leq s(K) - 1$.
- Hedden (2008): $\forall$ oriented (M, ξ) with $c_{\text{O-Sz}}(\xi) \neq 0$, $sl(K) \leq 2\tau_\xi(K) - 1$.

Slice-Bennequin Inequality and Friends

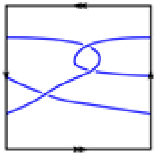
Rudolph: For a transverse knot in (S^3, ξ_{std}),

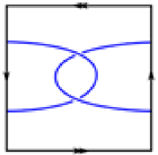
$$\boxed{sl(K) \leq 2g_4(K) - 1}$$

- Plamenevskaya (2004): In (S^3, ξ_{std}), $sl(K) \leq 2\tau(K) - 1$.
- Plamenevskaya (2006), Shumakovitch (2007): In (S^3, ξ_{std}), $sl(K) \leq s(K) - 1$.
- Hedden (2008): $\forall$ oriented (M, ξ) with $c_{\text{O-Sz}}(\xi) \neq 0$, $sl(K) \leq 2\tau_\xi(K) - 1$.
- And many more...

$s_{\mathbb{RP}^3}$ by Manolescu-Willis

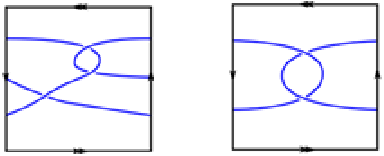
Links in $\mathbb{RP}^3$ can be presented as tangles over $\mathbb{RP}^2$, e.g.





$s_{\mathbb{RP}^3}$ by Manolescu-Willis

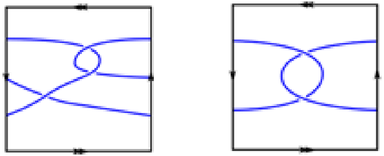
Links in $\mathbb{RP}^3$ can be presented as tangles over $\mathbb{RP}^2$, e.g.



- Asaeda-Przytycki-Sikora (2004): $\exists$ Khovanov homology $Kh_{\mathbb{RP}^3}$ from $\mathbb{RP}^2$ diagrams.

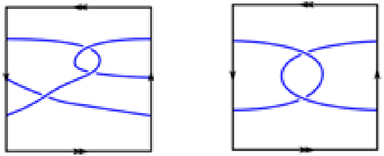
$s_{\mathbb{RP}^3}$ by Manolescu-Willis

Links in $\mathbb{RP}^3$ can be presented as tangles over $\mathbb{RP}^2$, e.g.



- Asaeda-Przytycki-Sikora (2004): $\exists$ Khovanov homology $Kh_{\mathbb{RP}^3}$ from $\mathbb{RP}^2$ diagrams.
- Manolescu-Willis (2023): $\exists$ $Kh'_{\mathbb{RP}^3}$ s.t. $\operatorname{rk} Kh'_{\mathbb{RP}^3}(L) = 2^{|L|}$.

Links in $\mathbb{RP}^3$ can be presented as tangles over $\mathbb{RP}^2$, e.g.



- Asaeda-Przytycki-Sikora (2004): $\exists$ Khovanov homology $Kh_{\mathbb{RP}^3}$ from $\mathbb{RP}^2$ diagrams.
- Manolescu-Willis (2023): $\exists$ $Kh'_{\mathbb{RP}^3}$ s.t. $\operatorname{rk} Kh'_{\mathbb{RP}^3}(L) = 2^{|L|}$.

  $\Rightarrow$ Max and min filtration levels of a knot $K \subset \mathbb{RP}^3$ defines $s_{\mathbb{RP}^3}(K)$.

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Proof idea: Imitate that of Shumakovitch

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Proof idea: Imitate that of Shumakovitch

- $s_{\mathbb{RP}^3}(K) - 1 \geq w(K) - r(K)$, $r(K) = \#\{\text{Seifert circles}\}$.

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Proof idea: Imitate that of Shumakovitch

- $s_{\mathbb{RP}^3}(K) - 1 \geq w(K) - r(K)$, $r(K) = \#\{\text{Seifert circles}\}$.
- $\mathbb{RP}^2$ diagram $\to$ Kirby diagram.

