Math 4318 - Fall 2019
Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 3, 5, 6, 11, 12, 16, 19, 20.* **Due: In class on September 3**

1. Compute the derivative of $f(x) = x^n$ in two ways. First use the binomial theorem and the definition of derivative. Then use the product rule and induction.
2. Compute the first 3 derivatives of $f(x) = |x|^3$.
3. Let
$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational.} \end{cases}$$
Show that f is differentiable at 0 and compute $f'(0)$.
4. For real numbers a and $b > 0$ define the function $f : [-1, 1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} x^a \sin |x|^{-b} & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Prove that
   (a) f is continuous if and only if $a > 0$.
   (b) $f'(0)$ exist if and only if $a > 1$.
   (c) f' is bounded if and only if $a \geq 1 + b$.
   (d) f' is continuous if and only if $a > 1 + b$.
5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and that $f(a) = 0$. If $g(x) = |f(x)|$ show that g is differentiable at a if and only if $f'(a) = 0$.
6. Recall that a function f is **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. If f is an even function then show that f' is an odd function.
7. Show that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.
8. Use the Mean Value Theorem to show that for all $x \in \mathbb{R}$ we have
$$e^x \geq 1 + x$$
with equality if and only if $x = 0$. Interpret this in terms of the graphs of the functions.
9. Let $h : \mathbb{R} \to \mathbb{R}$ be a function with bounded derivative (that is there is some constant M such that $|h'(x)| \leq M$ for all x). For a fixed $\epsilon > 0$ consider the function $f(x) = x + \epsilon h(x)$. Prove that f is injective if ϵ is small enough.
10. Suppose that f is defined and differentiable on some interval containing c. Show that
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}.$$
Give and example that shows that the limit on the right hand side might exist even if the derivative of f at c does not exits.
11. Suppose that f is defined and twice differentiable in some interval containing c. Show that
$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}.$$
Given and example that shows the limit on the right hand side might exist even if the second derivative of f at c does not exits. **Hint:** L'Hopital and the previous problem (which you should do but don't have to write up).

12. Suppose that $f : (a, b) \to \mathbb{R}$ and that f'' exists everywhere. Then show that $f''(x) \geq 0$ for all $x \in (a, b)$ if and only if f is **convex** by which we mean

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in (a, b)$ and $t \in [0, 1]$. **Hint**: Problem 11 for the forward implication and for the backward implication consider Taylor's theorem applied to $f(x)$ and $f(y)$ for the first our Taylor polynomial.

13. We call a function $f : (a, b) \to \mathbb{R}$ **Hölder continuous of order** α (or α-**Hölder continuous**) for $\alpha > 0$ if there is some constant M such that for all $x, y \in (a, b)$ we have

$$|f(x) - f(y)| \leq M|x - y|^\alpha.$$

Notice that function that is Hölder continuous of order 1 is Lipschitz.

(a) Show that a function that is α-Hölder continuous on (a, b) is uniformly continuous on (a, b).

(b) Show that a function that is α-Hölder continuous on $(a.b)$ for $\alpha > 1$ is constant. **Hint:** First show that such a function is differentiable.

14. Let f be a twice differentiable function on $(0, \infty)$. If f'' is bounded and $f(x) \to 0$ as $x \to \infty$ then show that $f'(x) \to 0$ as $x \to \infty$.

15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $f'(x) \neq 1$ for all x then show that f has at most one fixed point. (A **fixed point** of a function f is a point x such that $f(x) = x$.)

16. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If there is a constant $C \in (0, 1)$ such that $|f'(x)| < C$ for all x then show that f does have a fixed point. Show that it is not sufficient to assume that $|f'(x)| < 1$ to guarantee a fixed point by considering the function

$$f(x) = x + \frac{1}{1 + e^x}.$$

Hint: When C exists let x_1 be any point and set $x_n = f(x_{n-1})$. What can you say about $\lim x_n$? Does it exits?

17. Suppose that $f : [-1, 1] \to \mathbb{R}$ is three times differentiable. If $f(-1) = 0, f(0) = 0, f(1) = 1$ and $f'(0) = 0$, then show that there is some point $x \in (-1, 1)$ at which $f^{(3)}(x) \geq 3$. **Hint:** Look at the second order Taylor polynomial about 0 and consider the Remainder terms when you evaluate at ± 1.

18. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f(a) = 0$ and $|f'(x)| \leq C|f(x)|$ for some constant C then show that $f(x) = 0$ for all $x \in [a, b]$.

19. Let f be a twice differentiable function on the interval (a, ∞). Show that

$$\left(\sup_{x \in (a,\infty)} \{|f'(x)|\} \right)^2 \leq 4 \left(\sup_{x \in (a,\infty)} \{|f(x)|\} \right) \left(\sup_{x \in (a,\infty)} \{|f''(x)|\} \right).$$

Notice that this says we can bound the first derivative of f in terms of f and the second derivative. **Hint:** Consider Taylor polynomial expanded about x evaluated at $x + h$ to get a quadratic equation in h.

20. Suppose f is n times continuously differentiable on some interval (a, b) that contains c. If $f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$, then show that

(a) If n is even and $f^{(n)}(c) > 0$, then f has a relative minimum at c.

(b) If n is even and $f^{(n)}(c) < 0$, then f has a relative maximum at c.

(c) If n is odd, then f has neither a relative minimum or a relative maximum at c.

This, of course, is a large generalizations of the "second derivative test" you learned in calculus for determining if a critical point is a max or a min.