Math 4318 - Fall 2019
Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 3, 4, 11, 12, 13, 16, 17.* **Due: In class on October 31**

1. Consider the differential equation $f'(t) = 1 + \frac{1}{2}f(t)$ for $0 \le t \le 1$ and $f(0) = 1$. Finding a solution to this is the same as finding a fixed point for

$$\Phi(f) = 1 + \int_0^t (1 + \frac{1}{2}f(s))\, ds.$$

Let $f_0(t) = 1$ and defined $f_n(t) = \Phi(f_{n-1})(t) = \Phi^n(f_0)(t)$. Explicitly write out the f_n's. Show that this sequence converges uniformly to some function f. Show that f is a fixed point of Φ. Your expression for f_n should give a series expression for f that you should recognize as a simple function. Show that this function satisfies the original differential equation. (Notice that in this problem you are explicitly working out the solution to a differential equation using the ideas we developed to show existence of solutions using the contraction mapping theorem.)

2. For $b > 0$ and any a define

$$T(f)(x) = a + \int_0^x f(y)e^{-xy}\, dy.$$

Show that $T : C^0([0, b]) \to C^0([0, b])$ is a contraction (with the metric coming from the sup norm). Hence show that there is a unique solution to

$$f(x) = a + \int_0^x f(y)e^{-xy}\, dy$$

in $C^0([0, b])$.

3. Let $\phi : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a function that is C^n (that is if you fix either of the variables the function is C^n differentiable with respect to the other variable). Let

$$\Phi(f)(t) = c + \int_a^t \phi(s, f(s))\, ds$$

(recall this is the "integral operator" used in the proof that ODEs have solutions). Show that if f is a fixed point of Φ then f is has $n + 1$ continuous derivatives on $[a, b]$. (You may use the fact that $\frac{d}{dt}\phi(t, f(t)) = \phi_t(t, f(t)) + \phi_x(t, f(t))f'(t)$.)
Notice that this problem says that if f is a solution to the differential equation

$$y' = \phi(t, y) \quad y(t_0) = x_0$$

with ϕ a C^n function then $f \in C^{n+1}$ (where it is defined).

4. Continuing the previous problem consider the function

$$\phi(t, x) = \begin{cases} t & t \le 1 \\ 2 - t & t \ge 1. \end{cases}$$

Solve the differential equation $y' = \phi(t, y)$ with $y(0) = 1$ and show that it is C^1 but not twice differentiable.

5. Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ both converge uniformly on (a, b). Let $\{s_n\}$ be a sequence of distinct points in $[a, b]$ that converge to $s \in (a, b)$. If $f(s_n) = g(s_n)$ for all n and $f(s) = g(s)$ then show that $f(x) = g(x)$ for all $x \in [a, b]$. (In particular, show that $a_n = b_n$ for all n). Hint: Consider $h(x) = f(x) - g(x)$ and try to prove that h is zero. Look at the set Z of points where $h(x) = 0$ and S the limit points of Z. Show that Z is both relatively open and relatively closed in (a, b).
6. Suppose that f and g are analytic functions on (a, b) and there is some point $c \in (a, b)$ such that $f^{(k)}(c) = g^{(k)}(c)$ for all k. Prove that $f(x) = g(x)$ for all $x \in (a, b)$. Hint: Looking at the Taylor series around c does not immediately solve the problem.
7. Given a sequence of numbers $\{a_n\}$ show there is a function $f : \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable and has $f^{(k)}(0) = a_k$ for all k. (This is called Borel's Lemma.) Hint: Consider a series of the form $\sum \sigma_k(x) a_k \frac{1}{k!} x^k$. If you choose the functions σ_k so that they are, non-negative, 1 near 0 and zero outside the appropriate neighborhood of 0, then you can prove the series gives a smooth function.
8. Show that there are smooth functions whose Taylor series has radius of convergence 0. Hint: Use the last problem.
9. Let f and g be differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, Show that for any constants $a, b \in \mathbb{R}$ we have
$$D(af + bg) = aDf + bDg.$$
10. Let $f(x, y) = 0$ if $(x, y) = (0, 0)$ and if $(x, y) \neq (0, 0)$ then set
$$f(x, y) = \frac{xy}{x^2 + y^2}.$$
Compute the partial derivatives of f at $(0, 0)$. Compute the directional derivatives at $(0, 0)$ (when they exists). Determine if f is continuous at $(0, 0)$ and if it is differentiable at $(0, 0)$.
11. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that $\|f(x)\| \leq M\|x\|^2$ for all $x \in \mathbb{R}^n$. Prove that f is differentiable at $x = 0$ and $Df(0) = 0$.
12. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that L is differentiable at all $x \in \mathbb{R}^n$ and compute $DL(x)$.
13. Compute the derivative (that is Jacobian matrix) of
    (a) $f(x, y) = \sin(x^2 + y^3)$
    (b) $g(x, y, z) = (z \sin x, x \sin y)$
    (c) $h(x, y, z) = (x^2, xy)$
14. Compute ∇f and Df for $f(x, y, z) = xyz + x^2 - y^3 + z^4$.
15. Let $f(x, y) = (e^{2x+y}, 2y - \cos x, x^2 + y + 2)$ and $g(x, y, z) = (3x + 2y + z^2, x^2 - z + 1)$. Compute $D(f \circ g)(0)$ and $D(g \circ f)(0)$. Hint: Do not compute the compositions explicitly. Use the chain rule.
16. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ satisfy the conditions that $f(0) = (1, 2)$ and
$$Df(0) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$
If in addition $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g(x, y) = (x + 2y + 1, 3xy)$ then find $D(g \circ f)(0)$.
17. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. If $h(x, y) = f(x, y, g(x, y))$ then compute Dh in terms of the partial derivatives of f and g. If $h = 0$, then write $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in terms of the partial derivatives of f.