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Proof idea: Imitate that of Shumakovitch

- $s_{\mathbb{RP}^3}(K) - 1 \geq w(K) - r(K)$, $r(K) = \#\{\text{Seifert circles}\}$.
- $\mathbb{RP}^2$ diagram $\rightarrow$ Kirby diagram.
- Durst-Kegel's (2016): computation of $sl_{\mathbb{RP}^3}$.

Inequality for $\mathbb{RP}^3$

Theorem (S.)

Suppose $K \subset \mathbb{RP}^3$ is a transverse knot in $(\mathbb{RP}^3, \xi_{\text{std}})$ with $[K] = 0$, then

$$sl_{\mathbb{RP}^3}(K) \leq s_{\mathbb{RP}^3}(K) - 1.$$

Proof idea: Imitate that of Shumakovitch

- $s_{\mathbb{RP}^3}(K) - 1 \geq w(K) - r(K)$, $r(K) = \#\{\text{Seifert circles}\}$.
- $\mathbb{RP}^2$ diagram $\rightarrow$ Kirby diagram.
- Durst-Kegel's (2016): computation of $sl_{\mathbb{RP}^3}$.
- Combine the computations.

Some Corollaries

Corollary 1 (Milnor conjecture for transverse fibered knot)

Let K be a null-homologous, fibered transverse knot in $(\mathbb{RP}^3, \xi_{std})$, min. genus Σ with $\partial\Sigma = K$. If ξ_{std} is supported by (K, Σ), then

$$g_{DTS^2}(K) = g_3(K) = g_{\mathbb{RP}^3 \times I}(K).$$

Some Corollaries

Corollary 2 (Symplectic surface detection)

Suppose $K \subset \mathbb{RP}^3$ is a transverse, fibered, $[K] = 0$ that support $(\mathbb{RP}^3, \xi_{\text{std}})$ and $\Sigma' \subset DTS^2$ a smoothly embedded surface with $\partial\Sigma' = K$ but $s_{\mathbb{RP}^3}(K) - 1 \neq -\chi(\Sigma')$, then Σ' is not symplectic.

Some Corollaries

Corollary 3? (Combinatorial proof of Lisca-Matić inequality)

Let $\widetilde{\Sigma} \hookrightarrow DTS^2$ be a smoothly embedded oriented surface transversal to $\mathbb{RP}^3 = \partial(DTS^2)$ with null-homologous Legendrian boundary $K = \partial\Sigma \subset \mathbb{RP}^3$. Then,

$$|\operatorname{rot}(K, \widetilde{\Sigma})| + \operatorname{tb}(K, \widetilde{\Sigma}) \leq s_{\mathbb{RP}^3}(K) - 1 \leq 2g(\widetilde{\Sigma}) - 1.$$

Thank You!
Any Questions?

MICHIGAN STATE UNIVERSITY

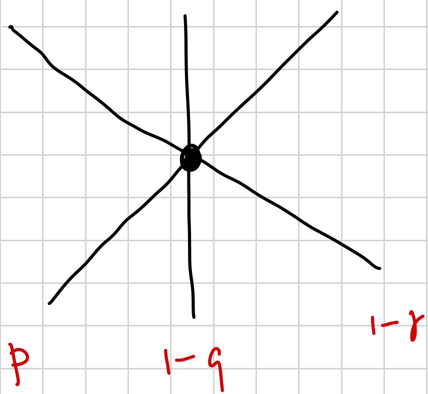
Stein Fillings of Triangle Singularities

Ethan (Yang) Zhou

UMass Amherst

For a triple of intergers (p, q, r) satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

⇝ triangle singularity w/ minimal resolution:



- They are singularities on 4-manifolds.
- Most of them are not complete intersection singularities

Goal: to study Stein fillings of the link L of $x \in X$.

