Math 4318 **Practice Problems for Midterm Exam 1** **Fall 2019**

1) Let $f : [a, b] \to \mathbb{R}$ and suppose that there is some M such that $|f'(x)| \leq M$. Prove using the definitions that f if Lipschitz and continuous on $[a, b]$.

2) Assuming that f' exists on $[a, b]$ and $\lim_{x \to c} f'(x) = L$ for some $c \in (a, b)$, prove that f' is continuous at c.

3) Let $f : [a, b] \to \mathbb{R}$ be an integrable function with $f(x) \geq 0$ for all $x \in [a, b]$.
a) If f is continuous at some $c \in (a, b)$ and $f(c) > 0$ show that

$$\int_a^b f(x)\, dx > 0.$$

b) If the set $C = \{x \in [a, b] : f(x) = 0\}$ has measure zero show that

$$\int_a^b f(x)\, dx > 0.$$

Hint: Does $[a, b]$ have measure zero?

4) Let $f : [0, 1] \to \mathbb{R}$ be the function that is 0 for all irrational numbers and $f(x) = x$ for all rational numbers. Prove that f is not integrable.
Hint: Show that the upper and lower Darboux integrals cannot be the same. Specifically show that any upper sum is bounded below by $\frac{1}{2}$.
5) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and set

$$F(x) = \int_a^x f(t)\, dt.$$

Prove that F is Lipschitz.

6) a) Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at all $x \in \mathbb{R}$ by computing the derivative. (Thought we did not prove it in class you may use, without proof, the fact that the derivative of $\sin x$ is $\cos x$ as well as other derivative rules you know.)
b) Show that f' is unbounded on $[0, 1]$.

7 a) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $f'(x) \geq 0$ for all $x \in [a, b]$. Show that f is increasing on $[a, b]$. (That is $f(x) \geq f(y)$ if $x > y$.)

b) If in addition $f'(x)$ is not identically zero on any sub-interval of $[a, b]$ then f is strictly increasing. (That is $f(x) > f(y)$ if $x > y$.)

8) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. For $c \in (a, b)$ we know that f restricted to $[a, c]$ and to $[c, b]$ gives an integrable function too (you do not have to prove this). Using the definition of the integral (either Riemann or Darboux) show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Hint: For any $\epsilon > 0$ choose a good partition that shows that both the right and left sides of the above equation are with ϵ of each other.

9) Are the following statements true or false (you don't have to justify your answer).

1. If $|f|$ is integrable on $[a, b]$ then f is integrable on $[a, b]$.
2. If f is not integrable on $[a, b]$ then there are partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$ such that $L(f, \mathcal{Q}) > U(f, \mathcal{P})$
3. If a function is differentiable on an open interval I then it is continuous on I.
4. Sets of measure zero must be countable.
5. If a function if differentiable on an open interval I then its derivative is continuous on I.
6. If a function has bounded derivative on an interval then it is uniformly continuous on the interval.
7. Every integrable function has an anti-derivative.
8. The set of integrable functions form a vector space.
9. The product of integrable functions is integrable.
10. The composition of integrable functions is integrable.