Math 4318 - Fall 2019
Homework 6

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 4, 7, 8, 9, 10, 11, 12.* **Due: In class on November 26**

1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called homogeneous of degree k if $f(tx) = t^k f(x)$. Show that for a such a function
$$Df(x)(x) = kf(x).$$
Hint: Take the derivative of $g(t) = f(tx)$.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy $\frac{\partial f}{\partial x_1}(x) = 0$ for all $x \in \mathbb{R}^n$. Show that $f(x)$ only depends on $x_2, \ldots, x_n$. (Recall we are writing x in coordinates as $x = (x_1, \ldots, x_n)$.)
3. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ satisfies $\|f(t)\| = 1$ for all t. Then show that $Df(t) \cdot f(t) = 0$ (where $\cdot$ is the dot product).
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $B = \{v \in \mathbb{R}^n : \|v\| \leq 1\}$. If f is differentiable on the interior of B and $f = 0$ on the boundary of B, then show that there is some point x_0 in the interior of B such that $Df(x_0) = 0$.
5. Compute the second order Taylor polynomial of $f(x, y) = e^x \cos y$ at $(0, 0)$.
6. Let $B : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear map. Show that the derivative of B at (x_0, y_0) is
$$DB(x_0, y_0)(x, y) = B(x_0, y) + B(x, y_0).$$
7. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Compute $D^2 L$.
8. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be two functions that are twice continuously differentiable. Show that
$$D^2(g \circ f)(x_0)(x, y) = D^2(g(f(x_0)))(Df(x_0)(x), Df(x_0)(y)) + Dg(f(x_0))D^2 f(x_0)(x, y).$$
9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x^2 - y^2, 2xy)$. Show that this function is locally invertible at all points $(x, y) \neq (0, 0)$. If we set $(u, v) = f(x, y)$ (that is $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$), then compute $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.
10. In an earlier homework assignment you showed that
$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
satisfies $f(0) = 0, f'(0) \neq 0$ but that f is not locally invertible near 0. How does this not contradict the inverse function theorem?
11. Given the system of equations
$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + 2x + 3z^2, \end{aligned}$$
can we always solve for x, y, z in terms of u, v, w near $(0, 0, 0)$? Explain.
12. Show that the equations
$$\begin{aligned} x^2 - y^2 - u^3 + v^2 + 4 &= 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0 \end{aligned}$$
determine functions $u(x, y)$ and $v(x, y)$ near $x = 2, y = -1$ such that $u(2, -1) = 2$ and $v(2, -1) = 1$. Compute $\frac{\partial u}{\partial x}$.