(link: 3-dim'l contact mfld, Seifert fibered space)

Sources of examples of Stein fillings from algebraic geometry:

① Milnor fibers ← $c_1 = 0$

② minimal resolution

Thm (Etnyre, Golla) if W is a Stein filling of triangle singularity given by $x^2 + y^3 + z^7 = 0$

then either

$$H_*(W) \cong H_*(\text{Milnor fiber})$$

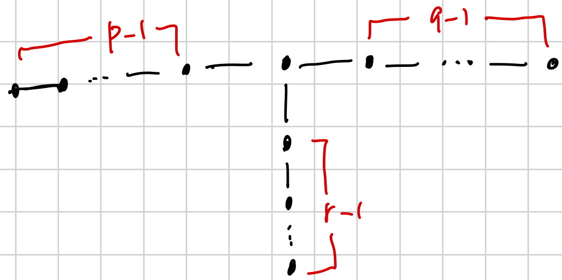
or

$$H_*(W) \cong H_*(\text{minimal resolution})$$

Milnor fibers

- For $p+q+r-2 < 20$, the triangle singularity can always be smoothed to a Milnor fiber M
- The Milnor fiber M can always be compactified to a K3 surface by adding $p+q+r-2$ $\mathbb{CP}^1$'s at ∞

T_{pqr}:



Expectation & Goals

If W is a Stein filling of L s.t. $c_1(W)=0$,

then $W \simeq$ Milnor fibers.

($\simeq$: under some equivalence)

Strategy

Cap W off w/ the plumbing of T_{par}

$\overline{W} := W \cup_L P(T_{par})$ $\Longrightarrow$ $\overline{W}$ is symplectic Calabi-Yau

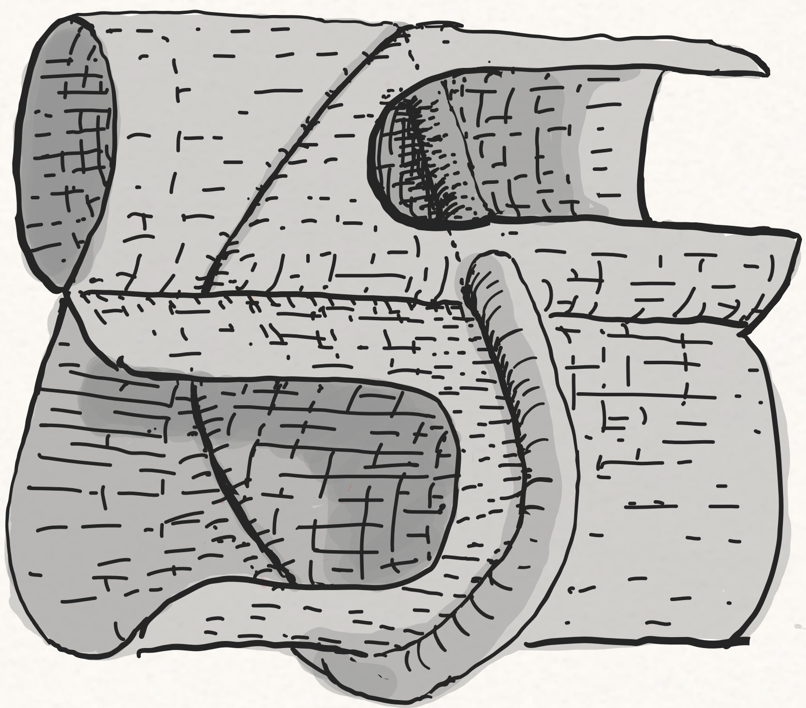
↑ CY cap

$\pi_1(\overline{W}) = 0$ $\Downarrow$

$\overline{W} \simeq_{homeo} K3$

Thanks!

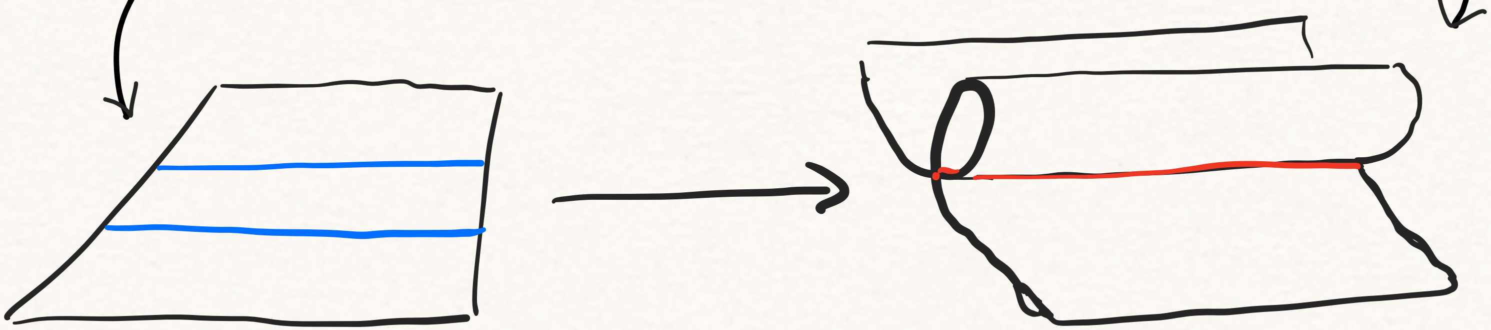
Constructing Immersions w/ Specified Self-Intersection



