Math 4318 - Fall 2019
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 4, 6, 11, 12, 13, 14, 15.* **Due: In class on October 17**

1. If $\{f_n\}$ and $\{g_n\}$ are two sequences of functions that converge uniformly on a set $D \subset \mathbb{R}$ then prove that $\{f_n + g_n\}$ converges uniformly on D too. If the sequences are bounded (that is each f_n and g_n is bounded) then $\{f_n g_n\}$ also converges uniformly on D. Show that it is necessary to assume that f_n and g_n are bounded in order to conclude that $\{f_n g_n\}$ converges uniformly.
2. Let
$$f_n(x) = \frac{x}{1 + nx^2}$$
be a sequence of functions. Show that $\{f_n\}$ converge uniformly to some function f and that
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
for all $x \neq 0$, but that the equality is not true for $x = 0$.
3. Let $\mathcal{P}$ be the set of polynomials thought of as functions on $[0, 1]$. So $\mathcal{P} \subset C^0([0, 1])$. The uniform norm $\|\cdot\|_\infty$ is a norm on $\mathcal{P}$. Is $(\mathcal{P}, \|\cdot\|_\infty)$ a Banach space?
Hint: Think about the Weierstrass theorem.
4. Let $f \in C^0([0, 1])$. Show that if
$$\int_0^1 x^n f(x)\, dx = 0$$
for all non-negative integers n then $f(x) = 0$.
Hint: Think about the Weierstrass theorem and try to show that $\int_0^1 f^2(x)\, dx = 0$.
5. Suppose that $\{f_n\}$ is an equicontinuous sequence of functions on a compact set $D \subset \mathbb{R}$ and that the sequence converges point-wise to f. Show that $\{f_n\}$ converges uniformly to f on D.
6. Given two functions $f, g \in \mathcal{R}([a, b])$ define the L^2-inner product to be
$$\langle f, g\rangle = \int_a^b f(x)g(x)\, dx.$$
Why is this not an inner product on $\mathcal{R}([a, b])$? Show that this does give an inner product on the set of continuous functions $C^0([a, b])$. (Notice that this also gives a norm on $C^0([a, b])$ by $\|f\|_2 = \sqrt{\langle f, f\rangle}$.)
7. Using the notation from the previous problem show that given any $f \in \mathcal{R}([a, b])$ there is a sequence of functions $\{f_n\}$ in $C^0([a, b])$ such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. Is $C^0([a, b])$ with the L^2-norm complete?
Hint: Of course you will have proven the result if you can show that for every $\epsilon > 0$ there is a continuous function g such that $\|f - g\|_2 < \epsilon$. To show this use the Riemann integral (instead of the Darboux integral) and for a well chosen partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ let
$$g(x) = \frac{x_i - x}{\Delta x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{\Delta x_i} f(x_i).$$

8. Given a function $f \in \mathcal{R}([a,b])$ show there is a sequence of polynomials p_n such that $\lim_{n\to\infty} \|f - p_n\|_2 = 0$.
Hint: Use the previous problem and Weierstrass.

Let V be a vector space and $\|v\|_a$ and $\|v\|_b$ be two norms on V. We say the norms are **equivalent** if there are positive constants C and C' such that

$$C\|v\|_a \leq \|v\|_b \leq C'\|v\|_a.$$

9. Consider the vector space $V = \mathbb{R}^2$. Let $\|(x,y)\| = \sqrt{x^2 + y^2}$ and let $\|(x,y)\|_1 = |x| + |y|$. These are two norms on V (you do not have to show they are norms). Show that these two norms are equivalent.

Interesting fact: On a finite dimensional vector space any two norms are equivalent.

10. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on V then a sequence $\{v_n\}$ converges to v in the norm $\|\cdot\|_a$ if and only if it converges to v in the norm $\|\cdot\|_b$.
11. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on V then a sequence $\{v_n\}$ is Cauchy in the norm $\|\cdot\|_a$ if and only if it is Cauchy in the norm $\|\cdot\|_b$. (Together with the previous problem we see that $(V, \|\cdot\|_a)$ is a Banach space if and only if $(V, \|\cdot\|_b)$ is a Banach space when the norms are equivalent.)
12. Let $\|\cdot\|_\infty$ be the sup-norm on $C^1([a,b])$. Is $(C^1([a,b]), \|\cdot\|_\infty)$ a Banach space? In class we saw that $(C^1([a,b]), \|\cdot\|_{C^1})$ is a Banach space (recall $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$). Are the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{C^1}$ equivalent on $C^1([a,b])$?
13. Consider $\mathcal{R}([a,b])$ with the sup norm $\|\cdot\|_\infty$ and $C^0([a,b])$ with the sup norm. Define a function
$$I : \mathcal{R}([a,b]) \to C^0([a,b])$$
by $I(f)(x) = \int_a^x f(s)\,ds$. Show that I is a uniformly continuous function (that is given any $\epsilon > 0$ there is a $\delta > 0$ such that for all $\|f - g\|_\infty < \delta$ we have $\|I(f) - I(g)\|_\infty < \epsilon$).
14. With the notation from the last problem assume that $\{f_n\}$ is a bonded sequence in $\mathcal{R}([a,b])$ with the sup norm. Show that $\{I(f_n)\}$ has a convergent subsequence (that is converges uniformly to some function f on $[a,b]$).
15. Let $\{f_n\}$ be a sequence of functions in $C^1([a,b])$ that are bounded in the norm $\|\cdot\|_{C^1}$. Show that there is a subsequence that converges to a function f in the $\|\cdot\|_\infty$ norm.
16. Suppose that $f \in C^1([a,b])$ and that $f(a) = 0$. Prove that
$$\|f\|_\infty \leq \sqrt{b-a}\,\|f'\|_2$$
where $\|\cdot\|_2$ is the L^2-norm defined in Problem 6.
Hint: Use the fundamental theorem of calculus and the Cauchy-Schwartz inequality.

This problem gives the idea behind the famous Sobolev embedding theorems. For example it does not take too much more work to show that if we set $\|f\|_{1,2} = \|f\|_2 + \|f'\|_2$ then this is a norm on $C^1([a,b])$ and there is some constant C such that $\|f\|_\infty \leq C\|f\|_{1,2}$. In other words, the averages (that is integrals) of a function and its derivative control the pointwise values of the function. Or more precisely the identity map from $C^1([a,b])$ with the $\|\cdot\|_{1,2}$ norm to $C^1([a,b])$ with the sup norm is continuous.