Def: Given $f : M \overset{\text{imm}}{\looparrowright} N$

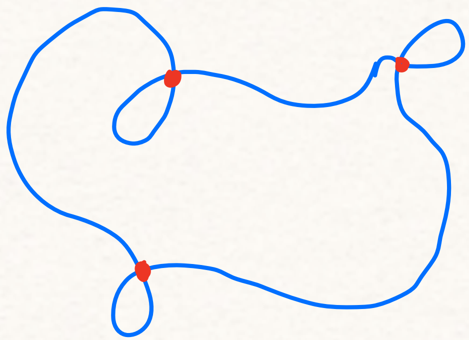
- $D(f) := \{ q \in N \mid |f^{-1}(q)| \geq 2 \}$ "double pt. set"

- $C(f) := f^{-1}(D(f))$ "crossing set"



Q1: Given $\mathrm{Imm}(M^n, \mathbb{R}^N) \neq \emptyset$ & an immersed submanifold $Y^{2n-N} \subset \mathbb{R}^N$, is there a stable $f \in \mathrm{Imm}(M, \mathbb{R}^N)$ w/

$$D(f) = Y$$

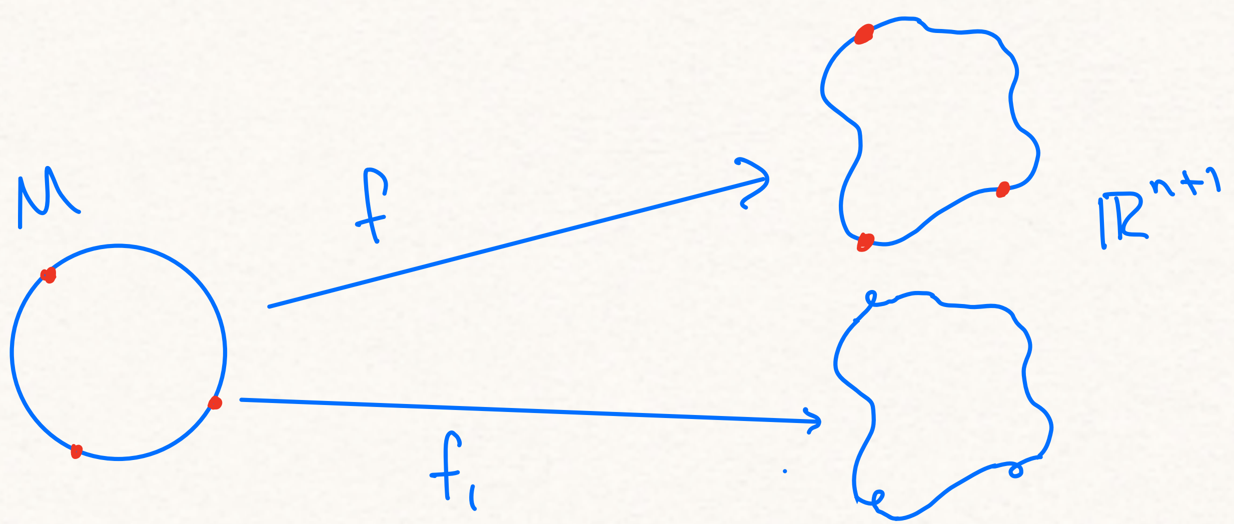


A1
- if $N = n+1$, "nearly"
- $n+1 < N \leq 2n$ is work in progress

Thm: Given $f: M^n \hookrightarrow \mathbb{R}^{n+1}$ & $j: Y^{n-1} \hookrightarrow f(M)$,

$\exists$ a family of smooth maps $f_t: M \longrightarrow \mathbb{R}^{n+1}$

- $f_0 = f$
- f_1 stable immersion w/ $D(f_1) = j(Y)$ outside $\mathcal{O}_p(D(j))$
- $f_t = f$ outside $\mathcal{O}_p(f^{-1}(j(Y)))$



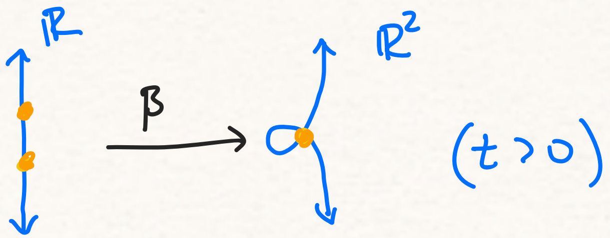
Pockets:

$t \in \mathbb{R}, \quad 0 \leq q \leq n-1$

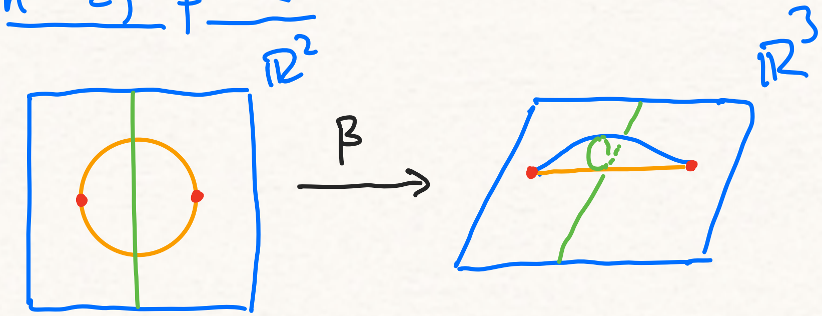
$$\beta_t^{n,q} : \mathbb{R} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{2n-q}$$

$$(x, y) \longmapsto \left(x^2, x^3 - \left(t - \sum_1^q y_i^2\right)x, xy_{q+1}, \ldots, xy_{n-1}, y\right)$$

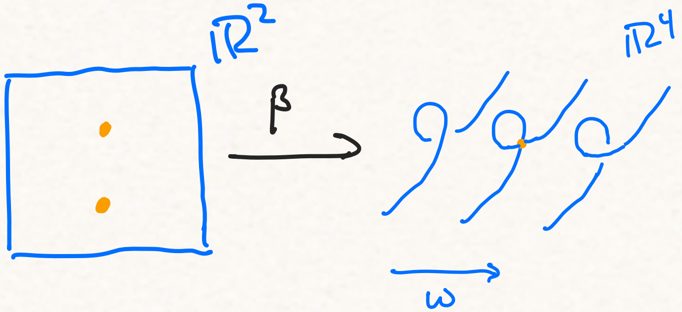
n = 1, q = 0:



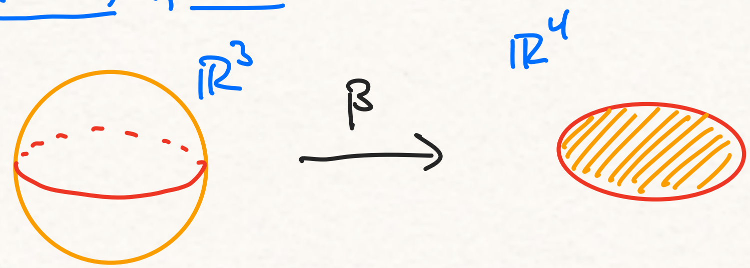
n = 2, q = 1:



n = 2, q = 0:



n = 3, q = 2:



$$\rightsquigarrow \text{``} \; \mathrm{Imm}(Y^q, M^n) \longrightarrow \mathrm{Path}(C^\infty(Y,M)) \text{''}, \quad f_t$$

$$\mathrm{Imm}(Y^q, M^n) \xrightarrow{\Phi} \mathrm{Imm}(M, \mathbb{R}^{2n-q}), \qquad \mathrm{Path}(C^\infty(Y,M)) \longrightarrow \mathrm{Imm}(M, \mathbb{R}^{2n-q}), \qquad f_t \longmapsto f_1$$

Q2: Does any $f_1 \in \mathrm{Imm}(S^n, \mathbb{R}^{n+1})$ obtained this way lift to a non-trivial knot in $\mathbb{R}^{n+2}$?

Q3: Can we obtain representatives for all regular homotopy classes this way?

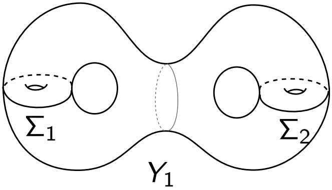
Thank You!

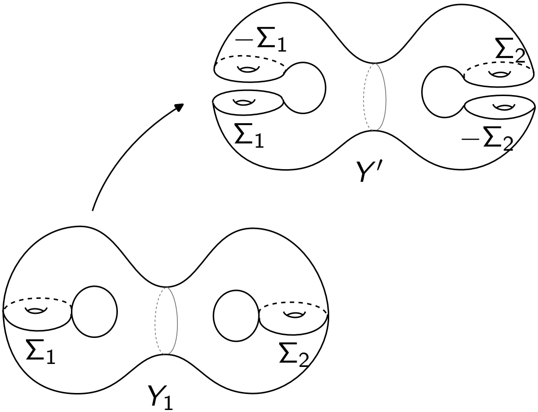
An Excision Theorem in Heegaard Floer Theory

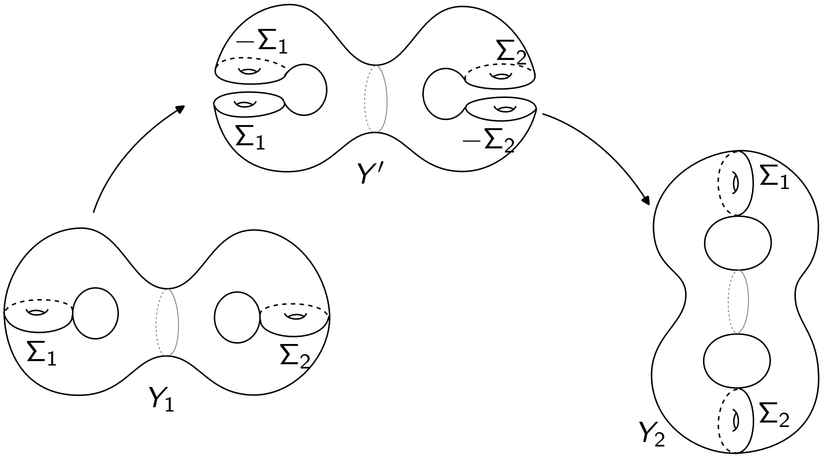
Neda Bagherifard

University of Oregon

December 7, 2024


Σ_1
Σ_2
Y_1


$-\Sigma_1$
Σ_2
Σ_1
$-\Sigma_2$
Y'
Σ_1
Σ_2
Y_1


$-\Sigma_1$
Σ_2
Σ_1
$-\Sigma_2$
Y'
Σ_1
Σ_2
Y_1
Σ_1
Σ_2
Y_2

Let Y be a closed oriented 3-manifold.

Let Y be a closed oriented 3-manifold.

- Let Λ be the Universal Novikov ring and $[\omega] \in H^2(Y;\mathbb{R})$.

Let Y be a closed oriented 3-manifold.

- Let Λ be the Universal Novikov ring and $[\omega] \in H^2(Y;\mathbb{R})$.
- Λ_ω: use ω to equip Λ with an $\mathbb{F}[H^1(Y)]$-module structure where $\mathbb{F} = \mathbb{Z}/2$.

Let Y be a closed oriented 3-manifold.

- Let Λ be the Universal Novikov ring and $[\omega] \in H^2(Y; \mathbb{R})$.
- Λ_ω: use ω to equip Λ with an $\mathbb{F}[H^1(Y)]$-module structure where $\mathbb{F} = \mathbb{Z}/2$.
- $Y \to \underline{HF}^+(Y; \Lambda_\omega)$.

Let Y be a closed oriented 3-manifold.

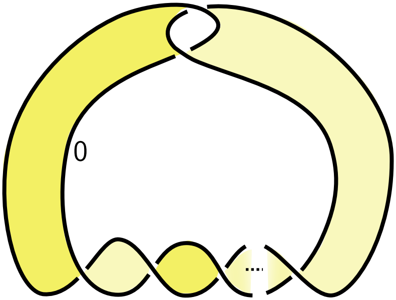
- Let Λ be the Universal Novikov ring and $[\omega] \in H^2(Y;\mathbb{R})$.
- Λ_ω: use ω to equip Λ with an $\mathbb{F}[H^1(Y)]$-module structure where $\mathbb{F} = \mathbb{Z}/2$.
- $Y \to \underline{HF}^+(Y;\Lambda_\omega)$.
- $\underline{HF}^+(Y;\Lambda_\omega)$ is a $\Lambda[U]$-module.

Theorem (B.)

Let Y_2 be obtained from Y_1 by excision along $\Sigma_1 \cup \Sigma_2$, where $g(\Sigma_i) = 1$. For a generic choice of $[\omega_i] \in H^2(Y_i; \mathbb{R})$, we have

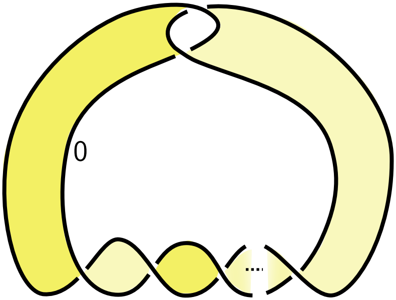
$$\underline{HF}^+(Y_1; \Lambda_{\omega_1}) \cong \underline{HF}^+(Y_2; \Lambda_{\omega_2})$$

Y_n :



$2n-1$-crossings

Y_n :

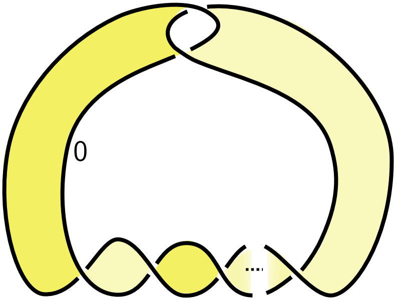


$2n-1$-crossings

Proposition (B.)

$\underline{HF}^+(Y_n;\Lambda_\omega)\cong\Lambda^{|n|}$.

Y_n :



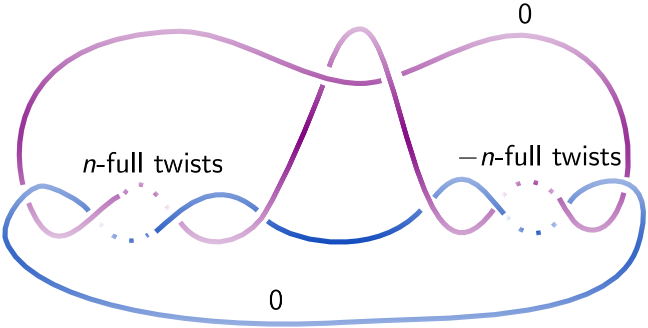
$2n-1$-crossings

Proposition (B.)

$\underline{HF}^+(Y_n;\Lambda_\omega) \cong \Lambda^{|n|}$.

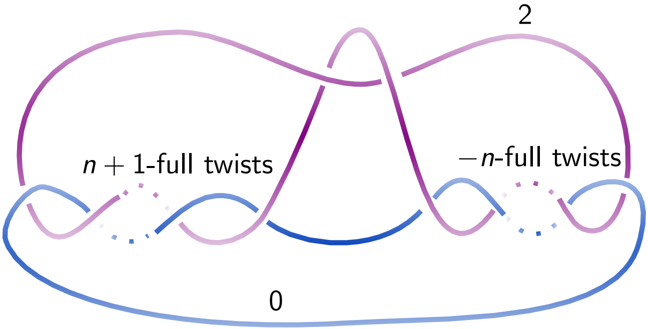
Corollary (B.)

If $|n| \neq |m|$, Y_n is not related to Y_m by excision along a genus one surface.



Corollary (B.)

$\underline{HF}^+(Y;\Lambda_\omega) \cong \Lambda^{n^2}$.



Corollary (B.)

$$0 \to \Lambda^{|n(n+1)|} \to \underline{HF}^{+}(Y;\Lambda_{\omega}) \to \Lambda[U^{-1}] \to 0.$$